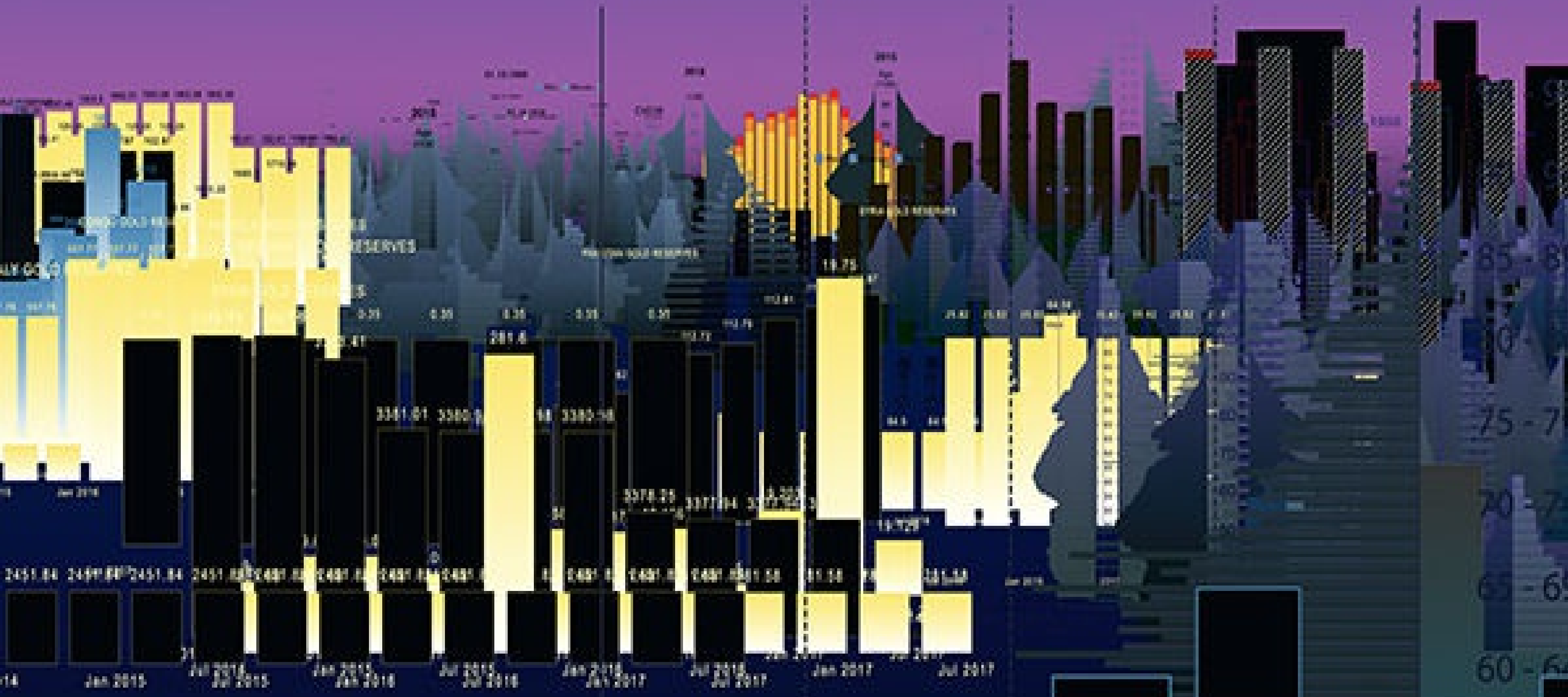
CHRIS BROOKS



INTRODUCTORY ECONOMETRICS FOR FINANCE

4TH EDITION

Introductory Econometrics for Finance

This bestselling and thoroughly classroom-tested textbook is a complete resource for finance students. A comprehensive and illustrated discussion of the most common empirical approaches in finance prepares students for using econometrics in practice, while detailed case studies help them understand how the techniques are used in relevant financial contexts. Learning outcomes, key concepts and end-of-chapter review questions (with full solutions online) highlight the main chapter takeaways and allow students to self-assess their understanding. Building on the successful data- and problem-driven approach of previous editions, this fourth edition has been updated with new examples, additional introductory material on mathematics and dealing with data, as well as more advanced material on extreme value theory, the generalised method of moments and state space models. A dedicated website, with numerous student and instructor resources including videos and a set of companion manuals for various statistical software – all available free of charge – completes the learning package.

CHRIS BROOKS is Professor of Finance at the ICMA Centre, Henley Business School, University of Reading, UK where he also obtained his PhD. Chris has diverse research interests and has published over a hundred articles in leading academic and practitioner journals, and six books. He is Associate Editor of several journals, including the *Journal of Business Finance and Accounting* and the *British Accounting Review*. He acts as consultant and advisor for various banks, corporations and regulatory and professional bodies in the fields of finance, real estate and econometrics.

Introductory Econometrics for Finance

FOURTH EDITION

CHRIS BROOKS

The ICMA Centre, Henley Business School, University of Reading

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Preface to the Fourth Edition

All of the motivations for the first edition, described below, seem just as important today. Given that the book seems to have gone down well with readers, I have left the style largely unaltered but added a lot of new material. The main motivations for writing the first edition of the book were:

- To write a book that focused on *using and applying* the techniques rather than deriving proofs and learning formulae.
- To write an accessible textbook that required no prior knowledge of econometrics, but which also covered more recently developed approaches usually only found in more advanced texts.
- To use examples and terminology from finance rather than economics since there are many introductory texts in econometrics aimed at students of economics but none for students of finance.
- To populate the book with case studies of the use of econometrics in practice taken from the academic finance literature.
- To include sample instructions, screen dumps and computer output from a popular econometrics package. This enabled readers to see how the techniques can be implemented in practice. In this fourth edition, the EViews instructions have been separated off and are available free of charge on the book's web site along with parallel manuals for other packages including Stata, Python and R.
- To develop a companion web site containing answers to end of chapter questions, a multiple choice question bank with feedback, PowerPoint slides and other supporting materials.

What is New in the Fourth Edition

The fourth edition includes a number of important new features

(1) Students of finance have enormously varying backgrounds, and in particular varying levels of training in elementary mathematics and statistics. In order to make the book more self-contained, the introductory chapter has again been expanded. So the material previously in Chapter 2 has been separated into introductory maths (Chapter 1) and introductory statistics/dealing with data (Chapter 2).

(2) More new material has been added on state space models and their estimation using the Kalman filter in Chapter 10.

(3) A chapter has been added which collects together a number of techniques often used in financial research, including event studies and the Fama MacBeth approach (previously elsewhere in the book) and new sections on using extreme value distribution to model the fat tails in financial series and on estimating models with the generalised method of moments.

(4) The incorporation of EViews directly into the core of the book may have been a distraction for those using other packages. Thus, as stated above, in the new edition the EViews instructions have been separated off and are available free of charge on the book's web site along with parallel manuals for other packages including Stata, Python and R. This package should ensure that the book fits the bill whatever the reader's preferred software.

Motivations for the First Edition

This book had its genesis in two sets of lectures given annually by the author at the ICMA Centre (formerly the ISMA Centre), Henley Business School, University of Reading and arose partly from several years of frustration at the lack of an appropriate textbook. In the past, finance was but a small sub-discipline drawn from economics and accounting, and therefore it was generally safe to assume that students of finance were well grounded in economic principles; econometrics would be taught using economic motivations and examples.

However, finance as a subject has taken on a life of its own in recent years. Drawn in by perceptions of exciting careers in the financial markets, the number of students of finance has grown phenomenally all around the world. At the same time, the diversity of educational backgrounds of students taking finance courses has also expanded. It is not uncommon to find undergraduate students of finance even without advanced high-school qualifications in mathematics or economics. Conversely, many with PhDs in physics or engineering are also attracted to study finance at the Masters level. Unfortunately, authors of textbooks failed to keep pace with the change in the nature of students. In my opinion, the currently available textbooks fall short of the requirements of this market in three main regards, which this book seeks to address

(1) Books fall into two distinct and non-overlapping categories: the introductory and the advanced. Introductory textbooks are at the appropriate level for students with limited backgrounds in mathematics or statistics, but their focus is too narrow. They often spend too long deriving the most basic results, and treatment of important, interesting and relevant topics (such as simulations methods, VAR modelling, etc.) is covered in only the last few pages, if at all. The more advanced textbooks, meanwhile, usually require a quantum leap in the level of mathematical ability assumed of readers, so that such books cannot be used on courses lasting only one or two semesters, or where students have differing backgrounds. In this book, I have tried to sweep a broad brush over a large number of different econometric techniques that are relevant to the analysis of financial and other data.

(2) Many of the currently available textbooks with broad coverage are too theoretical in nature and students can often, after reading such a book, still have no idea of how to tackle real-world problems themselves, even if they have mastered the techniques in theory. This book and the accompanying software manuals should assist students who wish to learn how to estimate models for themselves – for example, if they are required to complete a project or dissertation. Some examples have been developed especially for this book, while many others are drawn from the academic finance literature. In my opinion, this is an essential but rare feature of a textbook that should help to show students how econometrics is really applied. It is also hoped that this approach will encourage some students to delve deeper into the literature, and will give useful pointers and stimulate ideas for research projects. It should, however, be stated at the outset that the purpose of including examples from the academic finance print is not to provide a comprehensive overview of the literature or to discuss all of the relevant work in those areas, but rather to illustrate the techniques. Therefore, the literature reviews may be considered deliberately deficient, with interested readers directed to the suggested readings and the references therein.

(3) With few exceptions, almost all textbooks that are aimed at the introductory level draw their motivations and examples from economics, which may be of limited interest to students of finance or business. To see this, try motivating regression relationships using an example such as the effect of changes in income on consumption and watch your audience, who are primarily interested in business and finance applications, slip away and lose interest in the first ten minutes of your course.

Who Should Read this Book?

The intended audience is undergraduates or Masters/MBA and PhD students who require a broad knowledge of modern econometric techniques commonly employed in the finance literature. It is hoped that the book will also be useful for researchers (both academics and practitioners), who require an introduction to the statistical tools commonly employed in the area of finance. The book can be used for courses covering financial time-series analysis or financial econometrics in undergraduate or post-graduate programmes in finance, financial economics, securities and investments.

Although the applications and motivations for model-building given in the book are drawn from finance, the empirical testing of theories in many other disciplines, such as management studies, business studies, real estate, economics and so on, may usefully employ econometric analysis. For this group, the book may also prove useful.

Finally, while the present text is designed mainly for students at the undergraduate or Masters level, it could also provide introductory reading in financial modelling for finance doctoral programmes where students have backgrounds which do not include courses in modern econometric techniques.

Pre-Requisites for Good Understanding of This Material

In order to make the book as accessible as possible, no prior knowledge of statistics, econometrics or algebra is required, although those with a prior exposure to calculus, algebra (including matrices) and basic statistics will be able to progress more quickly. The emphasis throughout the book is on a valid application of the techniques to real data and problems in finance.

In the finance and investment area, it is assumed that the reader has knowledge of the fundamentals of corporate finance, financial markets and investment. Therefore, subjects such as portfolio theory, the capital asset pricing model (CAPM) and arbitrage pricing theory (APT), the efficient markets hypothesis, the pricing of derivative securities and the term structure of interest rates, which are frequently referred to throughout the book, are not explained from first principles in this text. There are very many good books available in corporate finance, in investments and in futures and options, including those by Brealey and Myers (2013), Bodie, Kane and Marcus (2014) and Hull (2017) respectively.

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I am grateful to Gita Persand, Olan Henry, James Chong and Apostolos Katsaris, who assisted with various parts of the software applications for the first edition. I am also grateful to Hilary Feltham for assistance with Chapters 1 and 2. I would also like to thank Simon Burke, James Chong and Con Keating for detailed and constructive comments on various drafts of the first edition, Simon Burke for suggestions on parts of the second edition, Mike Clements, Jo Cox, Eunyoung Mallet, Ogonna Nneji, Ioannis Oikonomou and Chardan Wese Simen for comments on part of the third edition and Marcel Prokopczuk for comments on part of the fourth edition.

I have additionally benefited from the comments, suggestions and questions of the following list of people, many of whom sent useful e-mails pointing out typos or inaccuracies: Zary Aftab, Panos Ballis-Papanastasiou, Mirco Balatti, Peter Burridge, Kyongwook Choi, Rishi Chopra, Araceli Ortega Diaz, Xiaoming Ding, Thomas Eilertsen, Waleid Eldien, Junjong Eo, Merlyn Foo, Andrea Gheno, Christopher Gilbert, Kimon Gomozias, Jan de Gooijer and his colleagues, Cherif Guermat, Abid Hameed, Ibrahim Jamali, Kejia Jia, Arty Khemlani, Margaret Lynch, David McCaffrey, Tehri Jokipii, Emese Lazar, Zhao Liuyan, Dimitri Lvov, Bill McCabe, Junshi Ma, Raffaele Mancuso, David Merchan, Yue Min, Victor Murinde, Kyoung Gook Park, Mikael Petitjean, Marcelo Perlin, Thai Pham, Jean-Sebastien Pourchet, Marcel Prokopczuk, Tao Qingmei, Satya Sahoo, Lisa Schopohl, Guilherme Silva, Jerry Sin, Andre-Tudor Stancu, Silvia Stanescu, Fred Sterbenz, Birgit Strikholm, Yiguo Sun, Li Qui, Panagiotis Varlagas, Jakub Vojtek, Henk von Eije, Jue Wang, Robert Wichmann and Meng-Feng Yen.

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Outline of the Remainder of this Book

Chapter 1

This covers the key mathematical techniques that readers will need some familiarity with to be able to get the most out of the remainder of this book. It starts with a discussion of what econometrics is about and how to set up an econometric model, then moves on to present the mathematical material on functions, and powers, exponents and logarithms of numbers. It then proceeds to explain the basics of differentiation and matrix algebra, which is illustrated via the construction of optimal portfolio weights.

Chapter 2

This chapter presents the statistical foundations of econometrics and the beginnings of how to work with financial data. It covers key results in statistics, discusses probability distributions, how to summarise data and different types of data. The chapter then moves on to discuss the calculation of present and future values, compounding and discounting, and how to calculate nominal and real returns in various ways.

Chapter 3

This introduces the classical linear regression model (CLRM). The ordinary least squares (OLS) estimator is derived and its interpretation discussed. The conditions for OLS optimality are stated and explained. A hypothesis testing framework is developed and examined in the context of the linear model. Examples employed include Jensen's classic study of mutual fund performance measurement and tests of the 'overreaction hypothesis' in the context of the UK stock market.

Chapter 4

This continues and develops the material of Chapter 3 by generalising the bivariate model to multiple regression – i.e., models with many variables. The framework for testing multiple hypotheses is outlined, and measures of how well the model fits the data are described. Case studies include modelling rental values and an application of principal components analysis (PCA) to interest rates.

Chapter 5

Chapter 5 examines the important but often neglected topic of diagnostic testing. The consequences of violations of the CLRM assumptions are described, along with plausible remedial steps. Model-building philosophies are discussed, with particular reference to the general-to-specific approach. Applications covered in this chapter include the determination of sovereign credit ratings.

Chapter 6

This presents an introduction to time-series models, including their motivation and a description of the characteristics of financial data that they can and cannot capture. The chapter commences with a presentation of the features of some standard models of stochastic (white noise, moving average, autoregressive and mixed ARMA) processes. The chapter continues by showing how the appropriate model can be chosen for a set of actual data, how the model is estimated and how model adequacy checks are performed. The generation of forecasts from such models is discussed, as are the criteria by which these forecasts can be evaluated. Examples include model-building for UK house prices, and tests of the exchange rate covered and uncovered interest parity hypotheses.

Chapter 7

This extends the analysis from univariate to multivariate models. Multivariate models are motivated by way of explanation of the possible existence of bi-directional causality in financial relationships, and the simultaneous equations bias that results if this is ignored. Estimation techniques for simultaneous equations models are outlined. Vector autoregressive (VAR) models, which have become extremely popular in the empirical finance literature, are also covered. The interpretation of VARs is explained by way of joint tests of restrictions, causality tests, impulse responses and variance decompositions. Relevant examples discussed in this chapter are the simultaneous relationship between bid–ask spreads and trading volume in the context of options pricing, and the relationship between property returns and macroeconomic variables.

Chapter 8

The first section of the chapter discusses unit root processes and presents tests for non-stationarity in time-series. The concept of and tests for cointegration, and the formulation of error correction models, are then discussed in the context of both the single equation framework of Engle–Granger, and the multivariate framework of Johansen. Applications studied in Chapter 8 include spot and futures markets, tests for cointegration between international bond markets and tests of the purchasing power parity (PPP) hypothesis and of the expectations hypothesis of the term structure of interest rates.

Chapter 9

This covers the important topic of volatility and correlation modelling and forecasting. This chapter starts by discussing in general terms the issue of non-linearity in financial time series. The class of ARCH (autoregressive conditionally heteroscedastic) models and the motivation for this formulation are then discussed. Other models are also presented, including extensions of the basic model such as GARCH, GARCH-M, EGARCH and GJR formulations. Examples of the huge number of applications are discussed, with particular reference to stock returns. Multivariate GARCH and conditional correlation models are described, and applications to the estimation of conditional betas and time-varying hedge ratios, and to financial risk measurement, are given.

Chapter 10

This begins by discussing how to test for and model regime shifts or switches of behaviour in financial series that can arise from changes in government policy, market trading conditions or microstructure, among other causes. This chapter then introduces the Markov switching approach to dealing with regime shifts. Threshold autoregression is also discussed, along with issues relating to the estimation of such models. Examples include the modelling of exchange rates within a managed floating environment, modelling and forecasting the gilt–equity yield ratio and models of movements of the difference between spot and futures prices. Finally, the second part of the chapter moves on to examine how to specify models with time-varying parameters using the state space form and how to estimate them with the Kalman filter.

Chapter 11

This chapter focuses on how to deal appropriately with longitudinal data – that is, data having both time-series and cross-sectional dimensions. Fixed effect and random effect models are explained and illustrated by way of examples on banking competition in the UK and on credit stability in Central and Eastern Europe. Entity fixed and time-fixed effects models are elucidated and distinguished.

Chapter 12

This chapter describes various models that are appropriate for situations where the dependent variable is not continuous. Readers will learn how to construct, estimate and interpret such models, and to distinguish and select between alternative specifications. Examples used include a test of the pecking order hypothesis in corporate finance and the modelling of unsolicited credit ratings.

Chapter 13

This presents an introduction to the use of simulations in econometrics and finance. Motivations are given for the use of repeated sampling, and a distinction is drawn between Monte Carlo simulation and bootstrapping. The reader is shown how to set up a simulation, and examples are given in options pricing and financial risk management to demonstrate the usefulness of these techniques.

Chapter 14

This chapter presents a collection of techniques that are particularly useful for conducting research in finance. It begins with detailed illustrations of how to conduct event studies, which are commonly used in corporate finance applications, and how to use the Fama-French factor model approach to asset pricing. The chapter then proceeds to present the families of extreme value models that are used to accurately capture the fat tails of asset return distributions and as the basis for value at risk calculations. Finally, the chapter covers the generalised method of moments (GMM) technique, which has become increasingly popular in recent years for estimating a range of different types of models in finance.

Chapter 15

This offers suggestions related to conducting a project or dissertation in empirical finance. It introduces the sources of financial and economic data available on the internet and elsewhere, and recommends relevant online information and literature on research in financial markets and financial time series. The chapter also suggests ideas for what might constitute a good structure for a dissertation on this subject, how to generate ideas for a suitable topic, what format the report could take, and some common pitfalls.

1

Introduction and Mathematical Foundations

LEARNING OUTCOMES

In this chapter, you will learn how to

- Describe the key steps involved in building an econometric model
- Work with powers, exponents and logarithms
- Plot, interpret and calculate the roots of functions
- Use sigma (Σ) and pi (Π) notation
- Apply rules to differentiate various types functions
- Work with matrices
- Calculate the trace, inverse and eigenvalues of a matrix
- Construct and interpret utility functions

Learning econometrics is in many ways like learning a new language. To begin with, nothing makes sense and it is as if it is impossible to see through the fog created by all the unfamiliar terminology. While the way of writing the models – the *notation* – may make the situation appear more complex, in fact it is supposed to achieve the exact opposite. The ideas themselves are mostly not so complicated, it is just a matter of learning enough of the language that everything fits into place. So if you have never studied the subject before, then persevere through this preliminary chapter and you will hopefully be on your way to being fully fluent in econometrics!

This chapter comprises two parts. The first sets the scene for the book by discussing in broad terms the questions of what econometrics is, and the kinds of problems that can be tackled using econometrics. The second part of the chapter covers the mathematical techniques that underpin approaches to modelling and dealing with data in finance. Those with some prior background in algebra and introductory mathematics may skip the second part of this chapter without loss of continuity, but hopefully the material will also constitute a useful refresher for those who have studied mathematics but a long time ago!

1.1 What is Econometrics?

The literal meaning of the word 'econometrics' is 'measurement in economics'. The first five letters of the word suggest correctly that the origins of econometrics are rooted in economics. However, the main techniques employed for studying economic problems are of equal importance in financial applications. As the term is used in this book, financial econometrics will be defined as the *application of statistical techniques to problems in finance.* Financial econometrics can be useful for testing theories in finance, determining asset prices or returns, testing hypotheses concerning the relationships between variables, examining the effect on financial markets of changes in economic conditions, forecasting future values of financial variables and for financial decision-making. A list of possible examples of where econometrics may be useful is given in Box 1.1.

BOX 1.1 Examples of the uses of econometrics

(1) Testing whether financial markets are weak-form informationally efficient

(2) Testing whether the capital asset pricing model (CAPM) or arbitrage pricing theory (APT) represent superior models for the determination of returns on risky assets

(3) Measuring and forecasting the volatility of bond returns

(4) Explaining the determinants of bond credit ratings used by the ratings agencies

(5) Modelling long-term relationships between prices and exchange rates

(6) Determining the optimal hedge ratio for a spot position in oil

(7) Testing technical trading rules to determine which makes the most money

(8) Testing the hypothesis that earnings or dividend announcements have no effect on stock prices

(9) Testing whether spot or futures markets react more rapidly to news

(10) Forecasting the correlation between the stock indices of two countries.

The list in Box 1.1 is of course by no means exhaustive, but it hopefully gives some flavour of the usefulness of econometric tools in terms of their financial applicability.

1.2 Is Financial Econometrics Different from 'Economic Econometrics'?

As previously stated, the tools commonly used in financial applications are fundamentally the same as those used in economic applications, although the emphasis and the sets of problems that are likely to be encountered when analysing the two sets of data are somewhat different. Financial data often differ from macroeconomic data in terms of their frequency, accuracy, seasonality and other properties.

In economics, a serious problem is often a *lack of data at hand* for testing the theory or hypothesis of interest – this is sometimes called a 'small samples problem'. It might be, for example, that data are required on government budget deficits, or population figures, which are measured only on an annual basis. If the methods used to measure these quantities changed a quarter of a century ago, then only at most twenty-five of these annual observations are usefully available.

Two other problems that are often encountered in conducting applied econometric work in the arena of economics are those of *measurement error* and *data revisions*. These difficulties are simply that the data may be estimated, or measured with error, and will often be subject to several vintages of subsequent revisions. For example, a researcher may estimate an economic model of the effect on national output of investment in computer technology using a set of published data, only to find that the data for the last two years have been revised substantially in the next, updated publication.

These issues are usually of less concern in finance. Financial data come in many shapes and forms, but in general the prices and other entities that are recorded are those at which trades *actually took place*, or which were *quoted* on the screens of information providers. There exists, of course, the possibility for typos or for the data measurement method to change (for example, owing to stock index re-balancing or re-basing). But in general the measurement error and revisions problems are far less serious in the financial context.

Similarly, some sets of financial data are observed at much *higher frequencies* than macroeconomic data. Asset prices or yields are often available at daily, hourly or minute-by-minute frequencies. Thus the number of observations available for analysis can potentially be very large – perhaps thousands or even millions, making financial data the envy of macro-econometricians! The implication is that more powerful techniques can often be applied to financial than economic data, and that researchers may also have more confidence in the results.

Furthermore, the analysis of financial data also brings with it a number of new problems. While the difficulties associated with handling and processing such a large amount of data are not usually an issue given recent and continuing advances in computer power, financial data often have a number of additional characteristics. For example, financial data are often considered very 'noisy', which means that it is more difficult to separate *underlying trends or patterns* from random and uninteresting features. Financial data are also almost always not normally distributed in spite of the fact that most techniques in econometrics assume that they are. High frequency data often contain additional 'patterns' which are the result of the way that the market works, or the way that prices are recorded. These features need to be considered in the model-building process, even if they are not directly of interest to the researcher.

One of the most rapidly evolving areas of financial application of statistical tools is in the modelling of market microstructure problems. 'Market microstructure' may broadly be defined as the process whereby *investors' preferences and desires are translated into financial market transactions*. It is evident that microstructure effects are important and represent a key difference between financial and other types of data. These effects can potentially impact on many other areas of finance. For example, market rigidities or frictions can imply that current asset prices do not fully reflect future expected cashflows (see the discussion in Chapter 10 of this book). Also, investors are likely to require compensation for holding securities that are illiquid, and therefore embody a risk that they will be difficult to sell owing to the relatively high probability of a lack of willing purchasers at the time of desired sale. Measures such as volume or the time between trades are sometimes used as proxies for market liquidity.

A comprehensive survey of the literature on market microstructure is given by Madhavan (2000). He identifies several aspects of the market microstructure literature, including price formation and price discovery, issues relating to market structure and design, information and disclosure. There are also relevant books by O'Hara (1995), Harris (2002) and Hasbrouck (2007). At the same time, there has been considerable advancement in the sophistication of econometric models applied to microstructure problems. For example, an important innovation was the auto-regressive conditional duration (ACD) model attributed to Engle and Russell (1998). An interesting application can be found in Dufour and Engle (2000), who examine the effect of the time between trades on the price-impact of the trade and the speed of price adjustment.

1.3 Steps Involved in Formulating an Econometric Model

Although there are of course many different ways to go about the process of model-building, a logical and valid approach would be to follow the steps described in Figure 1.1.

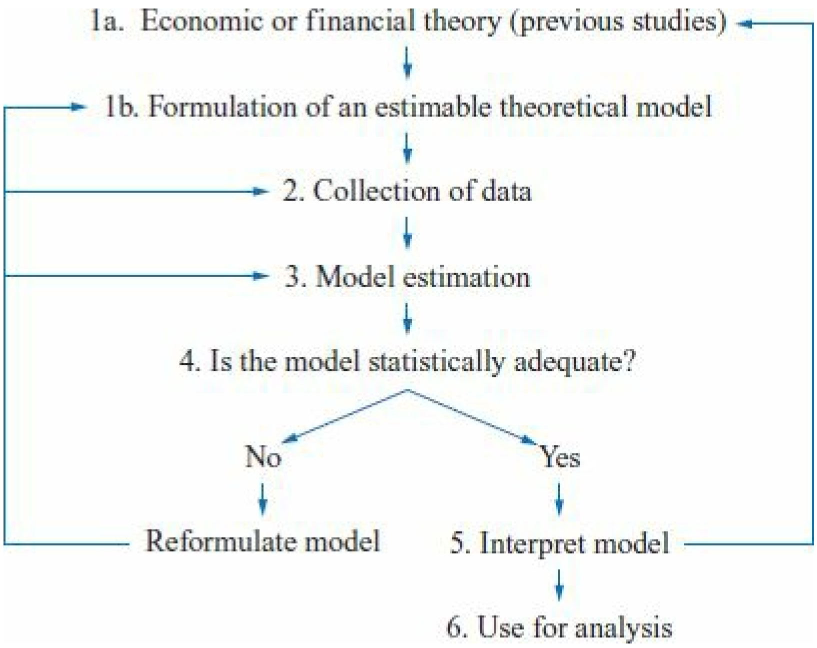


Figure 1.1 Steps involved in formulating an econometric model

The steps involved in the model construction process are now listed and described. Further details on each stage are given in subsequent chapters of this book.

- *Steps 1a and 1b: general statement of the problem* This will usually involve the formulation of a theoretical model, or intuition from financial theory that two or more variables should be related to one another in a certain way. The model is unlikely to be able to completely capture every relevant real-world phenomenon, but it should present a sufficiently good approximation that it is useful for the purpose at hand.
- *Step 2: collection of data relevant to the model* The data required may be available electronically through a financial information provider, such as Reuters or from published government figures. Alternatively, the required data may be available only via a survey after distributing a set of questionnaires, i.e., *primary* data.
- *Step 3: choice of estimation method relevant to the model proposed in step 1* For example, is a single equation or multiple equation technique to be used?
- *Step 4: statistical evaluation of the model* What assumptions were required to estimate the parameters of the model optimally? Were these assumptions satisfied by the data or the model? Also, does the model adequately describe the data? If the answer is 'yes', proceed to step 5; if not, go back to steps 1–3 and either reformulate the model, collect more data, or select a different estimation technique that has less stringent requirements.
- *Step 5: evaluation of the model from a theoretical perspective* Are the parameter estimates of the sizes and signs that the theory or intuition from step 1 suggested? If the answer is 'yes', proceed to step 6; if not, again return to stages 1–3.
- *Step 6: use of the model* When a researcher is finally satisfied with the model, it can then be used for testing the theory specified in step 1, or for formulating forecasts or suggested courses of action. This suggested course of action might be for an individual (e.g., 'if inflation and GDP rise, buy stocks in sector *X*'), or as an input to government policy (e.g., 'when equity markets fall, program trading causes excessive volatility and so should be banned').

It is important to note that the process of building a robust empirical model is an iterative one, and it is certainly not an exact science. Often, the final preferred model could be very different from the one originally proposed, and need not be unique in the sense that another researcher with the same data and the same initial theory could arrive at a different final specification.

1.4 Points to Consider When Reading Articles in Empirical Finance

As stated above, one of the defining features of this book relative to others in the area is in its use of published academic research as examples of the use of the various techniques. The papers examined have been chosen for a number of reasons. Above all, they represent (in this author's opinion) a clear and specific application in finance of the techniques covered in this book. They were also required to be published in a peer-reviewed journal, and hence to be widely available.

When I was a student, I used to think that research was a very pure science. Now, having had first-hand experience of research that academics and practitioners do, I know that this is not the case. Researchers often cut corners. They have a tendency to exaggerate the strength of their results, and the importance of their conclusions. They also have a tendency not to bother with tests of the adequacy of their models, and to gloss over or omit altogether any results that do not conform to the point that they wish to make. Therefore, when examining papers from the academic finance literature, it is important to cast a very critical eye over the research – rather like a referee who has been asked to comment on the suitability of a study for a scholarly journal. The questions that are always worth asking oneself when reading a paper are outlined in Box 1.2.

BOX 1.2 Points to consider when reading a published paper

(1) Does the paper involve the development of a theoretical model or is it merely a technique looking for an application so that the motivation for the whole exercise is poor?

(2) Are the data of 'good quality'? Are they from a reliable source? Is the size of the sample sufficiently large for the model estimation task at hand?

(3) Have the techniques been validly applied? Have tests been conducted for possible violations of any assumptions made in the estimation of the model?

(4) Have the results been interpreted sensibly? Is the strength of the results exaggerated? Do the results actually obtained relate to the questions posed by the author(s)? Can the results be replicated by other researchers?

(5) Are the conclusions drawn appropriate given the results, or has the importance of the results of the paper been overstated?

Bear these questions in mind when reading my summaries of the articles used as examples in this book and, if at all possible, seek out and read the entire articles for yourself.

This chapter now moves on to cover the fundamental mathematical framework that underpins financial econometrics. This material is intended as a refresher for readers who have covered these topics in the past but require a reminder; students who are seeing these concepts for the first time may find a more thorough treatment covering an entire book useful in addition to this text – see, for example Renshaw (2016) or Swift and Piff (2014), which are both detailed and very accessible.

1.5 Functions

1.5.1 Introduction to Functions

The ultimate objective of econometrics is usually to build a model, which may be thought of as a simplified version of the true relationship between two or more variables that can be described by a *function*. A function is simply a mapping or relationship between an input or set of inputs and an output. We usually write that y, the output, is a function f of x, the input, so $y = f(x)$. $f(.)$ is simply a general method of stating that y is related to x in some fashion. Another way to say this is that f provides a mapping between y and x so that it tells us, for every given value of x, what the corresponding value of y would be. f is a unique (1:1) mapping so that for each value of x there is only one corresponding value of y.

The *domain* of x is defined as the set of values that this variable can take; the *range* refers to the respective set of values that y can take. Usually, neither the domain nor the range are specified, in which case they can both be assumed to be allowed to take any real values.

1.5.2 Straight Lines

y could be a linear function of x, where the relationship can be expressed as a straight line on a graph, or y could be a non-linear function of x, in which case the relationship between the two variables would be represented graphically as a curve. If the relationship is linear, we could write the equation for this straight line as

$$y = a + bx \tag{1.1}$$

y and x are called *variables*, while a and b are *parameters*; a is termed the *intercept* and b is the *slope* or *gradient* of the line. The intercept is the point at which the line crosses the y-axis, while the slope measures the steepness of the line. Note that there will be only one value of a and one value of b, although there will be many values of x and of y. a and b could each be any combination of positive, negative or zero.

To illustrate, suppose we were trying to model the relationship between a student's grade-point average y (expressed as a percentage), and the number of hours that they studied throughout the year, x. Suppose further that the relationship can be written as a linear function with $y = 25 + 0.05x$.

Clearly it is unrealistic to assume that the link between grades and hours of study follows a straight line, but let us keep this assumption for now. So the intercept of the line, a, is 25, and the slope, b, is 0.05. What does this equation mean? It means that a student spending no time studying at all ($x = 0$) could expect to earn a 25% average grade, and for every hour of study time, their average grade should improve by 0.05% – in other words, an extra 100 hours of study through the year would lead to a 5% increase in the grade.

Suppose that a particular student wished to score a perfect 100% grade-point average. How many hours would (s)he need to study? To calculate this, we would need to set $y = 100$ and then to solve for x: $100 = 25 + 0.05x$, so $x = 1500$ hours. We could construct a table with several values of x and the corresponding value of y as in Table 1.1 and then plot them onto a graph (Figure 1.2).

Table 1.1 Sample data on hours of study and grades

Hours of study (*x*)	**Grade-point average in % (y)**
0	25
100	30
400	45

800	65
1000	75
1200	85

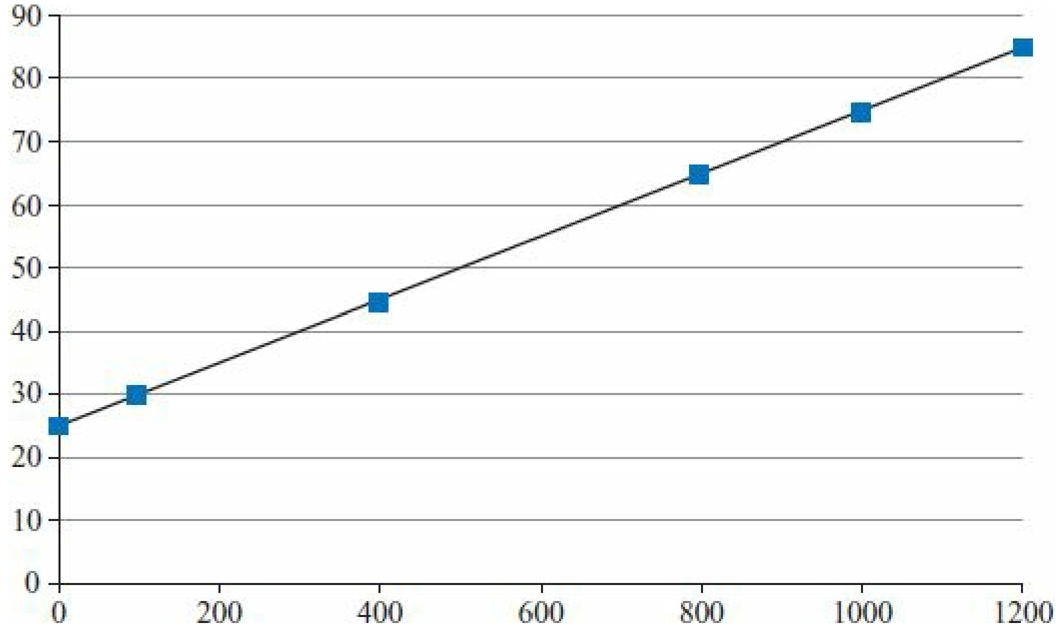


Figure 1.2 A plot of hours studied (x) against grade-point average (y)

We can see from the graph that the gradient of this line is positive (i.e., it slopes upwards from left to right). Note that for a straight line, the slope is the same along the whole line; this slope can be calculated from a graph by taking any two points on the line and dividing the change in the value of y by the change in the value of x between the two points.

In general, a capital delta, Δ, is used to denote a change in a variable. For example, suppose that we want to take the two points $x = 100$, $y = 30$ and $x = 1000$, $y = 75$. We could write these two points using a coordinate notation (x,y) and so (100,30) and (1000,75) in this example. We would calculate the slope of the line as

$$\frac{\Delta y}{\Delta x} = \frac{75 - 30}{1000 - 100} = 0.05 \quad \textbf{(1.2)}$$

So indeed, we have confirmed that the slope is 0.05 (although in this case we knew that from the start). Two other examples of straight line graphs are given in Figure 1.3. The gradient of the line can be zero or negative instead of positive. If the gradient is zero, the resulting plot will be a flat (horizontal) straight line. We could then write it as $y = 25 + 0x$, so that whatever the value of x, y will always be the same (25).

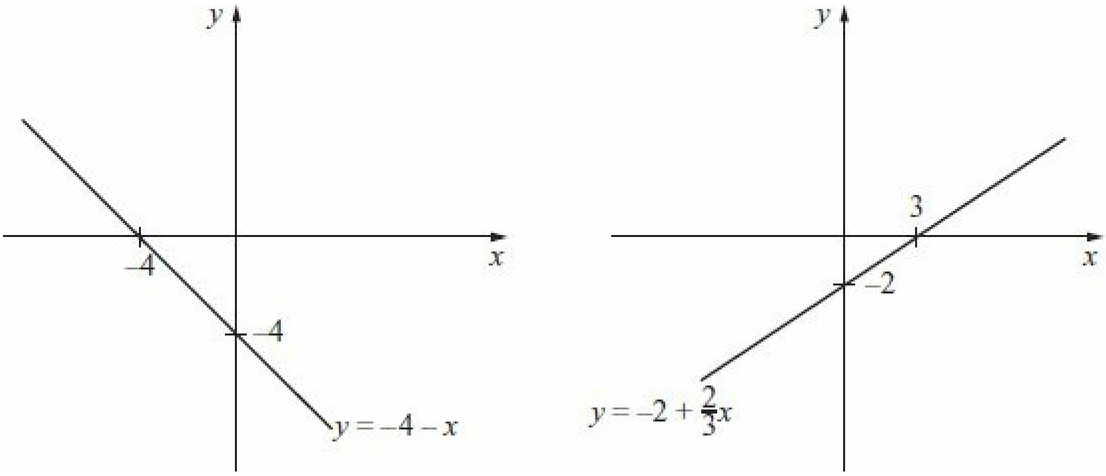


Figure 1.3 Examples of different straight line graphs

If there is a specific change in x, Δx, and we want to calculate the corresponding change in y, we would simply multiply the change in x by the slope, so $\Delta y = b\Delta x$.

As a final point, note that we stated above that the point at which a function crosses the y-axis is termed the intercept. The point at which the function crosses the x-axis is called its *root*. In the example above, if we take the function $y = 25 + 0.05x$, set y to zero and rearrange the equation, we would find that the root would be $x = -500$. In this case, the root of the equation does not have a useful interpretation (as the number of hours studied cannot be negative) but this will not always be the case.

The equation for a straight line has one root (except for a horizontal straight line such as $y = 4$, where there would be no root since it never crosses the x-axis). Further examples of how to calculate the roots of an equation will be given in Section 1.5.3.

1.5.3 Polynomial Functions

A linear function is often not sufficiently flexible to be able to accurately describe the relationship between two variables, and so a quadratic function may be used instead. A *polynomial* simply adds higher order powers of the variable x into the function. In the most general case, we would have an n^{th} order polynomial (a polynomial of order n)

$$y = a + b_1 x + b_2 x^2 + b_3 x^3 + \cdots + b_n x^n \quad \textbf{(1.3)}$$

If $n = 2$, we have a quadratic equation, if $n = 3$ a cubic, if $n = 4$ a quartic and so on. We use polynomials if y depends only on one variable x but in a non-linear way (and so it cannot be expressed as a straight line). An example of the shape of a general polynomial function is given in Figure 1.4.

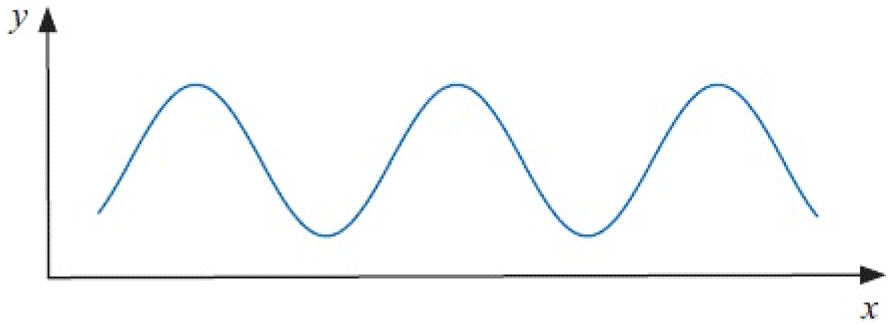


Figure 1.4 Example of a general polynomial function

Broadly, the higher the order of the polynomial, the more complex will be the relationship between y and x and the more twists and turns there will be in the plot like Figure 1.4. However, usually $n = 2$, a quadratic equation, is sufficient to describe the function as it seems unlikely that a real series y will rise with x then fall before rising again and so on, which would be the case if it was described by a higher order polynomial. So now we will focus on the quadratic case.

We could write the general expression for a quadratic function as

$$y = a + bx + cx^2 \tag{1.4}$$

where x and y are again the variables and a, b, c are the parameters that describe the shape of the function. Note that we have changed notation slightly for simplicity between equations (1.3) and (1.4), writing the slope parameters as b and c rather than b_1 and b_2. Either notation is equally acceptable so long as we are clear and explain what we mean.

A linear function only has two parameters (the intercept, a and the slope, b), but a quadratic has three and hence it is able to adapt to a broader range of relationships between y and x. The linear function is a special case of the quadratic where c is zero. As before, a is the intercept and defines where the function crosses the y-axis; the parameters b and c determine the shape.

Quadratic equations can be either ∪-shaped or ∩-shaped. As x becomes very large and positive or very large and negative, the x^2 term will dominate the behaviour of y and it is thus c that determines which of these shapes will apply. Figure 1.5 shows two examples of quadratic functions – in the first case c is positive and so the curve is ∪-shaped, while in the second c is negative so the curve is ∩-shaped. We discussed above that the root(s) of an equation is (are) the place(s) where the line crosses the x-axis. Box 1.3 discusses the features of the roots of a quadratic equation and shows how to calculate them.

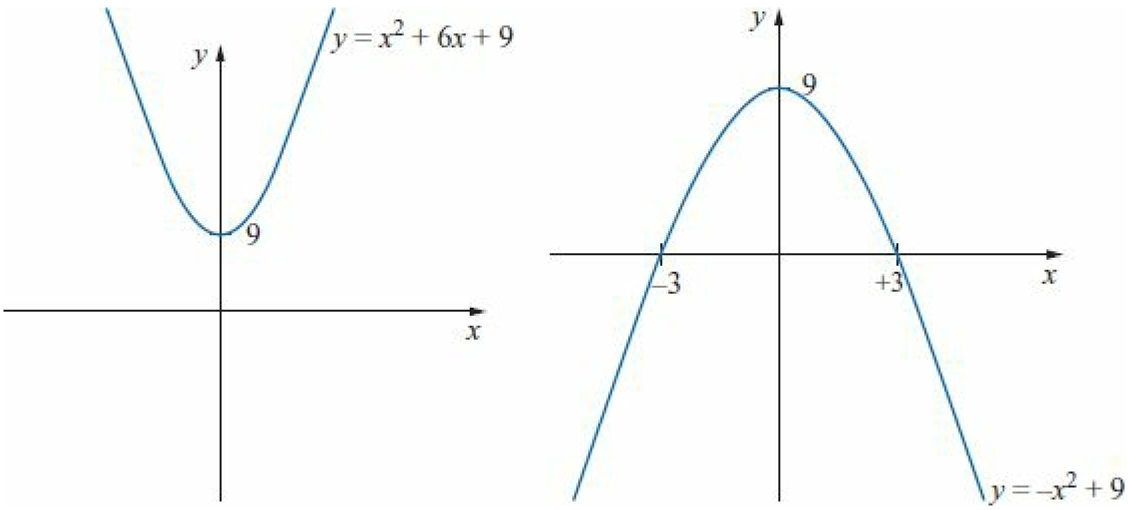


Figure 1.5 Examples of quadratic functions

BOX 1.3 The roots of a quadratic equation

- A quadratic equation has two roots
- The roots may be distinct (i.e., different from one another), or they may be the same (repeated roots); they may be real numbers (e.g., 1.7, −2.357, 4, etc.) or what are known as *complex numbers*
- The roots can be obtained either by *factorising* the equation – i.e., contracting it into parentheses, by ‘completing the square’ or by using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c} \quad \textbf{(1.5)}$$

- If $b^2 > 4ac$, the function will have two unique roots and it will cross the x-axis in two separate places; if $b^2 = 4ac$, the function will have two equal roots and it will only cross the x-axis in one place; if $b^2 < 4ac$, the function will have no real roots (only complex roots), it will not cross the x-axis at all and thus the function will always be above the x-axis.

EXAMPLE 1.1

Determine the roots of the following quadratic equations

1. $y = x^2 + x - 6$
2. $y = 9x^2 + 6x + 1$
3. $y = x^2 - 3x + 1$
4. $y = x^2 - 4x$

SOLUTION We would solve these equations by setting them in turn to zero. We could then use the quadratic formula from equation (1.5) in each case, although it is usually quicker to determine first whether they factorise (see Box 1.3).

1. $x^2 + x - 6 = 0$ factorises to $(x - 2)(x + 3) = 0$ and thus the roots are 2 and −3, which are the values of x that set the function to zero. In other words, the function will cross the x-axis at $x = 2$ and $x = -3$.
2. $9x^2 + 6x + 1 = 0$ factorises to $(3x + 1)(3x + 1) = 0$ and thus the roots are $-\frac{1}{3}$ and $-\frac{1}{3}$. This is known as repeated roots – since this is a quadratic equation there will always be two roots but in this case they are both the same. We call the expression $9x^2 + 6x + 1$ a *perfect square*. Here the plot of y against x would touch, but not cross, the x-axis at $x = -\frac{1}{3}$.
3. $x^2 - 3x + 1 = 0$ does not factorise and so the formula must be used with $a = 1$, $b = -3$, $c = 1$ and the roots are 0.38 and 2.62 to two decimal places.
4. $x^2 - 4x = 0$ factorises to $x(x - 4) = 0$ and so the roots are 0 and 4. The function crosses the x-axis at the points (0,0) and (4,0).

Note that all of these equations have two real roots. If we had an equation such as $y = 3x^2 - 2x + 4$, this would not factorise and would have complex roots since $b^2 - 4ac < 0$ in the quadratic formula. A similar situation is illustrated in the lefthand part of Figure 1.5, which does not cross the x-

axis anywhere.

1.5.4 Powers of Numbers or of Variables

A number or variable raised to a power is simply a way of writing repeated multiplication. So, for example, raising x to the power 2 means squaring it (i.e., $x^2 = x \times x$); raising it to the power 3 means cubing it ($x^3 = x \times x \times x$), and so on. The number that we are raising the number or variable to is called the *index*, so for x^3, 3 would be the index. There are a few rules for manipulating powers and their indices given in Box 1.4.

BOX 1.4 Manipulating powers and their indices

- Any number or variable raised to the power one is simply that number or variable, e.g., $3^1 = 3$, $x^1 = x$, and so on.
- Any number or variable raised to the power zero is one, e.g., $5^0 = 1$, $x^0 = 1$, etc., except that 0^0 is not defined (i.e., it does not exist).
- If the index is a negative number, this means that we divide one by that number – for example, $x^{-3} = \frac{1}{x^3} = \frac{1}{x \times x \times x}$.
- If we want to multiply together a given number raised to more than one power, we would add the corresponding indices together – for example, $x^2 \times x^3 = x^2x^3 = x^{2+3} = x^5$. The general rule is $x^a \times x^b = x^{a+b}$.
- If we want to calculate the power of a variable raised to a power (i.e., the power of a power), we would multiply the indices together – for example, $(x^2)^3 = x^{2\times 3} = x^6$. The general rule is $(x^a)^b = x^{a\times b}$.
- If we want to divide a variable raised to a power by the same variable raised to another power, we subtract the second index from the first – for example, $\frac{x^3}{x^2} = x^{3-2} = x$. The general rule is $\frac{x^a}{x^b} = x^{a-b}$.
- If we want to divide a variable raised to a power by a different variable raised to the same power, the following result applies

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}.$$

- The power of a product is equal to each component raised to that power – for example, $(x \times y)^3 = x^3 \times y^3$.
- It is important to note that the indices for powers do not have to be

integers. For example, $x^{\frac{1}{2}}$ is the notation we would use for taking the square root of *x*, sometimes written $\sqrt{x}$. Other, non-integer powers are also possible, but are harder to calculate by hand (e.g., $x^{0.76}$, $x^{-0.27}$, etc.) In general, $x^{1/n} = \sqrt[n]{x}$, the *n*th root of *x*.

1.5.5 The Exponential Function

It is sometimes the case that the relationship between two variables is best described by an *exponential* function – for example, when a variable *y* grows (or reduces) at a rate in proportion to its current value *x*, in which case we would write $y = e^x \cdot e$ is a simply number: 2.71828…In fact, *e* can be derived by letting *n* in the following expression tend towards infinity

$$e \approx \left(1 + \frac{1}{n}\right)^n \tag{1.6}$$

Alternatively, we can define *e* as the result from the following infinite sum

$$e = \sum_{i=0}^{\infty} \frac{1}{i!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{6} + \frac{1}{24} + \dots \tag{1.7}$$

where ! denotes a factorial (e.g., 4! = 4 × 3 × 2 × 1).

The exponential function has several useful properties, including that it is its own derivative (see Section 1.6.1 below) and thus the gradient of the function e^x at any point is also e^x; it is also useful for capturing the increase in value of an amount of money that is subject to compound interest. The exponential function can never be negative, so when *x* is negative, *y* is close to zero but positive. It crosses the *y*-axis at one and the slope increases at an increasing rate from left to right, as shown in Figure 1.6.

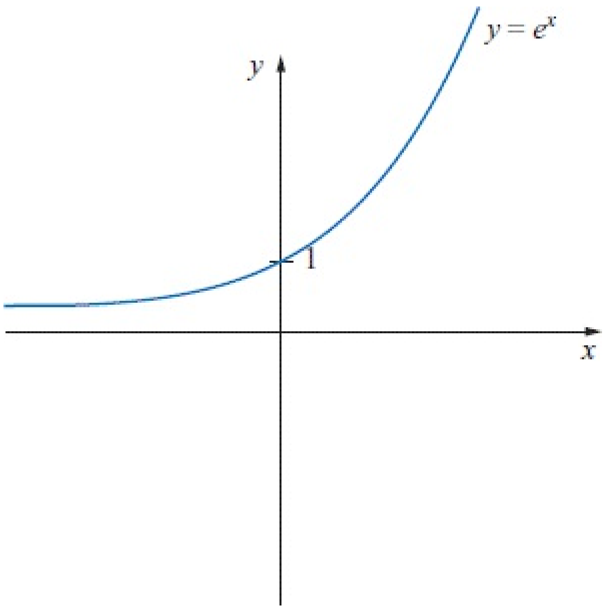


Figure 1.6 A plot of an exponential function

1.5.6 Logarithms

Logarithms were invented before computers and pocket calculators were widely available to simplify cumbersome calculations, since exponents can then be added or subtracted, which is easier than multiplying or dividing the original numbers. While logarithmic transformations are no longer necessary for computational ease, they still have important uses in algebra and in data analysis. For the latter, there are at least three reasons why log transforms may be useful. First, taking a logarithm can often help to rescale the data so that their variance is more constant, which overcomes a common statistical problem known as *heteroscedasticity*, discussed in detail in Chapter 5. Second, logarithmic transforms can help to make a positively skewed distribution closer to a normal distribution. Third, taking logarithms can also be a way to make a non-linear, multiplicative relationship between variables into a linear, additive one. These issues will also be discussed in some detail in Chapter 5.

To motivate how logs work, consider the power relationship $2^3 = 8$. Using logarithms, we would write this as $\log_2 8 = 3$, or 'the log to the base 2 of 8 is 3'. Hence we could say that a logarithm is defined as the power to which the base must be raised to obtain the given number. More generally, if $a^b = c$, then we can also write $\log_a c = b$.

Natural logarithms, also known as logs to base e, are more commonly used and more useful mathematically than logs to any other base. A log to base e is known as a *natural* or *Napierian* logarithm, denoted interchangeably by ln(y) or log(y). Taking a natural logarithm is the inverse of a taking an exponential, so sometimes the exponential function is called the *antilog*. The log of a number less than one will be negative, e.g., $\ln(0.5) \approx -0.69$. We cannot take the log of a negative number (so ln(-0.6), for example, does not exist). The properties of logarithmic functions or 'laws of logs' describe the way that we can work with logs or manipulate expressions using them. These are presented in Box 1.5.

BOX 1.5 The laws of logs

For variables x and y

- $\ln(x\,y) = \ln(x) + \ln(y)$
- $\ln(x/y) = \ln(x) - \ln(y)$
- $\ln(y^c) = c\ln(y)$
- $\ln(1) = 0$
- $\ln(1/y) = \ln(1) - \ln(y) = -\ln(y)$.
- $\ln(e^x) = e^{\ln(x)} = x$

If we plot a log function, $y = \ln(x)$, it would cross the x-axis at one, as in Figure 1.7. It can be seen that as x increases, y increases at a slower rate, which is the opposite to an exponential function where y increases at a faster rate as x increases.

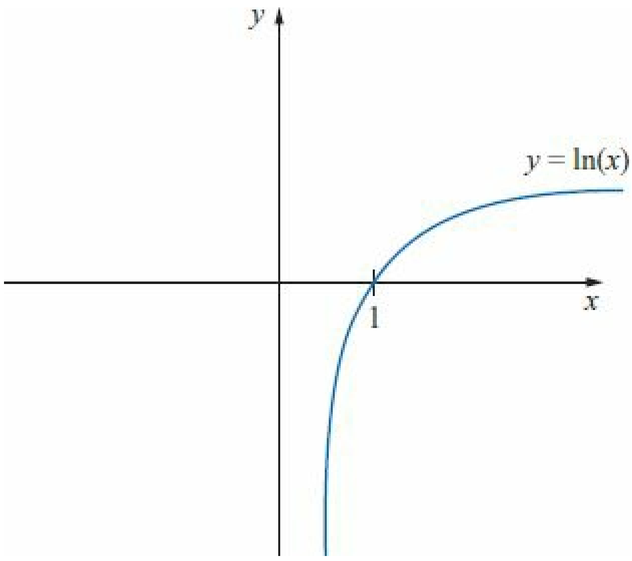


Figure 1.7 A plot of a logarithmic function

1.5.7 Inverse Functions

If we have a function such that $y = f(x)$, we would write the inverse as $x = f^{-1}(y)$. To give a simple example of a linear equation, if $y = 6x - 3$, the inverse function would be a rearrangement of the function to make x the subject: $x = (y + 3)/6$. For polynomials of order n, there could be up to n possible inverse functions, although the inverse of a function will not always exist.

1.5.8 Sigma Notation

If we wish to add together several numbers (or observations from variables), the *sigma* or summation operator can be very useful. Σ means 'add up all of the following elements'. For example, Σ(1, 2, 3) = 1 + 2 + 3 = 6. In the context of adding the observations on a variable, it is helpful to add 'limits' to the summation (although note that the limits are not always written out if the meaning is obvious without them). So, for instance, we might write

$$\sum_{i=1}^{4} x_i$$

where the i subscript is called an index, 1 is the lower limit and 4 is the upper limit of the sum. This would mean adding all of the values of x from x_1 to x_4.

It might be the case that one or both of the limits is not a specific number – for instance, $\sum_{i=1}^{n} x_i$, which would mean $x_1 + x_2 + \ldots + x_n$, or sometimes we simply write $\sum_i x_i$ to denote a sum over all the values of the index i. It is also possible to construct a sum of a more complex combination of variables, such as $\sum_{i=1}^{n} x_i z_i$, where x_i and z_i are two separate random variables.

It is important to be aware of a few properties of the sigma operator. For example, the sum of the observations on a variable x plus the sum of the observations on another variable z is equivalent to the sum of the observations on x and z first added together individually

$$\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} z_i = \sum_{i=1}^{n} (x_i + z_i) \quad \textbf{(1.8)}$$

The sum of the observations on a variable x each multiplied by a constant c is equivalent to the constant multiplied by the sum

$$\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i \tag{1.9}$$

But the sum of the products of two variables is not the same as the product of the sums

$$\sum_{i=1}^{n} x_i z_i \neq \sum_{i=1}^{n} x_i \sum_{i=1}^{n} z_i \tag{1.10}$$

We can write the left-hand side (LHS) of equation (1.10) as

$$\sum_{i=1}^{n} x_i z_i = x_1 z_1 + x_2 z_2 + \ldots + x_n z_n \tag{1.11}$$

whereas the right-hand side (RHS) of equation (1.10) is written

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} z_i = (x_1 + x_2 + \ldots + x_n)(z_1 + z_2 + \ldots + z_n) \tag{1.12}$$

We can see that equations (1.11) and (1.12) are different since the latter contains many ‘cross-product’ terms such as x_1z_2, x_3z_6, x_9z_2, etc., whereas the former does not.

If we sum n identical elements (i.e., we add a given number to itself n times), we obtain n times that number

$$\sum_{i=1}^{n} x = x + x + \ldots + x = nx \tag{1.13}$$

Suppose that we sum all of the n observations on a series, x_i – for example, the x_i could be the daily returns on a stock (which are not all the same), we would obtain

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \ldots + x_n = n\bar{x} \tag{1.14}$$

So the sum of all of the observations is, from the definition of the mean, equal to the number of observations multiplied by the mean of the series, $\bar{x}$. Notice that the difference between this situation in equation (1.14) and the previous one in equation (1.13) is that now the x_i are different from one another whereas before they were all the same (and hence no i subscript was necessary).

Finally, note that it is possible to have multiple summations, which can be conducted in any order, so for example

$$\sum_{i=1}^{n}\sum_{j=1}^{m}x_{ij}$$

would mean sum over all of the i and j subscripts, but we could either sum over the j's first for each value of i or sum over the i's first for each value of j. Usually, the convention is that the inner sum (in this case the one that runs over j from 1 to m would be conducted first – i.e., separately for each value of i).

1.5.9 Pi Notation

Similar to the use of sigma to denote sums, the pi operator (Π) is used to denote repeated multiplications. For example

$$\prod_{i=1}^{n}x_i = x_1x_2\ldots x_n \tag{1.15}$$

means 'multiply together all of the x_i for each value of i between the lower and upper limits'. It also follows that

$$\prod_{i=1}^{n}(cx_i) = c^n\prod_{i=1}^{n}x_i$$

For example, the product

$$\prod_{i=1}^{4}i^2$$

is equal to $1^2 \times 2^2 \times 3^2 \times 4^2 = 1 \times 4 \times 9 \times 16 = 576$.

Sometimes we need to calculate the *geometric mean* of a series. If the series contains n elements, this would mean taking the n^{th} root. For example, as we will see in Chapter 2, we would calculate the holding period return on an investment paying a return in each period (assume this is a year) i of r_i where there a total of n years as

$$\prod_{i=1}^{n}(1+r_i) = (1+r_1)(1+r_2)\ldots(1+r_n)$$

To calculate the average return in each year, we would take the geometric mean (i.e., the *n*th root) of this, as

$$\sqrt[n]{\prod_{i=1}^{n}(1+r_i)}$$

and then we would subtract one at the end. A detailed illustration will be given in Section 2.6 of Chapter 2.

1.5.10 Functions of More than one Variable

All the examples we have examined so far in this section involve situations where *y* is a function of a single variable *x*, but it is also possible for *y* to be a function of several variables. Returning to the example in Table 1.1 to illustrate, we might suppose that grades (*y*) depend on hours of study (x_1) and hours of tutoring (x_2), so we would write

$$y = a + b_1x_1 + b_2x_2 \qquad \textbf{(1.16)}$$

where *a* is still interpreted as an intercept, but there are now two slopes: b_1 measures how much *y* varies with changes in x_1 while b_2 measures how much *y* varies with changes in x_2. In order to plot such a function, we would need a three-dimensional representation. This notation will be very useful in later chapters when we examine relationships between many variables and we can continue to extend the model in exactly the same way according to how many variables we have included.

1.6 Differential Calculus

The effect of the *rate of change of one variable on the rate of change of another* is measured by a mathematical derivative. If the relationship between the two variables can be represented by a curve, the gradient of the curve will be this rate of change. Consider a variable *y* that is some function *f* of another variable *x*, i.e., $y = f(x)$. The derivative of *y* with respect to *x* is written

$$\frac{dy}{dx} = \frac{df(x)}{dx}$$

or sometimes written as

$$\frac{dy}{dx} = f'(x)$$

This term measures the instantaneous rate of change of y with respect to x, or in other words, the impact of an infinitesimally small change in x. Notice the difference between the notations Δy and dy – the former refers to a change in y of any size, whereas the latter refers specifically to an infinitesimally small change.

1.6.1 Differentiation: the Fundamentals

The basic rules of differentiation are as follows

1. The derivative of a constant is zero

$$\text{e.g., if } y = 10, \frac{dy}{dx} = 0$$

This is because $y = 10$ would be represented as a horizontal straight line on a graph of y against x, and therefore the gradient of this function is zero.

2. The derivative of a linear function is simply its slope

$$\text{e.g., if } y = 3x + 2, \frac{dy}{dx} = 3$$

But non-linear functions will have different gradients at each point along the curve. In effect, the gradient at each point is equal to the gradient of the tangent at that point – see Figure 1.8. Notice that the gradient will be zero at the point where the curve changes direction from positive to negative or from negative to positive – this is known as a *turning point* or equivalently as a *stationary point*.

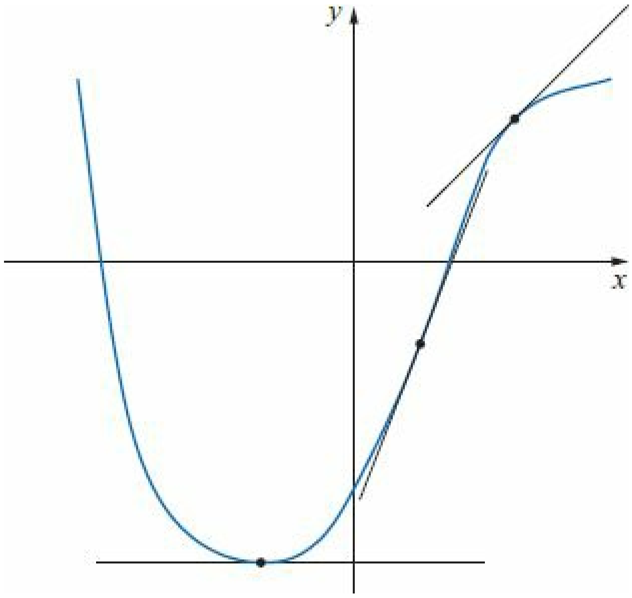


Figure 1.8 The tangents to a curve

3. The derivative of a power function *n* of *x*

$$\text{i.e., the derivative of } y = cx^n \text{ is given by } \frac{dy}{dx} = cnx^{n-1}$$

For example

$$y = 4x^3, \frac{dy}{dx} = (4 \times 3)x^2 = 12x^2$$

$$y = \frac{3}{x} = 3x^{-1}, \frac{dy}{dx} = (3 \times -1)x^{-2} = -3x^{-2} = \frac{-3}{x^2}$$

4. The derivative of a power of an entire function such as $[f(x)]^n$ is given by

$$\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$$

$$\text{e.g., if } y = (6x + x^4)^3, \frac{dy}{dx} = 3(6x + x^4)^2(6 + 4x^3)$$

5. The derivative of a sum is equal to the sum of the derivatives of the individual parts. Similarly, the derivative of a difference is equal to the difference of the derivatives of the individual parts

$$\text{e.g., if } y = f(x) + g(x), \frac{dy}{dx} = f'(x) + g'(x)$$

while

$$\text{if } y = f(x) - g(x), \frac{dy}{dx} = f'(x) - g'(x)$$

6. The derivative of the log of x is given by $1/x$

$$\text{i.e., } \frac{d(\ln(x))}{dx} = \frac{1}{x}$$

7. The derivative of the log of a function of x is the derivative of the function divided by the function

$$\text{i.e., } \frac{d(\ln(f(x)))}{dx} = \frac{f'(x)}{f(x)}$$

For example, the derivative of $\ln(x^3 + 2x - 1)$ is given by

$$\frac{d(\ln(x^3 + 2x - 1)}{dx} = \frac{3x^2 + 2}{x^3 + 2x - 1}$$

8. The derivative of an exponential of x is itself, so if $y = e^x$

$$\frac{dy}{dx} = e^x$$

More generally, the derivative of a function of an exponential is the derivative of the function multiplied by the exponential of the function, so if $y = e^{f(x)}$

$$\frac{dy}{dx} = f'(x)e^{f(x)}$$

So, to illustrate, if $y = e^{3x^2}$

$$\frac{dy}{dx} = 6xe^{3x^2}$$

1.6.2 Derivatives of Products and Quotients

Suppose that we have two functions multiplied together or one function divided by another function (recall that these are known as a *product* and a *quotient*, respectively). How would we differentiate these? Fortunately, both are fairly straight-forward.

For a product, which could be written as $y = f(x)g(x)$, the rule is

$$\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$$

For a quotient, written as $y = \frac{f(x)}{g(x)}$, the rule is

$$\frac{dy}{dx} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

Let us look at a couple of simple examples. Suppose that we have a product of two functions, $y = (3x^3 + 7x^2)(-2x^2 - 6)$. To differentiate this, product, we can view it as two functions, $y = f(x)g(x)$ and then we simply differentiate the first part, $f(x)$, multiplying that derivative by the second part, $g(x)$, unaltered, and then add the derivative of the second part multiplied by the first part unaltered

$$\frac{dy}{dx} = (9x^2 + 14x)(-2x^2 - 6) + (3x^3 + 7x^2)(-4x)$$

Again, it would be possible to simplify this expression but this is left as an exercise.

Now, suppose the quotient that we wish to differentiate is

$$y = \frac{(6x^4 - x)}{(3x^3 - 2x^2 + 4)}$$

Following the quotient rule, the derivative would be

$$\frac{dy}{dx} = \frac{(24x^3 - 1)(3x^3 - 2x^2 + 4) + (9x^2 - 4x)(6x^4 - x)}{(3x^3 - 2x^2 + 4)^2}$$

1.6.3 Higher Order Derivatives

It is possible to differentiate a function more than once to calculate the second order, third order, …, nth order derivatives. The notation for the second order derivative (which is usually just termed the second derivative, and which is the highest order derivative that we will need in this book) is

$$\frac{d^2y}{dx^2} = f''(x) = \frac{d(\frac{dy}{dx})}{dx}$$

To calculate second order derivatives, we simply differentiate the function with respect to x and then we differentiate it again. For example, suppose that we have the function

$$y = 4x^5 + 3x^3 + 2x + 6$$

The first order derivative is

$$\frac{dy}{dx} = \frac{d(4x^5 + 3x^3 + 2x + 6)}{dx} = f'(x) = 20x^4 + 9x^2 + 2$$

The second order derivative is

$$\frac{d^2y}{dx^2} = f''(x) = \frac{d(\frac{d(4x^5+3x^3+2x+6)}{dx})}{dx} = \frac{d(20x^4 + 9x^2 + 2)}{dx} = 80x^3 + 18x$$

The second order derivative can be interpreted as the gradient of the gradient of a function – i.e., the rate of change of the gradient.

We said above that at the turning point of a function its gradient will be zero. How can we tell, then, whether a particular turning point is a maximum or a minimum? In other words, is the shape of the function for that value of x a ∪ or a ∩? The answer is that to do this we would look at the second derivative. When a function reaches a maximum, its second derivative is negative, while it is positive for a minimum.

For example, consider the quadratic function $y = 5x^2 + 3x - 6$. We already know that since the squared term in the equation has a positive sign (i.e., it is 5 rather than, say, −5), the function will have a ∪-shape rather than an ∩-shape, and thus it will have a minimum rather than a maximum. But let us also demonstrate this using differentiation

$$\frac{dy}{dx} = 10x + 3, \frac{d^2y}{dx^2} = 10$$

Since the second derivative is positive, the function indeed has a minimum because the rate of change of the slope is positive as the gradient switches from negative on the left of the minimum to zero at the minimum to positive on the right of the minimum.

To find where this minimum is located, take the first derivative, set it to zero and solve it for x. So we have $10x + 3 = 0$, and thus $x = -\frac{3}{10} = -0.3$. If $x = -0.3$, the corresponding value of y is found by substituting −0.3 into the original function $y = 5x^2 + 3x - 6 = 5 \times (-0.3)^2 + (3 \times -0.3) - 6 = -6.81$. Therefore, the minimum of this function is found at (−0.3, −6.81).

What if the second derivative of a function is zero for a particular value of x? In such cases, the function is at a *point of inflection*. Turning points and points of inflection are both types of stationary point. At a point of inflection, the figure has neither a ∪-shape or a ∩-shape but something more like an 'S'.

To illustrate, consider the function $y = (x + 6)^3 - 5$. Its first derivative is $f'(x) = 3(x + 6)^2$. The second derivative is $f''(x) = 6(x + 6)$. Suppose that we are interested in evaluating the shape of the function at $x = -6$. At this point, $y = -5$, $f'(x) = 0$ and $f'' = 0$ so this is a point of inflection. We plot the original function, $y = f(x)$, the first derivative function, $y = f'(x)$, and the second derivative function, $y = f''(x)$, in Figure 1.9.

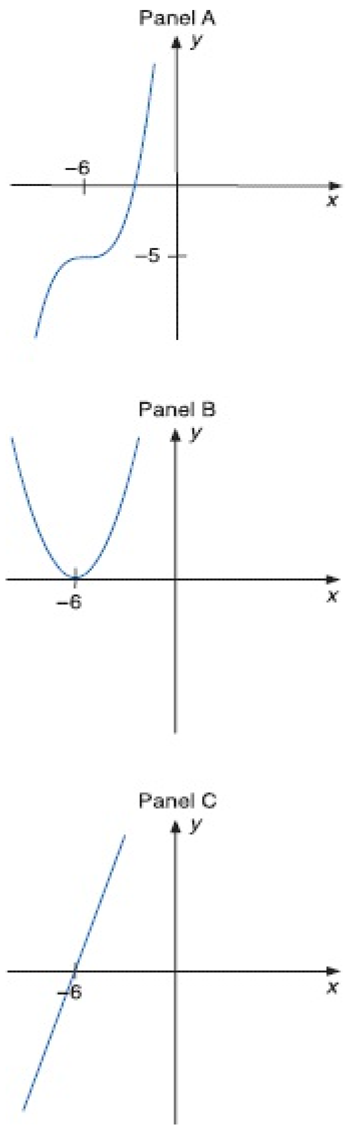


Figure 1.9 $y = f(x)$, its first derivative and its second derivative around the point $x = -6$

1.6.4 Differentiation of Functions of Functions Using the Chain Rule

In the section above we saw how to differentiate powers of functions and logarithms of functions. These are just special cases of a more general situation where we might want to differentiate a function of a function, $y = f(g(x))$. In such situations, we effectively split the process into two parts: we differentiate y with respect to g and then multiply it by the derivative of g with respect to x. We can write this as

$$\frac{dy}{dx} = \frac{dy}{dg}\frac{dg}{dx}$$

It is easy to see why this approach is often known as the *chain rule of differentiation*. As an illustration, suppose that we wish to differentiate the function $y = (4x^3 - 6x + 4)^4$. In this case we would have $g(x) = 4x^3 - 6x + 4$ and $y = g^4$. The derivative of y with respect to g is

$$\frac{dy}{dg} = 4g^3$$

and the derivative of g with respect to x is

$$\frac{dg}{dx} = 12x^2 - 6$$

Putting these together, the derivative of y with respect to x is:

$$\frac{dy}{dx} = \frac{dy}{dg}\frac{dg}{dx} = (4g^3)(12x^2 - 6) = 4(4x^3 - 6x + 4)^3(12x^2 - 6)$$

It may be possible to simplify this function but we leave it in its factorised form.

EXAMPLE 1.2: Utility Functions

In economics, utility provides a measure of the satisfaction that a consumer derives from a good or service that they have purchased. In finance, the concept is usually used to measure how satisfaction changes with differing levels of (terminal, i.e., end of period) wealth or with risk and return. Utilities constitute a useful illustration of how the concepts of functions and differentiation can be applied in finance.

Let us start by considering utility as a function of wealth. We would write the utility function as $U = f(W)$. Many such utility functions would be possible, e.g.

1. $U = 5 + 8W$
2. $U = 30 - e^{0.5W}$
3. $U = 100W + 0.5W^2$
4. $U = \ln(W)$

But would they all make sense as utility functions and what are the properties that we would want a utility function to have?

SOLUTION For a utility function to be plausible, we usually have two requirements. First, we would want to ensure that the investor has a positive marginal utility of wealth – in other words, utility always rises with wealth or mathematically, $dU/dW > 0$. We would also usually expect that investors are *risk averse* – i.e., they would reject a fair gamble or they prefer less risk to more. When utility is a function of wealth, it turns out that the condition for the investor to be risk averse is $d^2U/dW^2 < 0$. This condition also implies that marginal utility diminishes with wealth – in other words, I get more utility the more wealth I have, but each additional unit of wealth gives me less and less additional satisfaction. This makes intuitive sense.

For completeness, note that a *risk neutral* investor would be indifferent to a gamble and the second derivative of their utility function with respect to wealth would be: $d^2U/dW^2 = 0$. A *risk loving* investor who would prefer more risk to less and who would therefore accept a fair gamble has a second derivative greater than zero: $d^2U/dW^2 > 0$.

So to evaluate the plausibility of each of the four utility functions above, we would need to differentiate each of them twice and determine whether the first derivative is positive and the second derivative negative. These would be:

1. $U = 5 + 8W, \quad dU/DW = 8, \quad d^2U/DW^2 = 0$
2. $U = 30 - 30e^{0.5W}, \quad dU/DW = -15e^{0.5W}, \quad d^2U/DW^2 = -7.5e^{0.5W}$
3. $U = 100W + 0.5W^2, \quad dU/DW = 100 + W, \quad d^2U/DW^2 = 1$
4. $U = \ln(W), \quad dU/DW = 1/W, \quad d^2U/DW^2 = -1/W^2$

Utility function 1 is a linear equation, sloping upwards. The first derivative is positive for all values of W and so the investor having this utility function would have a positive marginal utility of wealth but the second derivative is zero and thus such an investor would be risk neutral.

Utility function 2 has a first derivative that is negative (so that the investor prefers less wealth to more) and a second derivative that is also negative for all values of W since e^{ax} will be positive for any (positive or negative) value of a and thus the investor would be risk averse.

The third utility function has a first derivative that is positive for any value of W greater than −100, and a second derivative that is positive everywhere and thus the investor would be risk loving.

Finally, the fourth utility function has a first derivative that is positive for all positive values of W but a negative second derivative for all values of W. So overall, we would conclude that the fourth utility function is the most appropriate of the four to describe a typical investor as it is the only one having the required properties of a positive first derivative and a negative second derivative.

1.6.5 Partial Differentiation

In the case where y is a function of more than one variable (e.g., $y = f(x_1, x_2, \ldots, x_n)$), it may be of interest to determine the effect that changes in each of the individual x variables would have on y. The differentiation of y with respect to only one of the variables, holding the others constant, is known as *partial differentiation*. The partial derivative of y with respect to a variable x_1 is usually denoted

$$\frac{\partial y}{\partial x_1}$$

All of the rules for differentiation explained above still apply and there will be one (first order) partial derivative for each variable on the RHS of the equation. We calculate these partial derivatives one at a time, treating all of the other variables as if they were constants. To give an illustration, suppose $y = 3x_1^3 + 4x_1 - 2x_2^4 + 2x_2^2$. The partial derivative of y with respect to x_1 would be

$$\frac{\partial y}{\partial x_1} = 9x_1^2 + 4$$

while the partial derivative of y with respect to x_2 would be

$$\frac{\partial y}{\partial x_2} = -8x_2^3 + 4x_2$$

As we will see in Chapter 3, the ordinary least squares (OLS) estimator gives formulae for the values of the parameters that minimise the function given by $L = \sum_t (y_t - \hat{\alpha} - \hat{\beta}x_t)^2$. The minimum of L (the residual sum of squares) is found by partially differentiating this function with respect to $\hat{\alpha}$ and then separately with respect to $\hat{\beta}$ and setting these partial derivatives to zero. Therefore, partial differentiation has a key role in deriving the main approach to parameter estimation that we use in econometrics – see Appendix 3.1 at the end of Chapter 3 for a demonstration of this application.

1.6.6 Functions that Cannot be Differentiated

Fortunately, it is possible to differentiate the majority of functions of interest to us in finance, but are there any formulations where it is not possible to calculate the gradient? The answer is that there are particular difficulties where a function is *discontinuous* or, in other words, it contains a jump (either up or down). For example, if we have a function $y = f(x)$ which takes a certain form when y is positive or zero and a different form when y is negative such as

$$y = \begin{cases} 2x + 4 & if\, x \geq 0 \\ -x + 3 & if\, x < 0 \end{cases} \quad \textbf{(1.17)}$$

It would not be possible to differentiate this function, which is known as a piecewise linear model, since each of the pieces (≥ 0 and < 0) are linear functions of x but overall it is non-linear. These models will be discussed in more detail in Chapter 10.

1.6.7 Derivatives in Use in Finance

What do we actually use differentiation for? A key use relates to the concept of what happens *at the margin* – in other words, what is the effect of an infinitesimally small change in x on y – this is exactly the interpretation of the slope of a function at a specific value of x. In reality, we usually weaken this slightly to say that the derivative of y with respect to x can be used to measure the effect of a unit change in x on y. This is a very useful concept that is widely used in measuring marginal utility, marginal propensity to save as income changes, etc. – for instance, what is the effect of a one-unit change in wealth upon the utility of an investor?

Differentiation relates *unit* changes in x to *unit* changes in y but it will often be of interest to consider what happens to y if x changes by one *percent* rather than one unit. This would be measured by an *elasticity*. The formula for calculating an elasticity of y with respect to x would be given by

$$\text{elasticity} = \frac{dy}{dx}\frac{x}{y} \quad \textbf{(1.18)}$$

EXAMPLE 1.3

Suppose that the demand for an on-line stock brokerage account is given by the following function

$$q = 100{,}000 - 500p$$

where q is the number of trades made per month and p is the fee charged per trade. If p =£20, calculate the price elasticity of demand.

SOLUTION To solve this, we first need to calculate the derivative of q with respect to p, which is very straightforward as it is a linear function: $dq/dp = -500$. To then calculate the elasticity, we need to calculate the value of q that corresponds to the value of p in the question (20). If $p = 20$, $q = 100{,}000 - (500 \times 20) = 90{,}000$. We then calculate the elasticity as

$$\text{elasticity} = \frac{dq}{dp}\frac{p}{q} = -500 \times \frac{20}{90{,}000} = -0.111$$

This would be interpreted as implying that a 1% increase in the fee per trade would reduce the number of trades by 0.111%. Since this figure is less than one in absolute value, we would conclude that demand for brokerage services is *inelastic* and thus the firm may have the opportunity to increase its revenue and profits by raising prices.

1.6.8 Integration

Integration is the opposite of differentiation, so that if we integrate a function and then differentiate the result, we get back the original function. Recall that derivatives give functions for calculating the slope of a curve; integration, on the other hand, is used to calculate the area under a curve (between two specific points). Further details on the rules for integration are beyond the scope of this book since the mathematical technique is not needed for any of the econometric approaches we will employ, but it will be useful to be familiar with the general concept. For further reading, see for example Renshaw (2016, Chapter 18).

1.7 Matrices

Before we can work with matrices, we need to define some terminology and to distinguish between a scalar, a vector and a matrix.

- A *scalar* is simply a single number (although it need not be a whole number – e.g., 3, −5, 0.5 are all scalars)
- A *vector* is a one-dimensional *array of numbers* (see below for

examples)

- A *matrix* is a two-dimensional *collection or array of numbers*. The size of a matrix is given by its numbers of rows and columns.

Matrices are very useful and important ways for organising sets of data together, which make manipulating and transforming them much easier than it would be to work with each constituent of the matrix separately. Matrices are widely used in econometrics and finance for solving systems of linear equations, for deriving key results and for expressing formulae in a succinct way. Sometimes **bold-faced type** is used to denote a vector or matrix (e.g., **A**), although in this book we will not do so – hopefully it should be obvious whether an object is a scalar, vector or matrix from the context, or this will be clearly stated. Some useful features of matrices and explanations of how to work with them are now described.

- The dimensions of a matrix are quoted as $R \times C$, which is the number of rows by the number of columns.
- Each element in a matrix is referred to using subscripts. For example, suppose a matrix M has two rows and four columns. The element in the second row and the third column of this matrix would be denoted m_{23}, so that more generally m_{ij} refers to the element in the ith row and the jth column. Thus a 2×4 matrix would have elements

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \end{pmatrix}$$

- Vectors are special cases of matrices where there is only one column or only one row.
- If a matrix has only one row, it is known as a *row vector*, which will be of dimension $1 \times C$, where C is the number of columns

$$\text{e.g.,} \quad (2.7 \quad 3.0 \quad -1.5 \quad 0.3)$$

- A matrix having only one column is known as a *column vector*, which will be of dimension $R \times 1$, where R is the number of rows

$$\text{e.g.,} \quad \begin{pmatrix} 1.3 \\ -0.1 \\ 0.0 \end{pmatrix}$$

- When the number of rows and columns is equal (i.e., $R = C$), it would be said that the matrix is square, as is the following 2×2 matrix

$$\begin{pmatrix} 0.3 & 0.6 \\ -0.1 & 0.7 \end{pmatrix}$$

- A matrix in which all the elements are zero is known as a *zero matrix*

$$\text{e.g.,} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- A *symmetric matrix* is a special type of square matrix that is symmetric about the leading diagonal (the diagonal line running through the matrix from the top left to the bottom right), so that $m_{ij} = m_{ji} \; \forall \; i, j$

$$\text{e.g.,} \quad \begin{pmatrix} 1 & 2 & 4 & 7 \\ 2 & -3 & 6 & 9 \\ 4 & 6 & 2 & -8 \\ 7 & 9 & -8 & 0 \end{pmatrix}$$

- A diagonal matrix is a square matrix which has non-zero terms on the leading diagonal and zeros everywhere else

$$\text{e.g.,} \quad \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- A diagonal matrix with 1 in all places on the leading diagonal and zero everywhere else is known as the *identity matrix*, denoted by *I*. By definition, an identity matrix must be symmetric (and therefore also square)

$$\text{e.g.,} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- The identity matrix is essentially the matrix equivalent of the number one. Multiplying any matrix by the identity matrix of the appropriate size results in the original matrix being left unchanged. So for any matrix *M*

$$MI = IM = M$$

1.7.1 Operations with Matrices

In order to perform operations with matrices (e.g., addition, subtraction or multiplication), the matrices concerned must be *conformable*. The dimensions of matrices required for them to be conformable depend on the operation.

- Addition and subtraction of matrices requires the matrices concerned to be of the same order (i.e., to have the same number of rows and the same number of columns as one another). The operations are then performed element by element

$$\text{e.g.,}\quad \text{if } A = \begin{pmatrix} 0.3 & 0.6 \\ -0.1 & 0.7 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0.2 & -0.1 \\ 0 & 0.3 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 0.3 + 0.2 & 0.6 - 0.1 \\ -0.1 + 0 & 0.7 + 0.3 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ -0.1 & 1.0 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 0.3 - 0.2 & 0.6 - -0.1 \\ -0.1 - 0 & 0.7 - 0.3 \end{pmatrix} = \begin{pmatrix} 0.1 & 0.7 \\ -0.1 & 0.4 \end{pmatrix}$$

- Multiplying or dividing a matrix by a scalar (that is, a single number), implies that every element of the matrix is multiplied by that number

$$\text{e.g.,}\quad 2A = 2\begin{pmatrix} 0.3 & 0.6 \\ -0.1 & 0.7 \end{pmatrix} = \begin{pmatrix} 0.6 & 1.2 \\ -0.2 & 1.4 \end{pmatrix}$$

- More generally, for two matrices *A* and *B* of the same order and for *c* a scalar, the following results hold

$$A + B = B + A$$

$$A + 0 = 0 + A = A$$

$$cA = Ac$$

$$c(A + B) = cA + cB$$

$$A0 = 0A = 0$$

- Multiplying two matrices together requires the number of columns of the first matrix to be equal to the number of rows of the second matrix. Note also that the ordering of the matrices is important when multiplying them, so that in general, $AB \neq BA$. When matrices are multiplied together, the resulting matrix will be of size (number of rows of first matrix × number of columns of second matrix), e.g., if we multiply a (3 × 2) matrix by a (2 × 4) matrix, the result is a (3 × 4) matrix: (3 × 2) × (2 × 4) = (3 × 4). In terms of determining the dimensions of the matrix, it is as if the number of columns of the first matrix and the number of rows of the second cancel out.[1] This rule also follows more generally, so that $(a \times b) \times (b \times c) \times (c \times d) \times (d \times e) = (a \times e)$, etc.

- The actual multiplication of the elements of the two matrices is done by multiplying along the rows of the first matrix and down the columns of the second

$$\text{e.g.,}\quad \underset{(3\times 2)}{\begin{pmatrix} 1 & 2 \\ 7 & 3 \\ 1 & 6 \end{pmatrix}} \underset{(2\times 4)}{\begin{pmatrix} 0 & 2 & 4 & 9 \\ 6 & 3 & 0 & 2 \end{pmatrix}}$$

$$= \underset{(3\times 4)}{\begin{pmatrix} ((1\times 0)+(2\times 6)) & ((1\times 2)+(2\times 3)) & ((1\times 4)+(2\times 0)) & ((1\times 9)+(2\times 2)) \\ ((7\times 0)+(3\times 6)) & ((7\times 2)+(3\times 3)) & ((7\times 4)+(3\times 0)) & ((7\times 9)+(3\times 2)) \\ ((1\times 0)+(6\times 6)) & ((1\times 2)+(6\times 3)) & ((1\times 4)+(6\times 0)) & ((1\times 9)+(6\times 2)) \end{pmatrix}}$$

$$= \underset{(3\times 4)}{\begin{pmatrix} 12 & 8 & 4 & 13 \\ 18 & 23 & 28 & 69 \\ 36 & 20 & 4 & 21 \end{pmatrix}}$$

- In general, matrices cannot be divided by one another. Instead, we achieve the same sort of outcome by multiplying by the inverse – see below.
- The transpose of a matrix, written A' or A^T, is the matrix obtained by transposing (switching) the rows and columns of a matrix

$$\text{e.g.,}\quad \text{if } A = \begin{pmatrix} 1 & 2 \\ 7 & 3 \\ 1 & 6 \end{pmatrix} \quad \text{then } A' = \begin{pmatrix} 1 & 7 & 1 \\ 2 & 3 & 6 \end{pmatrix}$$

If A is of dimensions $R \times C$, A' will be $C \times R$.

1.7.2 The Rank of a Matrix

The rank of a matrix A is given by the maximum number of linearly independent rows (or columns) contained in the matrix. For example,

$$\text{rank}\begin{pmatrix} 3 & 4 \\ 7 & 9 \end{pmatrix} = 2$$

since both rows and columns are (linearly) independent of one another, but

$$\text{rank}\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} = 1$$

as the second column is not independent of the first (the second column is simply twice the first and also the second row is two thirds of the first). A matrix with a rank equal to its dimension, as in the first of these two cases, is known as a *matrix of full rank*. A matrix that is less than of full rank is known as a *short rank matrix*, and such a matrix is also termed *singular*.

Three important results concerning the rank of a matrix are

- Rank(A) = Rank(A')
- Rank(AB) ≤ min(Rank(A), Rank(B))
- Rank($A'A$) = Rank(AA') = Rank(A)

1.7.3 The Inverse of a Matrix

The inverse of a matrix A, where defined, is denoted A^{-1}. It is that matrix which, when pre-multiplied or post-multiplied by A, will result in the identity matrix

$$\text{i.e.,}\quad AA^{-1} = A^{-1}A = I$$

The inverse of a matrix exists only when the matrix is square and non-singular – that is, when it is of full rank. The inverse of a 2 × 2 non-singular matrix whose elements are

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

will be given by

$$\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The expression in the denominator above to the left of the matrix ($ad - bc$) is the *determinant* of the matrix, and will be a scalar. If this determinant is zero, the matrix is *singular*, and thus not of *full rank* so that its inverse does not exist. For example, if

$$A = \begin{pmatrix} 1 & 6 \\ 2 & 12 \end{pmatrix}$$

$ad - bc = 12 - 12 = 0$ so this matrix is singular since the second column is six times the first (or looking at it another way, the second row is double the first). We usually write the determinant of a matrix using |.| (the same notation as for the absolute value of a variable). So $|A|$ is the determinant of matrix A.

EXAMPLE 1.4

If the matrix is

$$\begin{pmatrix} 2 & 1 \\ 4 & 6 \end{pmatrix}$$

the inverse will be

$$\frac{1}{8}\begin{pmatrix} 6 & -1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{8} \\ -\frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

As a check, multiply the two matrices together and it should give the identity matrix – the matrix equivalent of one (analogous to $\frac{1}{3} \times 3 = 1$)

$$\begin{pmatrix} 2 & 1 \\ 4 & 6 \end{pmatrix} \times \frac{1}{8}\begin{pmatrix} 6 & -1 \\ -4 & 2 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$= I$, as required.

The calculation of the inverse of an $N \times N$ matrix for $N > 2$ is more complex. Two of the most common approaches to finding the inverse of a larger matrix are known as the *method of determinants* and the *Gauss-Jordan elimination method*. These are beyond the scope of this text but see Wisniewski (2013), for example, for further details.

Properties of the inverse of a matrix include

- $I^{-1} = I$
- $(A^{-1})^{-1} = A$
- $(A')^{-1} = (A^{-1})'$
- $(AB)^{-1} = B^{-1}A^{-1}$

1.7.4 The Trace of a Matrix

The trace of a square matrix is the sum of the terms on its leading diagonal. For example, the trace of the matrix

$$A = \begin{pmatrix} 3 & 4 \\ 7 & 9 \end{pmatrix}$$

written Tr(A), is $3 + 9 = 12$. Some important properties of the trace of a matrix are

- $\text{Tr}(cA) = c\text{Tr}(A)$
- $\text{Tr}(A') = \text{Tr}(A)$
- $\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B)$
- $\text{Tr}(I_N) = N$

1.7.5 The Eigenvalues of a Matrix

The concept of the eigenvalues of a matrix is necessary for testing for long-run relationships between series using what is known as the Johansen cointegration test used in Chapter 8. Let Π denote a $p \times p$ square matrix, c denote a $p \times 1$ non-zero vector, and λ denote a set of scalars. λ is called a *characteristic root* or set of roots of the matrix if it is possible to write

$$\underset{p\times p}{\Pi}\underset{p\times 1}{c} = \underset{p\times 1}{\lambda c}$$

This equation can also be written as

$$\Pi c = \lambda I_p c$$

where I_p is an identity matrix, and hence

$$(\Pi - \lambda I_p)c = 0$$

Since $c \neq 0$ by definition, then for this system to have a non-zero solution, the matrix $(\Pi - \lambda I_p)$ is required to be singular (i.e., to have a zero determinant)

$$|\Pi - \lambda I_p| = 0$$

For example, let Π be the 2×2 matrix

$$\Pi = \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix}$$

Then the characteristic equation is

$$\begin{aligned}
&|\Pi - \lambda I_p| \\
&= \left| \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \\
&= \begin{vmatrix} 5-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} = (5-\lambda)(4-\lambda) - 2 = \lambda^2 - 9\lambda + 18
\end{aligned}$$

This gives the solutions $\lambda = 6$ and $\lambda = 3$. The characteristic roots are also known as *eigenvalues*. The eigenvectors would be the values of c

corresponding to the eigenvalues. Some properties of the eigenvalues of any square matrix A are

- the sum of the eigenvalues is the trace of the matrix
- the product of the eigenvalues is the determinant
- the number of non-zero eigenvalues is the rank

For a further illustration of the last of these properties, consider the matrix

$$\Pi = \begin{bmatrix} 0.5 & 0.25 \\ 0.7 & 0.35 \end{bmatrix}$$

Its characteristic equation is

$$\left| \begin{bmatrix} 0.5 & 0.25 \\ 0.7 & 0.35 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

which implies that

$$\begin{vmatrix} 0.5 - \lambda & 0.25 \\ 0.7 & 0.35 - \lambda \end{vmatrix} = 0$$

This determinant can also be written $(0.5 - \lambda)(0.35 - \lambda) - (0.7 \times 0.25) = 0$ or

$$0.175 - 0.85\lambda + \lambda^2 - 0.175 = 0$$

or

$$\lambda^2 - 0.85\lambda = 0$$

which can be factorised to $\lambda\ (\lambda - 0.85) = 0$.

The characteristic roots are therefore 0 and 0.85. Since one of these eigenvalues is zero, it is obvious that the matrix Π cannot be of full rank. In fact, this is also obvious from just looking at Π, since the second column is exactly half the first.

KEY CONCEPTS

The key terms to be able to define and explain from this chapter are

- functions
- turning points
- the chain rule
- powers

- exponentials
- sigma and pi notation
- quadratic equations
- inverse of a matrix
- eigenvalues
- roots
- derivatives and differentiation
- products and quotients
- indices
- polynomials
- logarithms
- conformable matrix
- rank of a matrix
- eigenvectors

SELF-STUDY QUESTIONS

1. (a) If $f(x) = 3x^2 - 4x + 2$, find $f(0)$, $f(2)$, $f(-1)$

(b) If $f(x) = 4x^2 + 2x - 3$, find $f(0)$, $f(3)$, $f(a)$, $f(3 + a)$

(c) Considering your answers to the previous question part, in general does $f(a) + f(b) = f(a + b)$? Explain.

2. Simplify the following as much as possible

(a) $4x^5 \times 6x^3$

(b) $3x^2 \times 4y^2 \times 8x^4 \times -2y^4$

(c) $(4p^2q^3)^3$

(d) $6x^5 \div 3x^2$

(e) $7y^2 \div 2y^5$

(f) $\dfrac{3(xy)^3 \times 6(xz)^4}{2(xy)^2x^3}$

(g) $(xy)^3 \div x^3y^3$

(h) $(xy)^3 - x^3y^3$

3. Solve the following

(a) $125^{1/3}$

(b) $64^{1/3}$

(c) $16^{1/4}$

(d) $9^{3/2}$

(e) $9^{2/3}$

(f) $81^{1/2} + 64^{1/2} + 64^{1/3}$

4. Write each of the following as a prime number raised to a power

(a) 9

(b) 625

(c) 125^{-1}

5. Solve the following equations

(a) $3x - 6 = 6x - 12$

(b) $2x - 304x + 8 = x + 9 - 3x + 4$

(c) $\frac{x+3}{2} = \frac{2x-6}{3}$

6. Write out all of the terms in the following and evaluate them

(a) $\sum_{j=1}^{3} j$

(b) $\sum_{j=2}^{5} (j^2 + j + 3)$

(c) $\sum_{i=1}^{n} x$ with $n = 4$ and $x = 3$

(d) $\prod_{j=1}^{3} x$ with $x = 2$

(e) $\prod_{i=3}^{6} i$

7. Write the equations for each of the following lines

(a) Gradient = 3, intercept = −1

(b) Gradient = −2, intercept = 4

(c) Gradient $= \frac{1}{2}$, crosses y-axis at 3

(d) Gradient $= \frac{1}{2}$, crosses x-axis at 3

(e) Intercept 2 and passing through (3,1)

(f) Gradient 4 and passing through (−2,−2)

(g) Passes through $x = 4$, $y = 2$ and $x = -2$, $y = 6$

8. Differentiate the following functions twice with respect to x

(a) $y = 6x$

(b) $y = 3x^2 + 2$

(c) $y = 4x^3 + 10$

(d) $y = \frac{1}{x}$

(e) $y = x$

(f) $y = 7$

(g) $y = 6x^{-3} + \frac{6}{x^3}$

(h) $y = 3 \ln x$

(i) $y = \ln(3x^2)$

(j) $y = \dfrac{3x^4 - 6x^2 - x - 4}{x^3}$

9. Differentiate the following functions partially with respect to x and (separately) partially with respect to y

(a) $z = 10x^3 + 6y^2 - 7y$

(b) $z = 10xy^2 - 6$

(c) $z = 6x$

(d) $z = 4$

10. Factorise the following expressions

(a) $x^2 - 7x - 8$

(b) $5x - 2x^2$

(c) $2x^2 - x - 3$

(d) $6 + 5x - 4x^2$

(e) $54 - 15x - 25x^2$

11. Express the following in logarithmic form

(a) $5^3 = 125$

(b) $11^2 = 121$

(c) $6^4 = 1296$

12. Evaluate the following (without using a calculator)

(a) $\ln_{10} 10{,}000$

(b) $\ln_2 16$

(c) $\ln_{10} 0.01$

(d) $\ln_5 125$

(e) $\ln_e e^2$

13. Express the following logarithms using powers

(a) $\ln_5 3125 = 5$

(b) $\ln_{49} 7 = \frac{1}{2}$

(c) $\ln_{0.5} 8 = -3$

14. Write the following as simply as possible as sums of logs of prime numbers

(a) $\ln 60$

(b) $\ln 300$

15. Simplify the following as far as possible

(a) $\ln 27 - \ln 9 + \ln 81$

(b) $\ln 8 - \ln 4 + \ln 32$

16. Solve the following

(a) $\ln x^4 - \ln x^3 = \ln 5x - \ln 2x$

(b) $\ln(x - 1) + \ln(x + 1) = 2 \ln(x + 2)$

(c) $\log_{10} x = 4$

17. Use the result that ln(8) is approximately 2.1 to estimate the following (without using a calculator):

(a) ln(16)

(b) ln(64)

(c) ln(4)

18. Solve the following using logs and a calculator

(a) $4^x = 6$

(b) $4^{2x} = 3$

(c) $3^{2x-1} = 8$

19. Find the minima of the following functions. In each case, state the value of the function at the minimum

(a) $y = 6x^2 - 10x - 8$

(b) $y = (6x^2 - 8)^2$

20. Construct an example not used elsewhere in this book to demonstrate that for two conformable matrices A and B, $(AB)^{-1} = B^{-1}A^{-1}$.

21. Suppose that we have the following four matrices

$$A = \begin{bmatrix} 1 & 6 \\ -2 & 4 \end{bmatrix}, B = \begin{bmatrix} -3 & -8 \\ 6 & 4 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, D = \begin{bmatrix} 6 & -2 \\ 0 & -1 \\ 3 & 0 \end{bmatrix}$$

(a) Which pairs of matrices can be validly multiplied together? For these pairs, perform the multiplications.

(b) Calculate $2A$, $3B$, $\frac{1}{2}D$

(c) Calculate Tr(A), Tr(B), Tr(A + B) and verify that Tr(A) + Tr(B) = Tr(A + B)

(d) What is the rank of the matrix A?

(e) Find the eigenvalues of the matrix (A + B)

(f) What will be the trace of the identity matrix of order 12?

22. (a) Add

$$\begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix} \text{ to } \begin{bmatrix} -3 & 0 \\ 7 & -4 \end{bmatrix}$$

(b) Subtract

$$\begin{bmatrix} -3 & 0 \\ 7 & -4 \end{bmatrix} \text{ from } \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$$

(c) Calculate the inverse of

$$\begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$$

(d) Does the inverse of the following matrix exist? Explain your answer

$$\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}$$

23. A researcher suggests that the US dollar – British pound exchange rate is a function of US and UK interest rates.

(a) Write an equation for this function

(b) What signs would we expect for the parameters in this function and why?

(c) Give an example of parameter values that would lead the US interest rate to have three times the effect on the exchange rate as the UK interest rate

24. Give an example of a function that cannot be differentiated and explain why.

[1] Of course, the actual elements of the matrices themselves do not cancel out – this is just a simple rule of thumb for calculating the dimensions of the matrix resulting from a multiplication.

2

Statistical Foundations and Dealing with Data

LEARNING OUTCOMES

In this chapter, you will learn how to

- Construct minimum variance and mean-variance efficient portfolios
- Compute summary statistics for a data series
- Manipulate expressions using the expectations, variance and covariance operators
- Compare nominal and real series
- Deflate series to allow for inflation
- Distinguish between different types of data
- Compound and discount cashflows
- Calculate present values and future values
- Use standard formulae to value stocks and bonds
- Calculate asset price returns

This chapter covers the statistical building blocks that are essential for a good understanding of the rest of the book and provides an introduction to random variables and to dealing with and summarising financial data. It also explains how to work with discounted cashflows, how to calculate present values and how to compute returns in both nominal and real terms using both discrete and continuous compounding.

2.1 Probability and Probability Distributions

This section discusses and presents the theoretical expressions for the mean and variance of a random variable. A *random variable* is one that can take on any value from a given set and where this value is determined at least in part by chance. By their very nature, random variables are not perfectly predictable. Most data series in economics and finance are best considered as random variables, although there might be some measurable structure underlying them as well so they are not purely random. It is often helpful to think of such series as being made up of a fixed part (which we can model and forecast) and a purely random part, which we cannot forecast.

The data that we use in building econometric models either come from experiments or, more commonly, are observed in the 'real world'. The outcomes from an experiment can often only take on certain specific values – i.e., they are *discrete random variables*. For example, the sum of the scores from throwing two dice could only be a number between two (if we throw two ones) and twelve (if we throw two sixes). We could calculate the probability of each possible sum occurring and plot it on a diagram, such as Figure 2.1. This would be known as a *probability distribution function*, which shows the various outcomes that are possible and how likely each one is to occur.

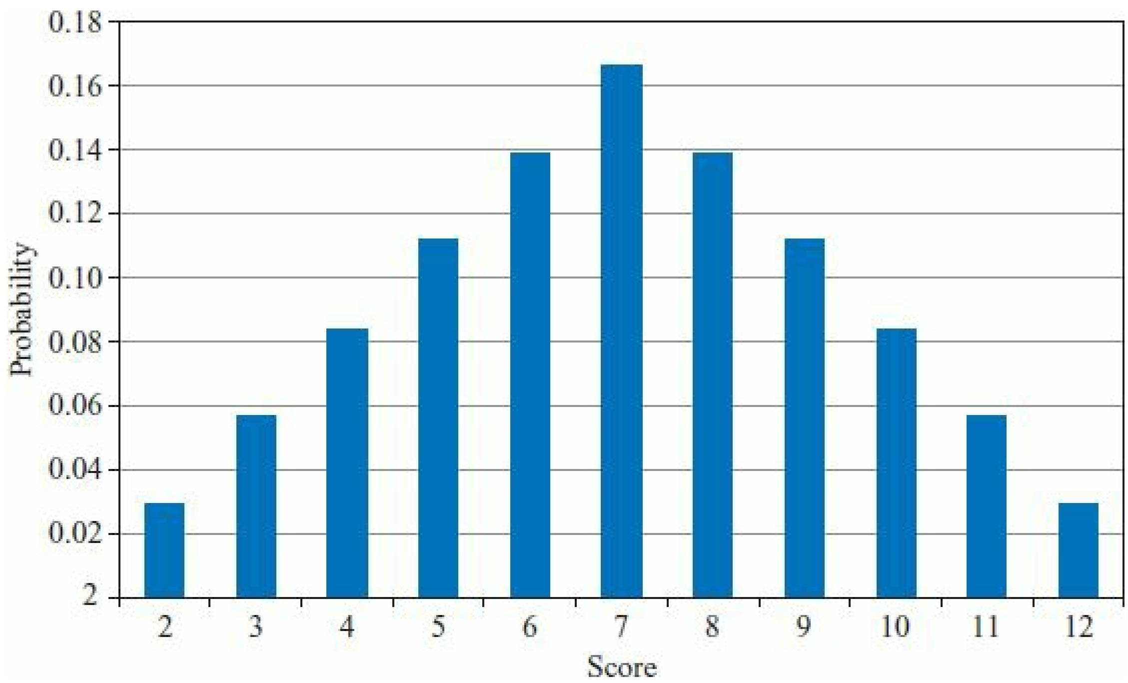


Figure 2.1 The probability distribution function for the sum of two dice

A *probability* is defined as the likelihood of a particular event happening. For example, we could calculate the probability that it will rain tomorrow, or the probability of scoring a total of seven when we throw two dice. All probabilities must lie between zero and one, with a probability of zero indicating an impossibility and one indicating a certainty. Notice that the sum of the probabilities in Figure 2.1 is, as always, one.

Most of the time in finance we work with continuous rather than discrete variables, in which case the plot above would be *probability density function* (pdf) rather than a distribution function. A continuous random variable can take any value (possibly only within a given range). For example, the amount of time a swimmer takes to complete one length of a pool or the return on a stock index. The time that the swimmer takes could be any positive value, depending on how fast they are! The return on a stock index could take any value greater than –100% – in other words, the most that an investor in the stock can lose is their entire investment (−100%), but there is no maximum to the amount that they can gain. At least in theory, the price could double, quadruple, quintuple, etc. In both cases, the value that the process takes can be defined to any arbitrary level of precision - e.g., the swimmer completing the length in 31 seconds, 31.2 seconds, 31.17 seconds, and so on. Hence, note that for a continuous random variable, the probability that it is *exactly* equal to a particular number is always zero by definition because the variable could take on any value.

There are many continuous distribution (density) functions. The simplest is the uniform distribution, where all of the possible outcomes are equally likely to occur. In this case, the pdf is a horizontal straight line. While conceptually simple, the uniform distribution is not very useful as it describes few variables of interest in economics and finance.

The distribution most commonly used to characterise a random variable is a *normal* or *Gaussian* (these terms are equivalent) distribution. The normal distribution is easy to work with since it is symmetric, it is unimodal (i.e., only has one peak) and the only pieces of information required to completely specify the distribution are its mean and variance, as discussed in Chapter 5. The normal distribution is particularly useful because many naturally occurring series follow it – for example, the heights, weights and IQ-levels of people in a given sample will in general roughly follow a normal distribution.

The normal distribution also has several useful mathematical properties. For example, any linear transformation of a normally distributed random variable will still be normally distributed. So, if $y \sim N(\mu, \sigma^2)$, that is, y is normally distributed with mean μ and variance σ^2, then $a + by \sim N(a + b\mu, b^2\sigma^2)$ where a and b are scalars. Furthermore, any linear combination of independent normally distributed random variables is itself normally distributed.

Suppose that we have a normally distributed random variable with mean μ and variance σ^2. Its probability density function is given by $f(y)$ in the following expression

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-\mu)^2/2\sigma^2} \tag{2.1}$$

Entering values of y into this expression would trace out the familiar 'bell-shape' of the normal distribution described in Figure 2.2. Notice that if a random variable follows a normal distribution, outcomes close to the mean of the series are more likely to occur than those in the extremes, as represented by the peak of the distribution being in the centre and its height declining further away from the mean. The x-axis on the left side and right side are both asymptotes to the normal distribution – in other words, it gradually gets closer and closer to the x-axis the further away from its mean the value of x becomes.

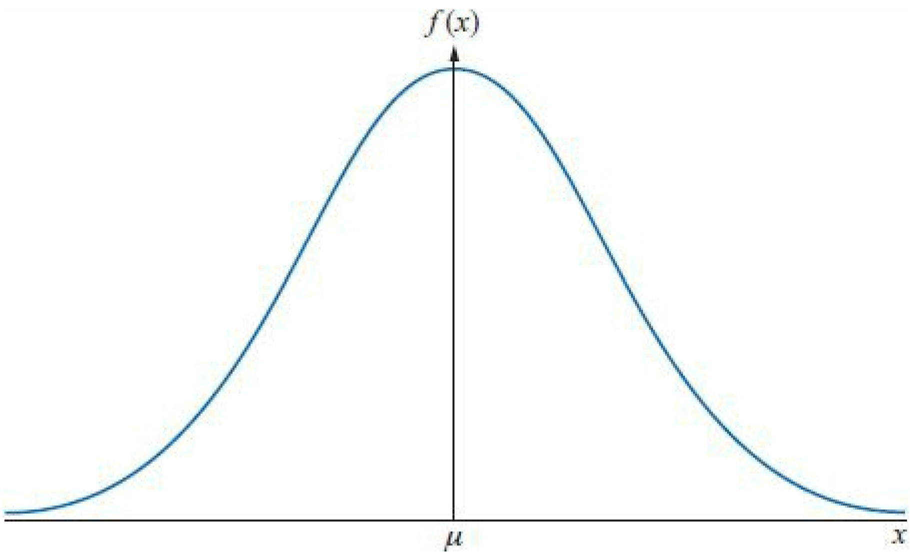


Figure 2.2 The pdf for a normal distribution

The area under a pdf measures the probabilities, and since the sum of the probabilities of all events occurring is one, the area under a pdf will always be one. Note that for a continuous random variable, we can only talk of the probability that it will take on values within a range (e.g., the probability that *y* will be between 1 and 2) and not the probability that *y* will be equal to some number (e.g., 2). Remember from above that the probability of *y* being an exact number is zero, since *y* can take any value to an arbitrary degree of precision (e.g., 2.0001, 2.0000001, etc.).

A *standard* normally distributed random variable can be obtained from any normal distribution by subtracting the mean and dividing by the standard deviation (the square root of the variance). A standard normally distributed random variable, usually denoted by *Z*, would then be written as

$$Z = \frac{y - \mu}{\sigma} \sim N(0, 1) \quad \textbf{(2.2)}$$

It is usually easier to work with the normal distribution in its standardised form, and only this normal distribution is tabulated (since there are an infinite number of normal distributions with different means and variances, we could not tabulate them all!).

Distributions are important in statistics because of their link with probabilities. If we know (or we can assume) the particular distribution that a series follows, then we can calculate the likelihood (probability) that values this series takes will fall within a certain range. For example, if we can assume that a series, *y*, follows a standard normal distribution, we can calculate the probability that it will take a value of +3 or more. This information can be calculated from the *cumulative density function*, also sometimes known as the *cumulative distribution function*, (cdf), which is written *F*(*y*). The cdf for a normally distributed random variable has a sigmoid shape, as in Figure 2.3.

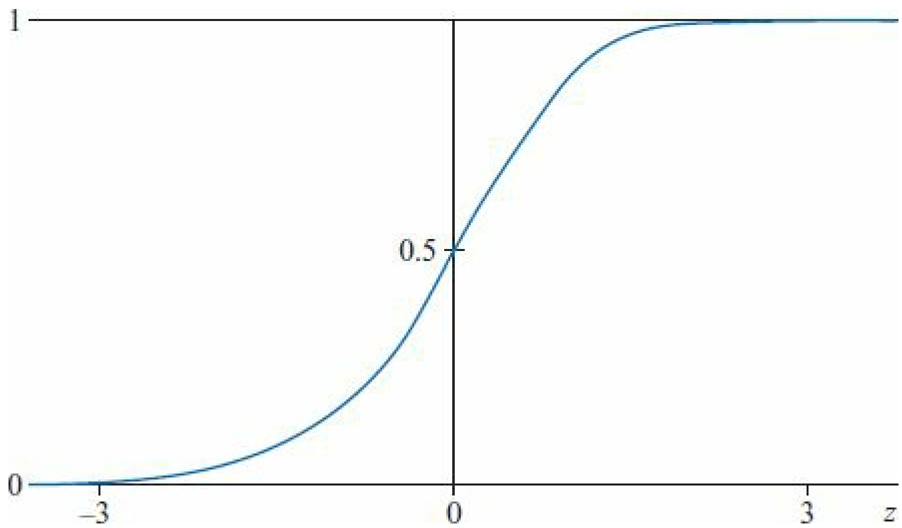


Figure 2.3 The cdf for a normal distribution

More specifically, we can use the cdf to calculate the probability that the random variable lies within a certain range – e.g., what is the probability that y lies between 0.2 and 0.3? This is equivalent to asking what is the area under the normal distribution pdf between 0.2 and 0.3? To obtain this, we would plug $y = 0.2$ and then separately $y = 0.3$ into the equation for the cdf and calculate the corresponding value of $f(y)$ in each case. Then the difference between these two values of $f(y)$ would give us the answer.

More often, rather than wanting to determine the probability that a random variable lies within a range, we instead want to know the probability that the variable is below a certain value (or above a certain value). So, for example, what is the probability that y is less than 0.4? Effectively, we want to know the probability that y lies between $-\infty$ and 0.4. Thus the probability that y is less than (or equal to) some specific value of y, y_0, is equal to the cdf of y evaluated where $y = y_0$

$$P(y \leq y_0) = F(y_0) \tag{2.3}$$

Note that there are also alternative versions of the normal distribution table that present the information the other way around. So they show values of Z_α and the corresponding values of α – i.e., for a given value of Z, say 1.5, they show the probability of a standard normally distributed random variable being bigger than this rather than less than as in equation (2.3). Table A2.2 in Appendix 2 at the back of this book presents what are known as the critical values for the normal distribution. Effectively, if we plotted the values on the first row, α, against the values in the second row, Z_α, then we would trace out the cdf.

Looking at the table, if $\alpha = 0.1$, $Z_\alpha = 1.2816$. So 10% (0.1 in proportion terms) of the normal distribution lies to the right of 1.2816. In other words, the probability that a standard normal random variable takes a value greater than 1.2816 is 10%. Similarly, the probability that it takes a value greater than 3.0902 is 0.1% (i.e., 0.001). We know that the standard normal distribution is symmetric about zero so if $P(Z \geq 1.2816) = 0.1$, $P(Z \leq -1.2816) = 0.1$ as well.

2.1.1 The Central Limit Theorem

If a random sample of size $N : y_1, y_2, y_3, \ldots, y_N$ is drawn from a population that is normally distributed with mean μ and variance σ^2, the sample mean, $\bar{y}$ is also normally distributed with mean μ and variance σ^2/N.

In fact, an important rule in statistics known as the *central limit theorem* states that the sampling distribution of the mean of any random sample of observations will tend towards the normal distribution with mean equal to the population mean, μ, as the sample size tends to infinity. This theorem is a very powerful result because it states that the sample mean, $\bar{y}$, will follow a normal distribution even if the original observations (y_1, y_2, …, y_N) did not. This means that we can use the normal distribution as a kind of benchmark when testing hypotheses, as discussed more fully in Chapter 3.

2.1.2 Other Statistical Distributions

There are many statistical distributions, including the binomial, Poisson, log normal, normal, exponential, *t*, chi-squared and *F*, and each has its own characteristic pdf. Different kinds of random variables will be best modelled with different distributions. Many of the statistical distributions are also related to one another, and most (except the normal) have one or more *degrees of freedom* parameters that determine the location and shape of the distribution. For example, the chi-squared (denoted χ^2) distribution can be obtained by taking the sum of the squares of independent normally distributed random variables. If we sum n independent squared normals, the result will be a χ^2 with n degrees of freedom. Since it comprises the sum of squares, the chi-squared distribution can only take positive values. Unlike the normal distribution, the chi-squared is not symmetric about its mean value.

The F-distribution, which has two degrees of freedom parameters, is the ratio of independent chi-squared distributions, each divided by their degrees of freedom. Suppose that $y_1 \sim \chi^2(n_1)$ and $y_2 \sim \chi^2(n_2)$ are two independent chi-squared distributions with n_1 and n_2 degrees of freedom, respectively. Then the ratio will follow an F distribution with (n_1, n_2) degrees of freedom,

$$\frac{y_1/n_1}{y_2/n_2} \sim F(n_1, n_2)$$

The final, and arguably most important, distribution used in econometrics is the t-distribution. The normal distribution is a special case of the t. The t-distribution can also be obtained by taking a standard normally distributed random variable, Z, and dividing it by the square root of an independent chi-squared distributed random variable (suppose that the latter is called y_1), itself divided by its degrees of freedom, n_1

$$\frac{Z}{\sqrt{y_1/n_1}} \sim t(n)$$

The t-distribution is symmetric about zero and looks similar to the normal distribution except that it is flatter and wider.

What do we use statistical distributions for? The normal, F, t and chi-squared distributions are all used predominantly to make *inferences* from the sample to the population. This implies making statements about the likely values of the corresponding unobservable population values from the sample values that we have. These ideas will be discussed in considerable detail in Chapter 3 onwards.

2.2 A Note on Bayesian versus Classical Statistics

The philosophical approach to model-building adopted in this entire book, as with the majority of others, is that of 'classical statistics'. Under the classical approach, the researcher postulates a theory and estimates a model to test that theory. Tests of the theory are conducted using the estimated model within the 'classical' hypothesis testing framework developed in Chapters 2 to 5. Based on the empirical results, the theory is either *refuted* or *upheld* by the data.

There is, however, an entirely different approach available for model construction, estimation and inference, known as *Bayesian statistics*. Under a Bayesian approach, the theory and empirical model work more closely together. The researcher would start with an assessment of the existing state of knowledge or beliefs, formulated into a set of probabilities. These prior inputs, or *priors*, would be combined with the observed data via a likelihood function. The beliefs and the probabilities would then be updated as a result of the model estimation, resulting in a set of *posterior probabilities*. Probabilities are thus updated sequentially, as more data become available. The central mechanism, at the most basic level, for combining the priors with the likelihood function, is known as Bayes' theorem.

The Bayesian approach to estimation and inference has found a number of important recent applications in financial econometrics, in particular in the context of volatility modelling (see Bauwens and Laurent, 2002, or Vrontos *et al.*, 2000 and the references therein for some examples), asset allocation (see, for example, Handa and Tiwari, 2006), and portfolio performance evaluation (Baks *et al.*, 2001).

The Bayesian setup is an intuitively appealing one, although the resulting mathematics is somewhat complex. Many classical statisticians are unhappy with the Bayesian notion of prior probabilities that are set partially according to judgement. Thus, if the researcher set very strong priors, an awful lot of evidence against them would be required for the notion to be refuted. Contrast this with the classical case, where the data are usually permitted to freely determine whether a theory is upheld or refuted, irrespective of the researcher's judgement.

2.3 Descriptive Statistics

When analysing a series containing many observations, it is useful to be able to describe its most important characteristics using a small number of summary measures. This section discusses the quantities that are most commonly used to describe financial and economic series, known as *summary statistics* or *descriptive statistics*. Descriptive statistics are calculated from a sample of data rather than assigned based on theory. Before discussing the most important summary statistics used in work with finance data, we define the terms *population* and *sample*, which have precise meanings in statistics, in Box 2.1.

BOX 2.1 The population and the sample

- The *population* is the total collection of all objects to be studied. For example, in the context of determining the relationship between risk and return for UK stocks, the population of interest would be all time-series observations on all stocks traded on the London Stock Exchange (LSE).
- The population may be either finite or infinite, while a sample is a selection of *just some items from the population*. A population is finite if it contains a fixed number of elements. In general, either all of the observations for the entire population will not be available, or they may be so many in number that it is infeasible to work with them, in which case a *sample* of data is taken for analysis.
- The sample is usually *random*, and it should be *representative* of the population of interest. A random sample is one in which each

individual item in the population is equally likely to be drawn.

- A *stratified sample* is obtained when the population is split into *layers* or *strata* and the number of observations in each layer of the sample is set to try to match the corresponding number of elements in those layers of the population.
- The *size of the sample* is the number of observations that are available, or that the researcher decides to use, in estimating the parameters of the model.

2.3.1 Measures of Central Tendency

The average value of a series is sometimes known as its *measure of location* or *measure of central tendency*. The average is usually thought to measure the 'typical' value of a series. There are a number of methods that can be used for calculating averages. The most well known of these is the *arithmetic mean* (usually just termed 'the mean'), denoted $\bar{r}_A$ for a series r_i of length N, which is simply calculated as the sum of all values in the series divided by the number of values

$$\bar{r}_A = \frac{1}{N}\sum_{i=1}^{N} r_i \tag{2.4}$$

EXAMPLE 2.1

Calculate the mean of the following numbers: 2, 4, −6, 7, 1, 0, 20.

The series has $N = 7$ items. The mean, $\bar{r}_A = (2 + 4 - 6 + 7 + 1 + 0 + 20)/7 = 10.86$ (to two decimal places).

The two other methods for calculating the average of a series are the *mode* and the *median*. The mode measures the most frequently occurring value in a series, which is sometimes regarded as a more representative measure of the average than the mean. Finally, the *median* is the middle value in a series when the elements are arranged in an ascending order.[1] If there is an even number of values in a series, then strictly there are two medians. For example, consider a variable that has taken the values listed in order: {3, 7, 11, 15, 22, 24}, the medians are 11 and 15. Sometimes we take the mean of the two medians, so that the median would be (11 + 15)/2 = 13.

Each of these three measures of average has its relative merits and demerits, which will now be discussed.

- The mean is the most familiar method to most researchers, is most easily used in algebraic formulae (see the discussion on expected values below), and has desirable econometric properties (most notably, it is, under some assumptions, unbiased and efficient as we will demonstrate in Chapter 3). But it can be unduly affected by extreme values (what are often termed *outliers*) and, in such cases, it may not be representative of most of the data. It should be evident for the example above that the mean of the series, 10.86, is larger than all but one of the data points. If the final data point had been −2 instead of 20, the mean would have been reduced to 7.71, and so just one data point can have a profound effect on the mean of a series.
- The mode is arguably the easiest to obtain, but is not suitable for continuous, non-integer data (e.g., returns or yields) or for distributions that incorporate two or more peaks (known as bimodal and multi-modal distributions, respectively). The mode has the advantage that, unlike the other two measures of average, the mode is guaranteed to be one of the observations. A commonly presented example of why the mode can be useful is that of a shoemaker who needs to know the number of pairs of shoes of each size to produce and asks his or her apprentice to give him or her one number that summarises the sizes of people's feet. The mean would be useless in this case, for what use is it to know that the mean shoe size is 8.9? On the other hand, if we know that the modal size is 7, this at least tells us that 7 is a more commonly occurring shoe size than any other. In different situations, especially if the distribution of the variable of interest is skewed (see below), the mode would be less useful. For example, if we were interested in knowing how much money the 'average' student gives to charity per month, it would not give us much information to know that the mode is zero.
- The median is often considered to be a useful representation of the 'typical' value of a series, and is robust to outliers, which is valuable if these are not of interest. But the median has the drawback that its calculation is based essentially on one observation. Thus if, say, we had a series containing ten observations and we were to double the values of the top three data points, the median would be unchanged. For example, the median of the set of data points: {1, 1, 1, 1, 100,

100, 100} is 1 and the median of {1, 1, 1, 1, 200, 200, 200} is also 1.

The Geometric Mean

There also exists another method that can be used to estimate the average of a series, known as the *geometric mean*. As briefly mentioned in Chapter 1, it involves calculating the *N*th root of the product of *N* numbers. In other words, if we want to find the geometric mean of six numbers, we multiply them together and take the sixth root (i.e., raise the product to the power of $\frac{1}{6}$).

In finance, we usually deal with *returns* or percentage changes (which could be positive, negative or zero) rather than prices or actual values, and the method for calculating the geometric mean described in the previous paragraph cannot handle zero or negative numbers. Therefore, we use a slightly different approach in such cases. To calculate the geometric mean of a set of N returns,[2] we express them as proportions (i.e., on a (−1, 1) scale) rather than percentages (on a (−100, 100) scale), and we would use the formula

$$\overline{R}_G = [(1 + r_1)(1 + r_2) \dots (1 + r_N)]^{1/N} - 1 \qquad \textbf{(2.5)}$$

where $r_1, r_2, \dots, r_N$ are the returns on a single asset or portfolio at each of N points in time and $\overline{R}_G$ is the calculated value of the geometric mean. Hence what we would do would be to add one to each return, then multiply the resulting expressions together, raise this product to the power $1/N$ and then subtract one right at the end. Return calculations will be discussed in considerable detail in Section 2.6 later in this chapter.

So which method for calculating mean returns (arithmetic or geometric) should we use? The answer is, as usual, that 'it depends'. Geometric returns give the fixed return on the asset or portfolio that would have been required to match the actual performance, which is not the case for the arithmetic mean. Thus, if you assumed that the arithmetic mean return had been earned on the asset every year, you would not reach the correct value of the asset or portfolio at the end. The reason is that the effect of compounding: the money that you have available to invest in year two will be the sum of the original investment and however much money was made or lost in year one, which implies that the investment values in each year are not independent of one another (even if the individual annual returns are). So if you invested £1000 at time zero, but the fund performed poorly and lost 20% in year one, you would only have £800 going into year two and would need a 25% positive return that year just to get back your original investment. The arithmetic averaging implicitly ignores this compounding effect and assumes that you always had the original investment amount at the start of each new year.

Note that if the individual annual returns are already continuously compounded (i.e., log returns – see Box 2.3 on p. 78), then it would be more appropriate to use the arithmetic average to calculate overall performance rather than the geometric average, since with log returns the effect of compounding has already been taken into account. An extensive discussion of compounding and its effects will be presented in Section 2.6 later in this chapter. In fact, the formula that links the simple return, R_t, with the continuously compounded return, r_t, is simply

$$R_t = e^{r_t} - 1 \quad \textbf{(2.6)}$$

If we plug some numbers into this equation, we can see that the continuously compounded return will be slightly smaller or more negative than the simple return, and the difference between the two will be bigger for large returns. For example, if the continuously compounded return r_t is 2, the equivalent simple return R_t will be 2.02; if r_t is 10, R_t will be 10.52, if $r_t = -4$, $R_t = -3.92$; $r_t = -20$, $R_t = -18.13$, etc.

But it can also be shown that the geometric return is always less than or equal to the arithmetic return, and so the geometric return is a downward-biased predictor of future performance. Hence, if the objective is to summarise historical performance, the geometric mean is more appropriate, but if we want to forecast future returns, the arithmetic mean is the one to use. Finally, it is worth noting that the geometric mean is evidently less intuitive and less commonly used than the arithmetic mean, but it is less affected by extreme outliers than the latter. There is an approximate relationship which holds between the arithmetic and geometric means, calculated using the same set of returns

$$\overline{R}_G \approx \overline{r}_A - \frac{1}{2}\sigma^2 \quad \textbf{(2.7)}$$

where $\overline{R}_G$ and $\overline{r}_A$ are the geometric and arithmetic means respectively and σ^2 is the variance of the returns. We can see from this formula that the arithmetic mean is higher than the geometric mean unless there is zero volatility and thus it is hardly surprising that it is more common for fund managers to report their arithmetic mean returns! We can also see that the higher the volatility, the greater will be the difference between the two measures of average returns, and thus the more the arithmetic average will overstate the investor experience and how much his or her money would have grown over time.

2.3.2 Measures of Spread

Usually, the average value of a series will be insufficient to adequately characterise a data series, since two series may have the same mean but very different profiles because the observations on one of the series may be much more widely spread about the mean than the other. Hence, another important feature of a series is how dispersed its values are. In finance theory, for example, the more widely spread are the returns around their mean value, the more risky the asset is usually considered to be.

Percentiles of a Distribution

The percentiles (sometimes also known as the *quantiles*) of a distribution provide information on where a particular observed value sits within the ordered set of all values. To illustrate, it is common to examine information on the weight of a baby compared with the weights of all other babies, and a parent might be given information that their baby is at the 80th percentile of the distribution of weights. This would imply that this baby was heavier than 80% of all other babies in the database. Or in other words, if we ordered all the babies in a line according to their weight with the lightest on the left and the heaviest on the right, this baby would have 80% of the others to his or her left. On the other hand, if the baby's weight is at the fifth percentile, this would mean that he/she was heavier than only 5% of other babies (or put equivalently, that 95% of other babies were heavier than him or her).

It should already be obvious that, by definition, the median is the 50th percentile, but providing information on other percentiles of an empirical distribution of real data can give us useful clues about its shape. The 0th percentile and the 100th percentile would, respectively, define the minimum and maximum values in the dataset, while the first and fifth percentiles have specific uses in financial risk management, as it is often the case that we want to focus on the lowest 1% or 5% of historical returns that have occurred.

The difference between two percentiles can be used as a measure of the spread of a distribution. The simplest such measure is arguably the *range*, which is calculated by subtracting the smallest observation from the largest. While the range has some uses, it is fatally flawed as a measure of dispersion by its extreme sensitivity to an outlying observation since it is effectively based only on the very lowest and very highest values in a series, and it ignores all of the other data points.

A more reliable measure of spread, although it is not widely employed by quantitative analysts, is the *semi-interquartile range*, sometimes known as the *quartile deviation*. Calculating this measure involves first ordering the data and then splitting the sample into four parts (*quartiles*) with equal numbers of observations.[3] The second quartile will be exactly at the half way point, and is the median, as described above. But the interquartile range focuses on the first and third quartiles, which will be at the quarter and three-quarter points in the ordered series, and which can be calculated, respectively, by the following

$$Q_1 = \left(\frac{N+1}{4}\right)^{\text{th}} \text{value} \qquad \textbf{(2.8)}$$

and

$$Q_3 = \frac{3}{4}(N+1)^{\text{th}} \text{value} \qquad \textbf{(2.9)}$$

The interquartile range is then given by the difference between the two

$$IQR = Q_3 - Q_1 \qquad \textbf{(2.10)}$$

This measure of spread is usually considered superior to the range since it is not so heavily influenced by one or two extreme outliers that by definition would be right at the end of an ordered series and so would affect the range. However, the semi-interquartile range still only incorporates two of the observations in the entire sample.

Variance and Standard Deviation

Another, more familiar, measure of the spread or dispersion of a set of data, the *variance*, is very widely used. It is interpreted as the average squared deviation of each data point about the mean value, and is calculated using the usual formula for the variance of a sample for a variable y

$$\sigma^2 = \frac{\sum (y_i - \bar{y})^2}{N - 1} \quad \textbf{(2.11)}$$

A further measure of spread, the standard deviation, is calculated by taking the square root of the variance formula given in equation (2.11)

$$\sigma = \sqrt{\frac{\sum (y_i - \bar{y})^2}{N - 1}} \quad \textbf{(2.12)}$$

The squares of the deviations from the mean are taken rather than the deviations themselves to ensure that positive and negative deviations (for points above and below the average, respectively) do not cancel each other out.

While there is little to choose between the variance and the standard deviation in terms of which is the best measure, the latter is sometimes preferred since it will have the same units as the variable whose spread is being measured, whereas the variance will have units of the square of the variable. So, for example, if y_i are observations on the prices of houses in a particular region in thousands of UK pounds, then σ^2 will have units of the square of prices (i.e., millions of pounds in this case) while σ will have units of thousands of pounds and so is more intuitive to interpret.

Both variance and standard deviation share the advantage that they encapsulate information from all the available data points, unlike the range and quartile deviation, although they can also be heavily influenced by outliers (but to a lesser degree than the range). The quartile deviation is an appropriate measure of spread if the median is used to define the average value of the series, while the variance or standard deviation will be appropriate if the arithmetic mean constitutes the measure of central tendency adopted.

Before moving on, it is worth discussing why the denominator in the formulae for the variance and standard deviation include $N - 1$ rather than N, the sample size. Subtracting one from the number of available data points is known as a *degrees of freedom correction*, and this is necessary since the spread is being calculated about the mean of the series, and this mean has had to be estimated from the sample data as well. Thus the spread measures described above are known as the *sample* variance and the *sample* standard deviation. Had we been observing the entire population of data rather than a mere sample from it, then the formulae would not need a degrees of freedom correction and we would divide by N rather than $N - 1$.

A further measure of dispersion is the *negative semi-variance*, which also gives rise to the *negative semi-standard deviation*. These measures use identical formulae to those described above for the variance and standard deviation, but when calculating their values, only those observations for which $y_i < \bar{y}$ are used in the sum, and N now denotes the number of such observations. This measure is sometimes useful if the observations are not symmetric about their mean value (i.e., if the distribution is *skewed* – see the next section),[4] and since they ignore deviations above the mean, they are sometimes used as measures of downside risk.

The Coefficient of Variation

A final statistic that has some uses for measuring dispersion is the *coefficient of variation, CV*. This is obtained by dividing the standard deviation by the arithmetic mean of the series (often multiplied by 100 to express it in percentage terms)

$$CV = \frac{\sigma}{\bar{y}} \qquad \textbf{(2.13)}$$

CV is useful where we want to make comparisons across series. Since the standard deviation has units of the series under investigation, it will scale with that series. Thus, if we wanted to compare the spread of monthly apartment rental values in London with those in Manchester, say, using the standard deviation would be misleading as the average rental value in London will be much bigger. By *normalising* the standard deviation, the coefficient of variation is a unit-free (*dimensionless*) measure of spread and so could be used more appropriately to compare series that have different scales.

2.3.3 Higher Moments

If the observations for a given set of data follow a normal distribution, then the mean and variance are sufficient to entirely describe the series. In other words, it is impossible to have two different normal distributions with the same mean and variance. However, most samples of data do not follow a normal distribution, and therefore we also need what are known as the *higher moments* of a series to fully characterise them. The mean and the variance are the first and second moments of a distribution, respectively, and the (standardised) third and fourth moments are known as the *skewness* and *kurtosis*, respectively.

Skewness defines the shape of the distribution, and measures the extent to which it is not symmetric about its mean value. When the distribution of data is symmetric and unimodal (i.e., it only has one peak rather than many), the three methods for calculating the average (mean, mode and median) of the sample will be equal. If the distribution is positively skewed (where there is a long right hand tail and most of the data are bunched over to the left), the ordering will be *mean* > *median* > *mode*, whereas if the distribution is negatively skewed (a long left hand tail and most of the data bunched on the right), the ordering will be the opposite. A normally distributed series has zero skewness (i.e., it is symmetric).

Kurtosis measures the fatness of the tails of the distribution and how peaked at the mean the series is. A normal distribution is defined to have a coefficient of kurtosis equal to three. It is possible to define a coefficient of excess kurtosis, equal to the coefficient of kurtosis minus three; a normal distribution will thus have a coefficient of excess kurtosis of zero. A normal distribution is said to be *mesokurtic*.

Denoting the observations on a series by y_i and their variance by σ^2, it can be shown that the coefficients of skewness and kurtosis can be calculated respectively as[5]

$$skew = \frac{\frac{1}{N-1}\sum (y_i - \bar{y})^3}{(\sigma^2)^{3/2}} \qquad \textbf{(2.14)}$$

and

$$kurt = \frac{\frac{1}{N-1}\sum (y_i - \bar{y})^4}{(\sigma^2)^2} \qquad \textbf{(2.15)}$$

It is worth noting that, given the way that they are constructed, the skewness can be positive or negative while the kurtosis can only be positive (or zero), in the same way that a variance cannot be negative.

To give some illustrations of what a series having specific departures from normality may look like, consider Figures 2.4 and 2.5. A normal distribution is symmetric about its mean, while a skewed distribution will not be, but will have one tail longer than the other (Figure 2.4).

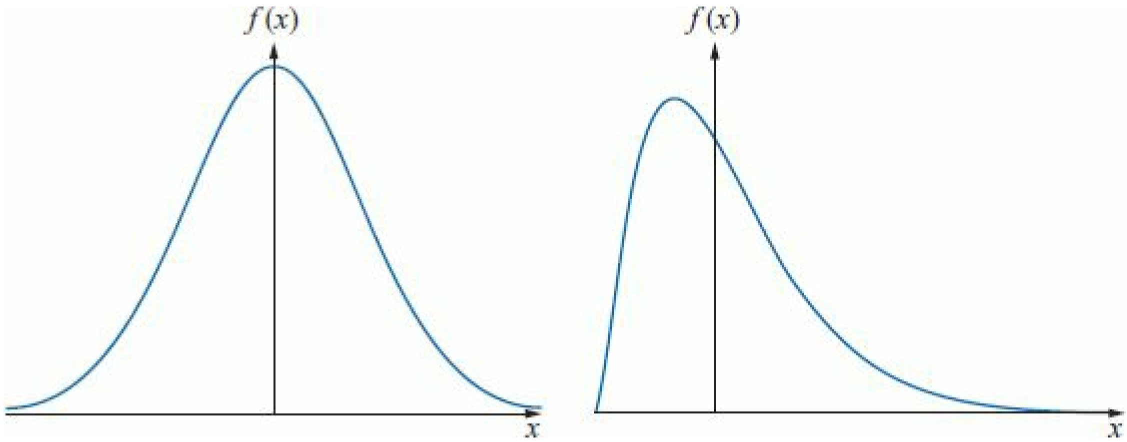


Figure 2.4 A normal versus a skewed distribution

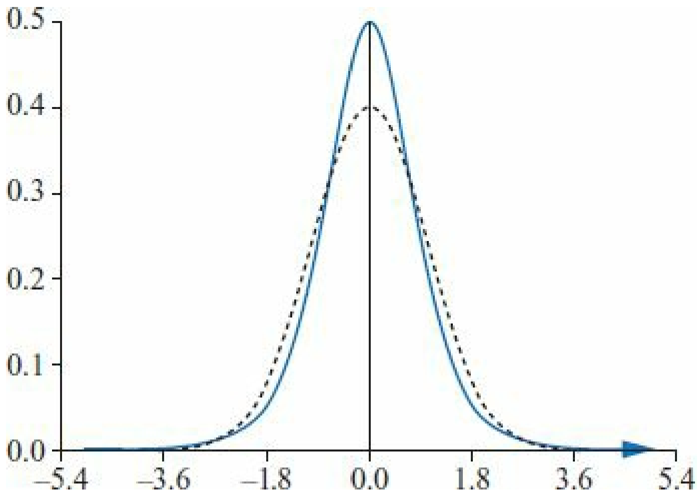


Figure 2.5 A normal versus a leptokurtic distribution

A leptokurtic distribution is one which has fatter tails and is more peaked at the mean than a normally distributed random variable with the same mean and variance, while a platykurtic distribution will be less peaked in the mean, will have thinner tails, and more of the distribution in the shoulders than a normal. In practice, a leptokurtic distribution is more likely to characterise real estate (and economic) time series, and to characterise the residuals from a time-series model. In Figure 2.5, the leptokurtic distribution is shown by the blue line, with the normal by the dotted line. There is a formal test for normality, and this will be described and discussed in Chapter 5.

2.3.4 Measures of Association

The summary measures we have examined so far have looked at each series in isolation. However, it is also very often of interest to consider the links between variables. There are two key descriptive statistics that are used for measuring the relationships between series: the covariance and the correlation.

Covariance

The *covariance* is a measure of linear association between two variables and represents the simplest and most common way to enumerate the relationship between them. It measures whether they on average move in the same direction (positive covariance), in opposite directions (negative covariance), or have no association (zero covariance). The formula for calculating the covariance, $\sigma_{x,y}$, between two series, x and y, is given by

$$\sigma_{x,y} = \frac{\sum (x_i - \overline{x})(y_i - \overline{y})}{(N-1)} \qquad \textbf{(2.16)}$$

Correlation

A fundamental weakness of the covariance as a measure of association is that it scales with the standard deviations of the two series, so it has units of $x \times y$. Thus, for example, multiplying all of the values of series y by ten will increase the covariance tenfold, but it will not really increase the true association between the series since they will be no more strongly related than they were before the rescaling. The implication is that the particular numerical value that the covariance takes has no useful interpretation on its own and hence is not particularly useful. Therefore, the *correlation* takes the covariance and standardises or normalises it so that it is unit free. The result of this standardisation is that the correlation is bounded to lie on the (−1,1) interval. A correlation of 1 (−1) indicates a perfect positive (negative) association between the series. The correlation measure, usually known as the *correlation coefficient*, is often denoted $\rho_{x,y}$, and is calculated as

$$\rho_{x,y} = \frac{\sum (x_i - \overline{x})(y_i - \overline{y})}{(N-1)\sigma_x \sigma_y} = \frac{\sigma_{x,y}}{\sigma_x \sigma_y} \qquad \textbf{(2.17)}$$

where σ_x and σ_y are the standard deviations of x and y, respectively.

This measure is more strictly known as *Pearson's product moment correlation*. To calculate Pearson's correlation validly requires that the series are linearly related to one another and any formal hypothesis tests involving this correlation measure would require the two series under study to be normally distributed. In cases where this does not apply, we can instead use *Spearman's rank correlation*. As the name suggests, using this measure involves calculating the ranks of each element of the two separate series and then computing the correlation between the two series of ranks in the usual way. Spearman's rank correlation is an example of a *nonparametric* test since it does not require any distributional assumptions (e.g., normality) to be validly applied.

Copulas

Covariance and correlation provide simple measures of association between series. However, as is well known, they are very limited measures in the sense that they are linear and are not sufficiently flexible to provide full descriptions of the relationship between financial series in reality. In particular, new types of assets and structures in finance have led to increasingly complex dependencies that cannot be satisfactorily modelled in this simple framework. *Copulas* provide an alternative way to link together the individual (*marginal*) distributions of series to model their joint distribution. One attractive feature of copulas is that they can be applied to link together any marginal distributions that are proposed for the individual series. The most commonly used copulas are the Gaussian and Clayton copulas. They are particularly useful for modelling the relationships between the tails of series, and find applications in stress testing and simulation analysis. For introductions to this area and applications in finance and risk management, see Nelsen (2006) and Embrechts *et al.* (2013).

2.3.5 An Example of How to Calculate Summary Statistics

We now build an example that pulls together all of the material above on computing summary statistics. Suppose that we have annual data on the performance (annual returns in per cent) of two fund managers working for the same company, Risky Ricky and Safe Steve – you can see where this is going – for 13 years between 2005 and 2017. There were some scandals involving the company and investors withdrew their funds in large numbers so unfortunately one of the fund managers has to be made redundant to save money. Which of the two fund managers has shown the best performance and so should be retained?

If we look at the data in Table 2.1, which shows the annual investment returns on each of the two managers’ portfolios (in the first two columns after the years), it is clear that Ricky’s returns are more volatile – that is, they move up and down more – but it is not clear just by looking which fund manager has performed better taking into account all of the years. Ignore the final two columns of ranks for now.

Table 2.1 Annual performance of two funds

Year	Risky Ricky	Safe Steve	Ricky Ranks	Steve Ranks
2005	24	9	2	3
2006	18	7	3	4
2007	4	5	7	6
2008	−23	−8	13	13
2009	−12	−3	11	12
2010	1	3	8	8
2011	7	4	6	7
2012	12	2	5	10
2013	−6	3	9	8
2014	−14	6	12	5
2015	−7	2	10	10
2016	56	19	1	1
2017	14	12	4	2

Note: The first two data columns give the annual performance of each fund in percentage points for the 13-year sample period, while the final two columns provide the ranks of each year’s returns within the 13 annual return figures for each fund manager.

Figure 2.6 includes a time-series plot of the performance of the two managers by year in panel A (left-hand side), while panel B (right-hand side) presents a scatter plot of the returns of the two managers with Ricky on the *x*-axis scale and Steve on the *y*-axis so that we can see if the two lie roughly on a straight line. Having a look at the data and getting a feel for it before conducting more sophisticated analysis is called *exploratory data analysis*. It is very important to always plot and summarise the data before building any models as the preliminary analysis can often inform the research agenda for more sophisticated models and can avoid the kinds of mistakes that can arise when researchers go straight into autopilot’.

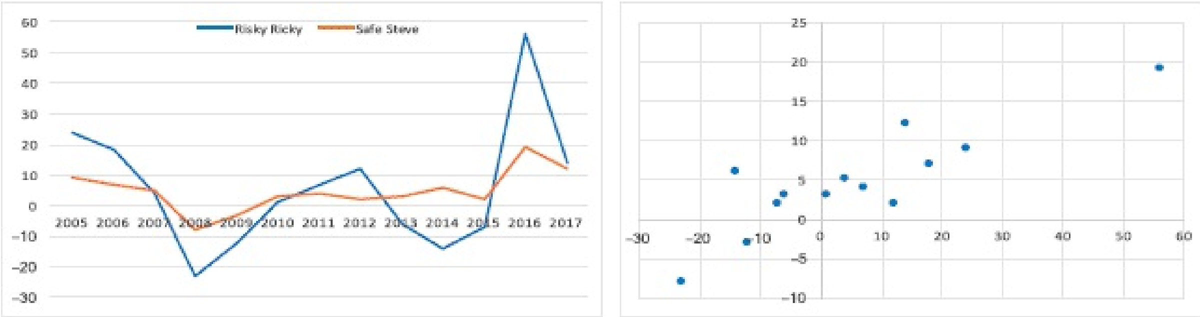


Figure 2.6 A time-series plot and scatter plot of the performance of two fund managers

We can compute the summary statistics using the formulae above, either manually by plugging the numbers in and using a pocket calculator or by using a spreadsheet or econometrics package. If we use Excel and the data on Ricky’s annual performance is in cells C4 to C16, we can use the following four built-in Excel functions to compute the mean, standard deviation, skewness and kurtosis of the annual returns in cells C17 to C20 respectively as

=AVERAGE(C4:C16)

=STDEV(C4:C16)

=SKEW(C4:C16)

=KURT(C4:C16)

Next, if the annual returns for Steve are in cells 4 to 16 of column D, we can drag the four resulting figures for the mean, standard deviation, skewness and kurtosis above across from column C into the next column D to calculate them for Steve. Note that the formula above will calculate the arithmetic average return rather than the geometric mean – whether this is the correct approach will depend on how the original annual return figures were calculated, as discussed earlier in this chapter. Excel has a function =GEOMEAN that can be used to calculate the geometric mean of a series, but this will only work when all of the numbers to be averaged are positive as it uses a different formula to equation (2.5) above.

If we look at the mean returns, we can see that Ricky’s is a full percentage point higher (5.69 versus 4.69, with all figures expressed to two decimal places). But this higher mean for Ricky comes at the expense of Ricky having a much higher standard deviation of returns and it is clear from the plot in Figure 2.6 that his performance is much more volatile: doing better in good years and worse in bad years.

What about the higher moments? It is possible to show (see Scott and Horvath, 1980) that investors care about all moments of the return distribution, not just the first two. We know that investors prefer higher values of the first moment (the mean) and lower values of the second moment (the standard deviation)[6] but Scott and Horvath demonstrate that investors also prefer higher values of all odd moments and lower values of all even moments, so that they want a higher skewness and a lower kurtosis.

If we look at the higher moment values for Ricky and Steve, the results on who is the better performer are again inconclusive. The skewness figures are, respectively, 1.12 and 0.29, and although they are both positive, this favours Ricky. On the other hand, the kurtosis values are 2.10 and 1.46, and therefore Steve's is better because it is lower and thus the distribution is more tightly centred around the mean (for given mean and variance).

So the company owner has a dilemma if he or she only wants to retain one of the staff: Ricky has a better (higher) mean and skewness while Steve has a better (lower) standard deviation and kurtosis. The way to pick one of the two fund managers would be to construct a composite performance measure that includes information from more than one moment of the distribution simultaneously. The Sharpe ratio, which is calculated as the mean fund return (minus the risk-free rate of return) divided by the standard deviation of returns, is one such performance measure that is very popular.

The Sharpe ratio is simple to calculate and very widely used but it only includes information from the first two moments of the portfolio return distribution. So strictly it is valid only if either investors care just about the first two moments (i.e., they ignore skewness and kurtosis) or if returns follow a normal distribution so that the skewness is zero and the kurtosis is the same as that of a normal distribution (equal to 3) – see above. In practice, however, neither of these two restricted cases are likely to apply and so it would be preferable to adopt a composite performance measure that incorporates information from the whole distribution rather than just the two moments, such as an appropriate utility function – see, for example, Brooks, Cerny and Miffre (2012) for a discussion.

We might also be interested to know the relationship between the returns on Ricky's fund and those of Steve's fund. Do the values of the two portfolios tend to move together over time, which would be the case if they were investing in similar asset classes, albeit Ricky's being more volatile? We can get an idea of this by computing the correlation between the two sets of returns.

In Excel, we can obtain covariances and correlations by using the =COVAR(C4:C16,D4:D16) and =CORREL(C4:C16,D4:D16) functions, respectively. As discussed above, covariances scale with the data and so are hard to interpret, but correlations must lie between −1 and +1. The Pearson correlation between Ricky’s and Steve’s sets of returns is 0.87, which is very close to one, indicating that the two series do move very closely together over time. This is also evident from the scatter plot on the right in Figure 2.6, where each return pairing lies close to a positive upward straight line.

For interest, we also calculate the Spearman rank correlation measure. This is achieved by calculating the ranks for each member of the two return series – for example, to calculate the rank of the first observation in Ricky’s series, we would use the function =RANK(C4,C$4:C$16). We then drag this command down the column and drag that column across to create a similar set of ranks for Steve’s returns. The final two columns in Table 2.1 present the ranks for Ricky and Steve, respectively. As can be seen, again these are highly related. Finally, to calculate the rank correlation measure, we simply use the CORREL formula on the two columns of ranks, and this results in a calculated figure of 0.76, which is a bit lower than for the Pearson correlation, but nonetheless suggestive that the two series very much move together.

2.3.6 Useful Algebra for Means, Variances and Covariances

There are several fairly straightforward equations that are useful for working with the expectations operator, the variance operator, and the covariance operator. In other words, these equations show how expressions for means, variances and covariances of random variables can be manipulated. The mean of a random variable y is also known as its expected value, written $E(y)$. The properties of expected values are used widely in econometrics, and are listed below, referring to a random variable y

- The expected value of a constant (or a variable that is non-stochastic) is the constant, e.g., $E(c) = c$.
- The expected value of a constant multiplied by a random variable is equal to the constant multiplied by the expected value of the variable: $E(c\, y) = c\, E(y)$. It can also be stated that $E(c\, y + d) = c\, E(y) + d$,

where d is also a constant.

- For two independent random variables, y_1 and y_2, $E(y_1 y_2) = E(y_1) E(y_2)$.

The variance of a random variable y is usually written var(y). The properties of the 'variance operator', var($\cdot$), are

- The variance of a random variable y is given by $\text{var}(y) = E[y - E(y)]^2$
- The variance of a constant is zero: $\text{var}(c) = 0$
- For c and d constants, $\text{var}(c\,y + d) = c^2\,\text{var}(y)$
- For two independent random variables, y_1 and y_2, $\text{var}(c\,y_1 + dy_2) = c^2\text{var}(y_1) + d^2\text{var}(y_2)$.

The covariance between two random variables, y_1 and y_2 may be expressed as $\text{cov}(y_1, y_2)$. The properties of the covariance operator are

- $\text{cov}(y_1, y_2) = E[(y_1 - E(y_1))(y_2 - E(y_2))]$
- For two independent random variables, y_1 and y_2, $\text{cov}(y_1, y_2) = 0$
- For four constants, c, d, e, and f, $\text{cov}(c + dy_1, e + fy_2) = df\,\text{cov}(y_1, y_2)$.

2.4 Types of Data and Data Aggregation

There are broadly three types of data that can be the employed in quantitative analysis of financial problems: time-series data, cross-sectional data and panel data. Each of these will be discussed in turn next, but first it is worth mentioning another feature of data to watch out for, its *degree of aggregation*. Many data forms begin as individual observations but for various reasons are then aggregated. For example, we could measure the selling price of a specific house in a specific street every time it is sold and observe how and why it changes over time. But usually, a particular house is unlikely to be sold more often than once every five or ten years. So it is common to form house price *indices*, which measure the 'average' value of houses sold during a specific time period such as a month. Thus, the individual sales prices of a number of houses would be combined (aggregated) in some way and transformed into an index. House prices could be aggregated to the street level, the town level, the county level or the country level. All of these indices would get around the problem that specific houses are not sold very often by combining the information from sales of many different houses. However, the researcher who constructs the index needs to be careful to compare like with like or to adjust the data in some way to account for the variations in the types of houses sold and to produce what are sometimes termed *constant quality house price indices*.

We might be interested in looking at the national pattern – and in such cases we would want a national index – for example, we might want to know what factors are causing house prices to decline in the UK as a whole over a particular time period and thus the prices of individual properties or of prices in specific towns are irrelevant.

Note that by using aggregate data, we can see the ‘big picture’, but we lose a lot of detail. To illustrate, it might be that prices are rising when averaged across the whole of the UK, but if we look at disaggregate data for England, Scotland, Wales and Northern Ireland, we might find that actually, only prices in England are rising while those in the other component countries are all falling. But since there are more houses sold in England than elsewhere, the overall average is rising.

2.4.1 Time-Series Data

Time-series data, as the name suggests, are data that have been collected over a period of time on one or more variables. Time-series data have associated with them a particular frequency of observation or frequency of collection of data points. The frequency is simply a measure of the *interval over*, or the *regularity with which*, the data are collected or recorded. Box 2.2 shows some examples of time-series data.

BOX 2.2 Time-series data

Series	*Frequency*
Industrial production	Monthly or quarterly
Government budget deficit	Annually
Money supply	Weekly
The value of a stock	As transactions occur

A word on ‘As transactions occur’ in Box 2.2 is necessary. Many financial data do not start their life as being *regularly spaced*. For example, the price of common stock for a given company might be recorded to have changed whenever there is a new trade or quotation placed by the financial information recorder. Such recordings are very unlikely to be evenly distributed over time – for example, there may be no activity between, say, 5 p.m. when the market closes and 8.30 a.m. the next day when it reopens; there is also typically less activity around the opening and closing of the market, and around lunch time. Although there are a number of ways to deal with this issue, a common and simple approach is to select an appropriate frequency, and use as the observation for that time period the last prevailing price during the interval.

It is also generally a requirement that all data used in a model be of the *same frequency of observation*. So, for example, regressions that seek to estimate an arbitrage pricing model using monthly observations on macroeconomic factors must also use monthly observations on stock returns, even if daily or weekly observations on the latter are available.

The data may be *quantitative* (e.g., exchange rates, prices, number of shares outstanding), or *qualitative* (e.g., the day of the week, a survey of the financial products purchased by private individuals over a period of time, a credit rating, etc.).

Problems that could be tackled using time-series data

- How the value of a country's stock index has varied with that country's macroeconomic fundamentals
- How the value of a company's stock price has varied when it announced the value of its dividend payment
- The effect on a country's exchange rate of an increase in its trade deficit

In all of the above cases, it is clearly the time dimension which is the most important, and the analysis will be conducted using the values of the variables over time.

2.4.2 Cross-Sectional Data

Cross-sectional data are data on one or more variables collected at a single point in time. For example, the data might be on:

- A poll of usage of internet stockbroking services
- A cross-section of stock returns on the New York Stock Exchange (NYSE)
- A sample of bond credit ratings for UK banks

Problems that could be tackled using cross-sectional data

- The relationship between company size and the return to investing in its shares
- The relationship between a country's GDP level and the probability that the government will default on its sovereign debt

2.4.3 Panel Data

Panel (also sometimes known as longitudinal) data have the dimensions of both time series and cross-sections, e.g., the daily prices of a number of blue chip stocks over two years. The estimation of panel regressions is an interesting and developing area, and will be examined in detail in Chapter 11.

Fortunately, virtually all of the standard techniques and analysis in econometrics are equally valid for time-series and cross-sectional data. For time-series data, it is usual to denote the individual observation numbers using the index t, and the total number of observations available for analysis by T. For cross-sectional data, the individual observation numbers are indicated using the index i, and the total number of observations available for analysis by N. Note that there is, in contrast to the time series case, no natural ordering of the observations in a cross-sectional sample. For example, the observations i might be on the price of bonds of different firms at a particular point in time, ordered alphabetically by company name. So, in the case of cross-sectional data, there is unlikely to be any useful information contained in the fact that Barclays follows Banco Santander in a sample of bank credit ratings, since it is purely by chance that their names both begin with the letter 'B'. On the other hand, in a time-series context, the ordering of the data is relevant since the data are usually ordered chronologically.

In this book, the total number of observations in the sample will be given by T even in the context of regression equations that could apply either to cross-sectional or to time-series data.

A final type of data, which we could argue is slightly different to any of the above, is *pooled* cross-section and time-series data. This occurs when the variable of interest has both time-series and cross-sectional dimensions, but for some reason we do not use these features and instead simply combine all of the observations together. For example, we might have monthly data for ten years on the amount of profit that six different traders in a team made, but if we ignore the time ordering of the data and we also ignore which traders generated which profits and simply put all of the profit figures into a single unordered column this would be a pooled sample. This would not be a panel of data since we would not be able to observe the performance of each individual trader in different months. Effectively, pooled data are treated as if they were simply a larger cross-sectional sample.

2.4.4 Continuous and Discrete Data

As well as classifying data as being of the time-series or cross-sectional type, we could also distinguish them as being either continuous or discrete, exactly as their labels would suggest. *Continuous* data can take on any value and are not confined to take specific numbers; their values are limited only by precision. For example, the rental yield on a property could be 6.2%, 6.24% or 6.238%, and so on. On the other hand, *discrete* data can only take on certain values, which are usually integers (whole numbers), and are often defined to be count numbers.[7] For instance, the number of people in a particular underground carriage or the number of shares traded during a day. In these cases, having 86.3 passengers in the carriage or $5857^1/_2$ shares traded would not make sense. The simplest example of a discrete variable is a *Bernoulli* or binary random variable, which can only take the values 0 or 1 – for example, if we repeatedly tossed a coin, we could denote a head by 0 and a tail by 1.

2.4.5 Cardinal, Ordinal and Nominal Numbers

Another way in which we could classify numbers is according to whether they are cardinal, ordinal or nominal. *Cardinal* numbers are those where the actual numerical values that a particular variable takes have meaning, and where there is an equal distance between the numerical values. On the other hand, *ordinal* numbers can only be interpreted as providing a position or an ordering. Thus, for cardinal numbers, a figure of 12 implies a measure that is ‘twice as good’ as a figure of 6. Examples of cardinal numbers would be the price of a share or of a building, and the number of houses in a street. On the other hand, for an ordinal scale, a figure of 12 may be viewed as ‘better’ than a figure of 6, but could not be considered twice as good. Examples of ordinal numbers would be the position of a runner in a race (e.g., second place is better than fourth place, but it would make little sense to say it is ‘twice as good’) or the level reached in a computer game.

The final type of data that could be encountered would be where there is no natural ordering of the values at all, so a figure of 12 is simply different to that of a figure of 6, but could not be considered to be better or worse in any sense. Such data often arise when numerical values are arbitrarily assigned, such as telephone numbers or when codings are assigned to qualitative data (e.g., when describing the exchange that a US stock is traded on, '1' might be used to denote the NYSE, '2' to denote the NASDAQ and '3' to denote the AMEX). Sometimes, such variables are called *nominal* variables. Cardinal, ordinal and nominal variables may require different modelling approaches or at least different treatments, as should become evident in the subsequent chapters.

2.5 Arithmetic and Geometric Series

A *series* or a *sequence* is simply a list of numbers in a particular order. An arithmetic series, also known as an *arithmetic progression*, is a sequence where a specific entry in that series is formed by adding a fixed number, known as the *common difference*, to the previous one. For example

$$2, 5, 8, 11, \ldots$$

$$-10, -30, -50, \ldots$$

The first of these is an arithmetic series with an initial value of 2 and adds 3 each time we move from one entry in the sequence to the next; the second row is an arithmetic series with an initial value of −10 and a common difference of −20. Arithmetic series do not have many uses in finance so we will not consider them further.

A geometric series (*geometric progression*), on the other hand, is a series where instead of adding a fixed amount to move from one entry to the next, we multiply by a fixed amount (the *common ratio*). For example

$$4, 8, 16, 32, \ldots$$

$$2, 1, 0.5, 0.25, \ldots$$

The first of these is a geometric series with an initial value of 4 and a common ratio of 2, while the second row is a geometric series with an initial value of 2 and a common ratio of 0.5. Geometric series are very useful in finance as they describe the situation where a sum of money is invested and earns a certain percentage of interest in each time period.

To develop some notation, let a denote the initial value of a geometric series (starting with the term numbered 0 and ending with term numbered $n - 1$), and let d denote the common ratio. Then we could write a geometric series containing n terms as

$$a, ad, ad^2, ad^3, \ldots, ad^{n-1}$$

There is an expression that can be used to calculate the sum of the first n terms, denoted S_n, of the series (running from a to ad^{n-1})

$$S_n = \frac{a(1 - d^n)}{1 - d} \tag{2.18}$$

For instance, if a geometric series begins with 2 and has a common ratio of 3, the sum of the first 8 terms would be

$$S_n = \frac{2 \times (1 - 3^8)}{1 - 3} = 6560 \tag{2.19}$$

As an exercise, calculate each of the first eight terms in this series and confirm that the sum is indeed 6560.

Of particular use in financial applications is the infinite sum of a geometric progression, denoted S_∞. We can see what will happen to $S_n = \frac{a(1-d^n)}{1-d}$ as n tends to infinity, since the term d^n will tend towards zero (so long as $0 < d < 1$), in which case the expression can be simplified as

$$S_\infty = \frac{a}{1 - d} \tag{2.20}$$

Here, even though there is an infinite number of terms in the series, their sum is finite. Note that if $d \geq 1$, the series would not 'converge' (i.e., successive terms would not become smaller and smaller) and therefore the sum would be infinite.

2.6 Future Values and Present Values

A fundamental concept in economics and finance is the notion that money has *time value*. This means that receipt of a given amount of money is worth a different amount depending on when it is received. In general, money has positive time value. This means that £100, for instance, is worth more if received today than next week, and worth more if received next week than if received next year. This arises for several reasons: cashflows are usually considered more risky the further in the future they are to be received (more time for something to go wrong!), inflation will erode the value of a fixed amount received in the future, and people have *positive time preference*, which means simply that they are impatient and would rather have stuff now than waiting until the future.

As a result of the time value of money, we cannot simply combine cashflows in their raw form into financial calculations if they are received at different points in time. The way that we ensure we are comparing like-with-like is to transform the cashflows to what they would be worth if they were all received at the same point in time. So we either transform all cashflows to the amount that they would be worth at some given point in the future (the *future value*) or we transform all future cashflows into the equivalent amount that they would be worth if received today (the *present value*). If we transform current values into future ones we are *compounding,* while if we transform future values into present ones we are *discounting*. We will now look at each of these two concepts in turn and examine how they are used.

2.6.1 Future Values

Suppose that we place £100 in a bank savings account for five years, paying an annual interest rate of 2%. The sum of money in the account at the end of the period would be given by

$$P_T = P_0 \times (1 + r)^T \qquad \textbf{(2.21)}$$

where P_T denotes the terminal (future) value of the account, r is the interest rate (expressed as a proportion – e.g., 0.02, rather than a percentage), P_0 is the amount placed in the account now, and T is the number of time periods for which the money is invested.

In this example, the *future value* of the investment at the end of the first year would be P_T = £100 × (1 + 0.02) = £102, while at the end of the second year it would have grown to P_T = £100 × $(1 + 0.02)^2$ = £102 × (1 + 0.02) = £104.04. The savings balance would continue to grow in this way until, at the end of the fifth year, it would have reached P_T = £100 × $(1 + 0.02)^5$ = £110.41.

We would say in this case that the interest is *compounded annually* – in other words, interest is paid this year on the total value of this year’s end savings, which will comprise both last year’s savings value and last year’s interest. So after the first year, the saver earns interest on their previous interest as well as on the amount invested. This is why the saver earned £2 in interest in year one but £2.04 in year two: the extra 4p in year two is the additional interest on the £2 that had been earned in year one.

We can rearrange the future value formula in equation (2.21) above to make r the subject and this would enable us to calculate the rate of interest required to secure a specific future value, P_T, given the initial investment P_0

$$r = \left[\frac{P_T}{P_0}\right]^{1/T} - 1 \tag{2.22}$$

For example, if we make an initial investment of £1000 and no further investments, and we leave the funds for ten years, what rate of interest is required to enable us to achieve a sum of £1500 by the end of the decade? The calculation is

$$r = \left[\frac{1500}{1000}\right]^{1/10} - 1 = 0.0414 \tag{2.23}$$

So an annual interest rate of about 4.14% is required.

A further re-arrangement of equations (2.21) and (2.22) enables us to make the term of the investment, T, the subject of the formula

$$T = \frac{\ln(P_T/P_0)}{\ln(1+r)} \tag{2.24}$$

So, for instance, if we can invest £1000 initially and wish it to grow to £2000, assuming an interest rate of 10% (we should be so lucky!), how many years do we need to wait? We would have

$$T = \frac{\ln(2000/1000)}{(\ln(1+0.1)} = 7.273 \tag{2.25}$$

Notice that we can use the formula (2.24) above to determine how many years it would take for an investment to grow by a factor of Z

$$T = \frac{\ln(Z)}{\ln(1+r)} \tag{2.26}$$

where $P_T = ZP_0$. So, for example, if we wanted to triple the initial investment, assuming that the interest rate remained at 10%, we would set $Z = 3$ and we would have

$$T = \frac{\ln(3)}{\ln(1+0.1)} = 11.527 \tag{2.27}$$

Thus about eleven an a half years are required – clearly a long time unless we could achieve an even higher interest rate!

All of the above examples assume that interest is paid annually at the end of the year. But now suppose instead that the account paid an annual rate of interest of 2% with the payments made every six months (i.e., 1% paid every six months). Many companies pay dividends and many bonds pay coupons semi-annually so this is empirically relevant. In this case, the compounding would be semi-annual rather than annual. We would be better off since we would receive interest in the second six months of each year on the interest paid in the first six months. This effect would be very small with such a low interest rate, but we would calculate the future value for the first example above in this sub-section by now using an interval of six months, an interest rate r of 1%, and a number of periods T of 10 (i.e., ten periods each of six months): $P_T = £100 \times (1 + 0.01)^{10} = £110.46$ – not much extra interest to get excited about.

If the interest was paid (compounded) monthly, the terminal value would be $P_T = £100 \times (1 + (0.02/12))^{60} = £110.51$. We can see that the higher the compounding frequency, the more interest would actually be received for a given *nominal* interest rate of 2%. We would call all of these situations where the compounding takes place discretely a *simple interest* calculation.

In the above example, the *nominal interest rate* is 2% per annum but if the interest is compounded more frequently than annually, the actual interest rate received, known as the *effective interest rate*, will be higher. We can calculate this interest rate on an annualised basis simply as

$$\text{effective rate} = \left[1 + \frac{r}{n}\right]^n - 1 \qquad \textbf{(2.28)}$$

where r is the nominal rate and n is the number of compounding periods per year.

EXAMPLE 2.2

What is the effective rate when the nominal rate, r, is 2% and interest is compounded monthly ($n = 12$)?

It would be: effective $\text{rate} = \left[1 + \frac{0.02}{12}\right]^{12} - 1 = 2.02\%$ (to two decimal places).

Another useful formula is one that calculates the future value of an investment (P_T) when interest of r per year in total is paid n times per year for a T years on the original amount P_0

$$P_T = P_0\left[1 + \frac{r}{n}\right]^{nT} \quad \textbf{(2.29)}$$

In the limit, as the compounding frequency increases and so we have more and more shorter and shorter time periods (i.e., we move from annual to monthly to weekly to daily to hourly compounding and so on), we would eventually reach a situation where the time period was infinitesimally small. We would term this *continuous compounding*. If interest is compounded continuously at an annual equivalent rate r, we would write

$$P_T = P_0 e^{rT} \quad \textbf{(2.30)}$$

where e is the exponential number discussed in section 1.5.5 of Chapter 1. If T = 5 and r = 2%, then the terminal value if interest is continuously compounded is $P_T = e^{0.02\times 5}$ = £110.52. This is barely any different to the terminal value when interest is earned monthly but the difference would be more noticeable if r was higher, as the examples in Table 2.2 show.

Table 2.2 Impact of different compounding frequencies on the effective interest rate and terminal value of an investment

Compound frequency	Number of periods per year (n)	Equivalent annual rate and terminal value					
		r=5%		r=10%		r=20%	
		EAR	P_T	EAR	P_T	EAR	P_T
Annual	1	5.00%	105.00	10.00%	110.00	20.00%	120.00
Quarterly	4	5.09%	105.09	10.38%	110.38	21.55%	121.55
Monthly	12	5.12%	105.12	10.47%	110.47	21.94%	121.94
Weekly	52	5.12%	105.12	10.51%	110.51	22.09%	122.09
Daily	365	5.13%	105.13	10.52%	110.52	22.13%	122.13
Continuously	∞	5.13%	105.13	10.52%	110.55	22.14%	122.14

Analogous to the formulae (2.22) and (2.24) above for simple interest calculations, we can re-arrange the expression (2.30) to make the continuously compounded interest rate the subject of the formula

$$r = \frac{1}{T} \ln \left[\frac{P_T}{P_0} \right] \quad \textbf{(2.31)}$$

And we can make the number of years of investment the subject of the formula

$$T = \frac{1}{r} \ln \left[\frac{P_T}{P_0} \right] \quad \textbf{(2.32)}$$

Table 2.2 shows the effects of different interest rates and compounding frequencies on the terminal value of a £100 investment. Two results can clearly be seen from the table. The first is that the effect of compounding is stronger the higher the nominal interest rate since the additional benefit from getting some of the interest payment early and reinvesting it is greater in such cases. The second observable feature is that the incremental effect of further increasing the compounding frequency gradually reduces. For example, going from annual to quarterly compounding has a bigger effect even than going from quarterly to continuous compounding.

2.6.2 Present Value

The reverse of calculating the future value of an amount of money that is earning interest in a bank account would be where we calculate the *present value* of an amount of money to be received at some point in the future. Instead of an interest rate as we would have for a future value calculation, in the case of present values we use a *discount rate*, which is the rate at which we would reduce the future payment into today's terms. We would write

$$P_0 = \frac{P_T}{(1+r)^T} \quad \textbf{(2.33)}$$

where P_0 is the present value, r is now the discount rate, P_T is the sum to be paid or received in the future, and T is the number of periods into the future that it will be paid or received.

EXAMPLE 2.3

What is the present value of £100 to be received in five years' time if the discount rate is 2%?

This would be P_0 = £100/(1 + 0.02)^5 = £90.57. This shows that £100 in five years' time is worth £90.57 in today's money terms. Such present value calculations underpin most of the valuation models employed in finance as the situation is that investors purchase assets now and receive cashflows in the future, which then need to be converted into today's terms (i.e., discounted back to the present) so that the amount to be paid for the asset now and the amount received in the future can be compared in equivalent terms.

EXAMPLE 2.4

To illustrate a finance application, suppose we have a bond that pays a £5 coupon annually with the next coupon due immediately, the bond has exactly five years left to maturity when it will be redeemed at its par value of £100 and an appropriate discount rate is 10%. What would be a fair price to pay today for the bond?

We would calculate the fair price as the discounted sum of the six coupon payments (one now and one at the end of each of the next five years) and plus the discounted value of the par amount of the bond. If we let the fair price in pounds be denoted by P_0, the calculation would be

$$P_0 = 5 + \frac{5}{(1+0.1)} + \frac{5}{(1+0.1)^2} + \frac{5}{(1+0.1)^3} + \frac{5}{(1+0.1)^4} + \frac{5}{(1+0.1)^5} + \frac{100}{(1+0.1)^5} \quad \textbf{(2.34)}$$

We can think of the discounted values of the coupons as a geometric progression where each term is multiplied by (1/(1 + 0.1)) to get the next term. The sum of the coupons (noting that there are n = 6 of them rather than 5 since one is due immediately and d = 1/(1 + 0.1)) would be

$$S_n = \frac{a(1-d^n)}{1-d} = \frac{5(1-(1/1.1)^6)}{1-(1/1.1)} = £23.95 \quad \textbf{(2.35)}$$

We then need to calculate the present value of the redemption amount, which is $100/(1.1)^5$ = £62.09. Thus the fair price to pay for the bond is P_0 = £23.95+£62.09 = £86.04.

In fact, the coupon payments made on the bond are an example of an *annuity*, which is a financial product paying a fixed amount every period for a fixed length of time. Many people choose (or are required by law) to purchase a specific type of annuity with their pension savings. This sort of annuity involves an insurance element, which guarantees to continue to pay the fixed amount for as long as the person lives (and thus provides insurance against the possibility that the individual will live a long time and run out of money), and the payments made might also increase with inflation during the time that the annuity is paying out. The formula to calculate the present value of an annuity paying a fixed amount a every period for T periods assuming a discount rate of r is

$$P_0 = \frac{a}{r}\left[1 - \frac{1}{(1+r)^T}\right] \tag{2.36}$$

To demonstrate that this formula works, we can calculate the present value of the annuity represented by the coupon payments in the bond example above and show that it is indeed £23.95.

$$P_0 = \frac{5}{0.1}\left[1 - \frac{1}{(1+0.1)^5}\right] + 5 = £23.95 \tag{2.37}$$

Note that the formula implicitly assumes that the first payment is made at the end of the first period – hence there are five of them, and then we need to add the first immediate £5 payment, which is not discounted.

If the bond under study was *irredeemable* (so that it had an infinite lifetime like a stock), we would need to use the S_∞ formula (2.20) to calculate its present value. As an illustration, what should an investor be willing to pay today for a perpetual (irredeemable) bond that pays a coupon of £5 every six months if the appropriate rate to discount future cashflows is 4% and the next coupon will be paid immediately? In this case, we could discount each cashflow with the discount rate for six months, which would be 2%. We could write this as an infinite series beginning with £5:

$$5, \frac{5}{(1+0.02)}, \frac{5}{(1+0.02)^2}, \frac{5}{(1+0.02)^3}, \frac{5}{(1+0.02)^4}, \ldots$$

We need to be slightly careful as the common difference here is 1/(1 + 0.02) Then we would have the value as

$$S_\infty = \frac{£5}{1 - [1/(1+0.02)]} = £255 \tag{2.38}$$

Note that if the first coupon were not to be paid until the end of the first period, i.e., in six months' time rather than immediately, the series would be:

$$\frac{5}{(1+0.02)}, \frac{5}{(1+0.02)^2}, \frac{5}{(1+0.02)^3}, \frac{5}{(1+0.02)^4}, \ldots$$

and we could calculate the value as

$$S_\infty = \frac{a}{r} = \frac{£5}{0.02} = £250 \tag{2.39}$$

So the £5 to be received immediately has now been removed, which reduces the present value by exactly £5 as this cashflow is not discounted. In the UK, irredeemable government bonds are sometimes known as *consols*.

Sometimes, the payment amount may be growing over time rather than being fixed. For example, if we purchase shares in a company, the dividend that it pays will usually rise over time. If we assume that the rate of increase in the value of the dividend is some constant proportion g, this makes valuing the company much easier. So if we buy one share in the company now, we would receive a dividend in every period (assume that this is one year and that the first payment of D is due in exactly one year) in perpetuity (i.e., for ever), so the present value formula would be

$$P_0 = \frac{D}{(1+r)} + \frac{D(1+g)}{(1+r)^2} + \frac{D(1+g)^2}{(1+r)^3} + \frac{D(1+g)^3}{(1+r)^4} + \ldots \tag{2.40}$$

Examining this formula, we can see that the value of the dividend is growing at a rate g but being reduced (discounted) at a rate r. If $g > r$, the present value of the future dividends would be growing over time and the share would have infinite value. Thus, for this sum to be convergent and for the share to have a finite value, we require $g < r$. If that is the case, we can calculate the infinite sum from equation (2.20) as

$$P_0 = \frac{\frac{D}{(1+r)}}{1 - \frac{(1+g)}{(1+r)}} = \frac{D}{(r-g)} \tag{2.41}$$

This is often known as the *Gordon growth model* of equity valuation based on the expected future stream of dividend payments.

Finally, to complete this section, we should also note that analogous to continuous compounding, cashflows can be continuously discounted. The formula would be

$$P_0 = P_T \times e^{-rT} \tag{2.42}$$

So, for example, we could calculate the present value of £100 to be received in five years' time with an annual discount rate of 2% and continuous discounting as P_0 = £100 × $e^{-0.02\times 5}$ = £90.48

2.6.3 Internal Rate of Return

It is sometimes the case that we know both the present value of a particular set of cashflows, and we know all of the future cashflows, but we do not know the discount rate, or in other words the rate of interest which implicitly the financial product would generate for us if we purchased it today. The value of r that would equate the amount to be paid today P_0 to the present value of all of the cashflows we would receive if we purchased the asset is known as the *internal rate of return* or IRR. We could calculate that by solving the annuity formula above for r. More generally, if the future cashflow payments were not fixed but varied over time, we would have a more flexible formula

$$P_0 = a_0 + \frac{a_1}{(1+r)} + \frac{a_2}{(1+r)^2} + \frac{a_3}{(1+r)^3} + \frac{a_4}{(1+r)^4} + \ldots + \frac{a_T}{(1+r)^T} \tag{2.43}$$

So the situation is that we know the value of P_0, all values of a_i (i = 1, …, T) and T but we want to find r. In general, there will be more than one solution to this equation – in other words, more than one value of r that sets the left- and righthand sides of this equation to be equal. If T = 1 or 2, then we would have a linear or a quadratic equation, respectively, to solve for r, which we could do analytically using the formulae we have learned in Section 1.5 of Chapter 1 on functions. But for T bigger than 2, the equation would be solved numerically.

EXAMPLE 2.5: Calculating an Internal Rate of Return with Excel

It is very straightforward to calculate internal rates of return using recent versions of Microsoft Excel. One way to do this would be to set out the cashflows in a spreadsheet, calculate their discounted values with a particular interest rate r and then use Solver to estimate the IRR. An alternative approach is to employ the IRR function that is built into Excel. For example, suppose that we purchase a bond today for £110 which has exactly five years to maturity when it will be redeemed at its par value of £100 and which will provide an annual coupon of £5. What is the internal rate of return of the bond investment (which is effectively the yield to maturity of the bond)?

If we set up the spreadsheet so that the following entries are in cells A1 to B7:

Year	Cashflow
0	-107
1	5
2	5
3	5
4	5
5	105

Then in any other cell, we simply write the command

=IRR(B2:B7,0.1)

where the second term in parentheses, 0.1, is an initial guess of the expected interest rate. The initial guess is required for situations where there are multiple IRRs in order that Excel chooses the most plausible value from among them. Then hitting ENTER leads Excel to calculate the IRR, which is 3.45% in this case.

Multiple IRRs will occur for projects where the cashflows change sign more than once during the lifetime of the project. In the example just presented, the cashflow is only negative (an outflow) during year 0 and then positive (i.e., cash inflows only) always thereafter. But if we had further outflows during the project, the IRR would be non-unique. More specifically, we would have as many IRRs as there are cashflow sign changes. To illustrate, suppose that we now have the following cashflows for a project

Year	Cashflow
0	-100
1	240
2	-143

There are cash outflows in years 0 and 2 with an inflow in year 1. If we use the formula as above but reducing the cell range as we now only have three cashflows and still with a guess of 0.1

= IRR(B2:B4,0.1)

then we find the interest rate as 10.00%, but if instead we use an initial guess of 0.5 (so typing =IRR(B2:B4,0.5), then we would end up with an interest rate of 30.00%. What has happened here is that both interest rate values would solve the equation and would set the net present value (NPV) of the project to zero and so Excel will converge upon the answer that is closest to the initial value. More generally, it is possible for one or more of the calculated IRR values to be negative.

2.7 Returns in Financial Modelling

In many of the problems of interest in finance, the starting point is a time series of prices – for example, the prices of shares in Ford, taken at 4 p.m. each day for 200 days. For a number of statistical reasons, it is preferable not to work directly with the price series, so that raw price series are usually converted into series of returns. Additionally, returns have the added benefit that they are unit-free. So, for example, if an annualised return were 10%, then investors know that they would have got back £110 for a £100 investment, or £1,100 for a £1,000 investment, and so on.

There are two methods used to calculate returns from a series of prices, and these involve the formation of simple returns, and continuously compounded returns, which are, respectively,

Simple returns

$$R_t = \frac{p_t - p_{t-1}}{p_{t-1}} \times 100\% \quad \textbf{(2.44)}$$

Continuously compounded returns

$$r_t = 100\% \times \ln\left(\frac{p_t}{p_{t-1}}\right) \quad \textbf{(2.45)}$$

where: R_t denotes the simple return at time t, r_t denotes the continuously compounded return at time t, p_t denotes the asset price at time t and ln denotes the natural logarithm.

In the limit, as the frequency of the sampling of the data is increased so that they are measured over a smaller and smaller time interval, the simple and continuously compounded returns will be identical.

If the asset under consideration is a stock or portfolio of stocks, the total return to holding it is the sum of the capital gain and any dividends paid during the holding period. Usually, the holding period is one year but it could be any amount of time. However, researchers often ignore any dividend payments. This is unfortunate, and will lead to an underestimation of the total returns that accrue to investors. This is likely to be negligible for very short holding periods, but will have a severe impact on cumulative returns over investment horizons of several years. Ignoring dividends will also have a distortionary effect on the cross-section of stock returns. For example, ignoring dividends will imply that 'growth' stocks with large capital gains will be inappropriately favoured over income stocks (e.g., utilities and mature industries) that pay high dividends.

Alternatively, it is possible to adjust a stock price time series so that the dividends are added back to generate a *total return index*. If p_t were a total return index, returns generated using either of the two formulae presented above thus provide a measure of the total return that would accrue to a holder of the asset during time t.

The academic finance literature generally employs the log-return formulation (also known as log-price relatives since they are the log of the ratio of this period's price to the previous period's price). Box 2.3 shows two key reasons for this.

BOX 2.3 Log returns

(1) Log returns have the nice property that they can be interpreted as *continuously compounded returns* – so that the frequency of compounding of the return does not matter and thus returns across assets can more easily be compared.

(2) Continuously compounded returns are *time-additive*. For example, suppose that a weekly returns series is required and daily log returns have been calculated for five days, numbered 1 to 5, representing the returns on Monday through Friday. It is valid to simply add up the five daily returns to obtain the return for the whole week:

Monday return	$r_1 = \ln(p_1/p_0) = \ln p_1 - \ln p_0$
Tuesday return	$r_2 = \ln(p_2/p_1) = \ln p_2 - \ln p_1$
Wednesday return	$r_3 = \ln(p_3/p_2) = \ln p_3 - \ln p_2$
Thursday return	$r_4 = \ln(p_4/p_3) = \ln p_4 - \ln p_3$
Friday return	$r_5 = \ln(p_5/p_4) = \ln p_5 - \ln p_4$

Return over the week	$\ln p_5 - \ln p_0 = \ln (p_5/p_0)$

There is, however, also a disadvantage of using the log returns. The simple return on a portfolio of assets is a weighted average of the simple returns on the individual assets

$$R_{pt} = \sum_{i=1}^{N} w_i R_{it} \tag{2.46}$$

But this does not work for the continuously compounded returns, so that they are not additive across a portfolio. The fundamental reason why this is the case is that the log of a sum is not the same as the sum of a log, since the operation of taking a log constitutes a *non-linear transformation*. Calculating portfolio returns in this context must be conducted by first estimating the value of the portfolio at each time period and then determining the returns from the aggregate portfolio values. Or alternatively, if we assume that the asset is purchased at time $t - K$ for price p_{t-K} and then sold K periods later at price p_t, then if we calculate simple returns for each period, $R_t, R_{t+1}, \ldots, R_K$, the aggregate return over all K periods is

$$\begin{aligned} R_{Kt} &= \frac{p_t - p_{t-K}}{p_{t-K}} = \frac{p_t}{p_{t-K}} - 1 = \left[\frac{p_t}{p_{t-1}} \times \frac{p_{t-1}}{p_{t-2}} \times \ldots \times \frac{p_{t-K+1}}{p_{t-K}}\right] - 1 \\ &= \left[(1 + R_t)(1 + R_{t-1}) \ldots (1 + R_{t-K+1})\right] - 1 \\ &= \left[\prod_{i=0}^{K-1} (1 + R_{t-i})\right] - 1 \end{aligned} \tag{2.47}$$

This is known as the holding period return, where the last line uses the ∏ notation presented in Section 1.5.9 from Chapter 1. The annualised holding period return (call it R_H) is given by the Kth root of equation (2.47) minus one

$$(1 + R_H) = (R_{Kt} + 1)^{1/K} \quad \text{or} \quad R_H = (R_{Kt} + 1)^{1/K} - 1 \tag{2.48}$$

EXAMPLE 2.6

Given the data in the following table, calculate the returns for each year on a single share in a company that is purchased on 31 December 2012 for a price of 100 pence and then held for four years. Next, calculate the total return over the whole period and the annualised average return.

Date	Price	Dividend
31 Dec 2012	100p	-
31 Dec 2013	120p	10p
31 Dec 2014	130p	10p
31 Dec 2015	140p	10p
31 Dec 2016	167p	10p

SOLUTION Assume here that we use the simple return formula rather than continuously compounded returns. The first step is to calculate each year's return separately for the calendar years 2013 (31 Dec 2012 – 31 Dec 2013), 2014, 2015 and 2016. Using the notation R_{13} to denote the 2013 return and so on, we have

$R_{13} = 100 \times (120 - 100 + 10)/100 = 30\%$,

$R_{14} = 100 \times (130 - 120 + 10)/120 = 16.7\%$,

$R_{15} = 100 \times (140 - 130 + 10)/130 = 15.4\%$, and

$R_{16} = 100 \times (167 - 140 + 10)/140 = 26.4\%$.

Next, we calculate the holding period return for the whole four years as

$$\begin{aligned}[(1 + R_{K,t}) &= (1 + R_{13})(1 + R_{14})(1 + R_{15})(1 + R_{16}) \\ &= 1.30 \times 1.167 \times 1.154 \times 1.264 = 2.213]\end{aligned}$$

So the return over the whole period, $R_{K,t}$, is 2.213 − 1 = 1.213 or 121.3%. This figure can be annualised by taking the fourth root of one plus it and then subtracting one – in other words, by calculating the geometric mean of the series of individual returns

$$(1 + R_H) = (1 + R_{K,t})^{1/N} = (2.213)^{0.25} = 1.21968 \approx 1.22$$

R_H, the average annual holding period return, is thus around 0.22 or 22%.

2.7.1 Real versus Nominal Series and Deflating Nominal Series

If a newspaper headline suggests that 'house prices are growing at their fastest rate for more than a decade. A typical 3-bedroom house is now selling for £280,000, whereas in 2005 the figure was £120,000', it is important to appreciate that this figure is almost certainly in *nominal* terms. That is, the article is referring to the actual prices of houses that existed at those points in time. The general level of prices in most economies around the world has a general tendency to rise almost all of the time, so we need to ensure that we compare prices on a like-for-like basis. We could think of part of the rise in house prices being attributable to an increase in demand for housing, and part simply arising because the prices of all goods and services are rising together. It would be useful to be able to separate the two effects, and to be able to answer the question, 'how much have house prices risen when we remove the effects of general inflation?' or equivalently, 'how much are houses worth now if we measure their values in 1990-terms?' We can do this by *deflating* the nominal house price series to create a series of *real* house prices, which is then said to be in *inflation-adjusted terms* or *at constant prices*.

Deflating a series is very easy indeed to achieve: all that is required (apart from the series to deflate) is a *price deflator series*, which is a series measuring general price levels in the economy. Series like the consumer price index (CPI), producer price index (PPI) or the GDP Implicit Price Deflator, are often used. A more detailed discussion of which is the most relevant general price index to use is beyond the scope of this book, but suffice to say that if the researcher is only interested in viewing a broad picture of the real prices rather than a highly accurate one, the choice of deflator will be of little importance.

The real price series is obtained by taking the nominal series, dividing it by the price deflator index, and multiplying by 100 (under the assumption that the deflator has a base value of 100)

$$real\ series_t = \frac{nominal\ series_t}{deflator_t} \times 100 \tag{2.49}$$

It is worth noting that deflation is only a relevant process for series that are measured in money terms, so it would make no sense to deflate a quantity-based series such as the number of shares traded or a series expressed as a proportion or percentage, such as the rate of return on a stock.

EXAMPLE 2.7: DEFLATING HOUSE PRICES

Let us use for illustration a series of average UK house prices, measured annually for 2006 – 18 and given in column 2 of Table 2.3. Some figures for the general level of prices as measured by the CPI are given in the column 3. So first, suppose that we want to convert the figures into constant (real) prices. Given that 2009 is the 'base' year (i.e., it has a value of 100 for the CPI), the easiest way to do this is simply to divide each house price at time t by the corresponding CPI figure for time t and then multiply it by 100, as per equation (2.49). This will give the figures in column 4 of the table.

Table 2.3 How to construct a series in real terms from a nominal one

Year	Nominal house prices	CPI (2009 levels)	House prices (2009 levels)	House prices (2018) levels
2006	83,450	97.6	85,502	105,681
2007	93,231	98.0	95,134	117,585
2008	117,905	98.7	119,458	147,650
2009	134,806	100.0	134,806	166,620
2010	151,757	101.3	149,810	185,165
2011	158,478	102.1	155,218	191,850
2012	173,225	106.6	162,500	200,850
2013	180,473	109.4	164,966	203,898
2014	150,501	112.3	134,017	165,645
2015	163,481	116.7	140,086	173,147
2016	161,211	119.2	135,244	167,162
2017	162,228	121.1	133,962	165,577
2018	162,245	123.6	131,266	162,245

Notes: All prices in British pounds; house price figures and CPI are for illustration only.

If we wish to convert house prices into a particular year's figures, we would apply equation (2.49), but instead of 100 we would have the CPI value that year. Consider that we wished to express nominal house prices in 2018 terms (which is of particular interest as this is the last observation in the table). We would thus base the calculation on a variant of (2.49)

$$real\ series_t = \frac{nominal\ series_t}{CPI_t} CPI_{reference\ year} \quad \textbf{(2.50)}$$

So, for example, to get the 2006 figure (i.e., t is 2006) of 105,681 for the average house price in 2018 terms, we would take the nominal figure of 83,450, multiply it by the CPI figure for the year that we wish to make the price for (the reference year, 123.6) and then divide it by the CPI figure for the year 2006 (97.6). Thus $105{,}681 = \frac{83{,}450}{97.6} \times 123.6$, etc.

2.8 Portfolio Theory Using Matrix Algebra

Probably the most important application of matrix algebra in finance is solving portfolio allocation problems. Although these can be solved in a perfectly satisfactory fashion with sigma notation rather than matrix algebra, use of the latter does considerably simplify the expressions and makes it easier to solve them when the portfolio includes more than two assets. This book is not the place to learn about portfolio theory *per se* – interested readers are referred to Bodie, Kane and Marcus (2014) or the many other investment textbooks that exist – rather, the purpose of this section is to demonstrate how matrix algebra is used in practice, drawing together the material in Chapter 1 together with what we have learned in this chapter on computing means, variances and covariances in a practical application.

So, let us pick up from the material we covered in Section 1.7 in Chapter 1 now that we have also covered the construction of means, variances, covariances and returns. To start, suppose that we have a set of N stocks that are included in a portfolio P with weights $w_1, w_2, \ldots, w_N$ and suppose that their expected returns are written as $E(r_1), E(r_2), \ldots, E(r_N)$. We could write the $N \times 1$ vectors of weights, w, and of expected returns, $E(r)$, as

$$w = \begin{pmatrix} w_1 \\ w_2 \\ \cdots \\ w_N \end{pmatrix} \quad E(r) = \begin{pmatrix} E(r_1) \\ E(r_2) \\ \cdots \\ E(r_N) \end{pmatrix}$$

For instance, w_3 and $E(r_3)$ are the weight attached to stock three and its expected return, respectively. The expected return on the portfolio, $E(r_P)$ can be calculated as $E(r)'w$ – that is, we multiply the transpose of the expected return vector by the weights vector.

We then need to set up what is called the variance–covariance matrix of the returns, denoted V. This matrix includes all of the variances of the components of the portfolio returns on the leading diagonal and the covariances between them as the off-diagonal elements. We will also discuss such a matrix extensively in Chapter 4 in the context of the parameters from regression models. The variance–covariance matrix of the returns may be written

$$V = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \dots & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \dots & \sigma_{2N} \\ \vdots & & & \vdots & \\ \sigma_{N1} & \sigma_{N2} & \sigma_{N3} & \dots & \sigma_{NN} \end{pmatrix}$$

The elements on the leading diagonal of V are the variances of each of the component stocks' returns - so, for example, σ_{11} is the variance of the returns on stock one, σ_{22} is the variance of returns on stock two and so on. The off-diagonal elements are the corresponding covariances – so, for example, σ_{12} is the covariance between the returns on stock one and those on stock two, σ_{58} is the covariance between the returns on stock five and those on stock eight, and so on. Note that this matrix will be symmetrical about the leading diagonal since cov(a, b) = cov(b, a) where a and b are random variables, and hence it is possible to write $\sigma_{12} = \sigma_{21}$ and so forth.

In order to construct a variance–covariance matrix, we would need to first set up a matrix containing observations on the actual returns (not the expected returns) for each stock where the mean, $\bar{r}_i$ $(i = 1, \dots, N)$, has been subtracted away from each series i. If we call this matrix R, we would write

$$R = \begin{pmatrix} r_{11} - \bar{r}_1 & r_{21} - \bar{r}_2 & r_{31} - \bar{r}_3 & \dots & r_{N1} - \bar{r}_N \\ r_{12} - \bar{r}_1 & r_{22} - \bar{r}_2 & r_{32} - \bar{r}_3 & \dots & r_{N2} - \bar{r}_N \\ \vdots & & & \vdots & \\ r_{1T} - \bar{r}_1 & r_{2T} - \bar{r}_2 & r_{3T} - \bar{r}_3 & \dots & r_{NT} - \bar{r}_N \end{pmatrix}$$

So each column in this matrix represents the deviations of the returns on individual stocks from their means and each row represents the mean-adjusted return observations on all stocks at a particular point in time. The general entry, r_{ij}, is the jth time-series observation on the ith stock. The variance–covariance matrix would then simply be calculated as $V = (R'R)/(T - 1)$ where T is the total number of time-series observations available for each series.

Suppose that we wanted to calculate the variance of returns on the portfolio P (a scalar which we might call V_P). We would do this by calculating

$$V_P = w'Vw \tag{2.51}$$

Checking the dimension of V_P, w' is $(1 \times N)$, V is $(N \times N)$ and w is $(N \times 1)$ so V_P is $(1 \times N \times N \times N \times N \times 1)$, which is (1×1) as required.

We could also define a correlation matrix of returns, C, which would be

$$C = \begin{pmatrix} 1 & C_{12} & C_{13} & \dots & C_{1N} \\ C_{21} & 1 & C_{23} & \dots & C_{2N} \\ \vdots & & & \vdots & \\ C_{N1} & C_{N2} & C_{N3} & \dots & 1 \end{pmatrix}$$

This matrix would have ones everywhere on the leading diagonal (since the correlation of something with itself is always one) and the off-diagonal elements would give the correlations between each pair of returns – for example, C_{35} would be the correlation between the returns on stock three and those on stock five. Note again that, as for the variance–covariance matrix, the correlation matrix will always be symmetrical about the leading diagonal so that $C_{31} = C_{13}$ etc. Using the correlation instead of the variance–covariance matrix, the portfolio variance given in equation (2.51) would be

$$V_P = w'SCSw \tag{2.52}$$

where C is the correlation matrix, w is again the vector of portfolio weights, and S is a diagonal matrix with each element containing the standard deviations of the portfolio returns.

Selecting Weights for the Minimum Variance Portfolio

Although in theory investors can do better by selecting the optimal portfolio on the efficient frontier, in practice a variance minimising portfolio often performs well when used out-of-sample. Thus we might want to select the portfolio weights *w* that minimise the portfolio variance, V_P. In matrix notation, we would write

$$\min_{w} \; w'Vw$$

We also need to be slightly careful to impose at least the restriction that all of the wealth has to be invested $(\sum_{i=1}^{N} w_i = 1)$, otherwise this minimisation problem can be trivially solved by setting all of the weights to zero to yield a zero portfolio variance. This restriction that the weights must sum to one is written using matrix algebra as $w' \cdot 1_N = 1$, where 1_N is a column vector of ones of length *N*.[8]

The minimisation problem can be solved to

$$w_{MVP} = \frac{1_N \cdot V^{-1}}{1'_N \cdot V^{-1} \cdot 1_N} \tag{2.53}$$

where *MVP* stands for minimum variance portfolio.

Selecting Optimal Portfolio Weights

In order to trace out the mean–variance efficient frontier, we would repeatedly solve this minimisation problem, but in each case set the portfolio's expected return equal to a different target value, $\bar{R}$. So, for example, we set $\bar{R}$ to 0.1 and find the portfolio weights that minimise V_P, then set $\bar{R}$ to 0.2 and find the portfolio weights that minimise V_P, and so on. We would write this as

$$\min_{w} \quad w'Vw \quad \text{subject to} \quad w' \cdot 1_N = 1, w'E(r) = \bar{R}$$

This problem is sometimes called the *Markowitz portfolio allocation problem*, and can be solved analytically as expressed above. That is, we can derive an exact solution using matrix algebra. However, it is often the case that we want to place additional constraints on the optimisation – for instance we might want to restrict the portfolio weights so that none are greater than 10% of the overall wealth invested in the portfolio, or we might want to restrict them to all be positive (i.e., long positions only with no short selling allowed). In such cases, the Markowitz portfolio allocation problem cannot be solved analytically and thus a numerical procedure must be used such as the Solver function in Microsoft Excel.

Note that it is also possible to write the Markowitz problem the other way around – that is, where we select the portfolio weights that maximise the expected portfolio return subject to a target maximum variance level.

If the procedure above is followed repeatedly for different return targets, it will trace out the efficient frontier. In order to find the tangency point where the efficient frontier touches the capital market line, we need to solve the following problem

$$\max_{w} \frac{w'E(r) - r_f}{(w'Vw)^{\frac{1}{2}}} \quad \text{subject to} \quad w' \cdot 1_N = 1$$

If no additional constraints are required on the stock weights, this can be solved fairly simply as

$$w = \frac{V^{-1}[E(r) - r_f \cdot 1_N]}{1_N' V^{-1}[E(r) - r_f \cdot 1_N]} \tag{2.54}$$

2.8.1 The Mean–Variance Efficient Frontier in Excel

This section will now describe how to construct an efficient frontier and draw the capital market line using a three-stock portfolio with Microsoft Excel. It is assumed that the reader knows the standard functions of Excel – for those who need a refresher, see the excellent book by Benninga (2017).

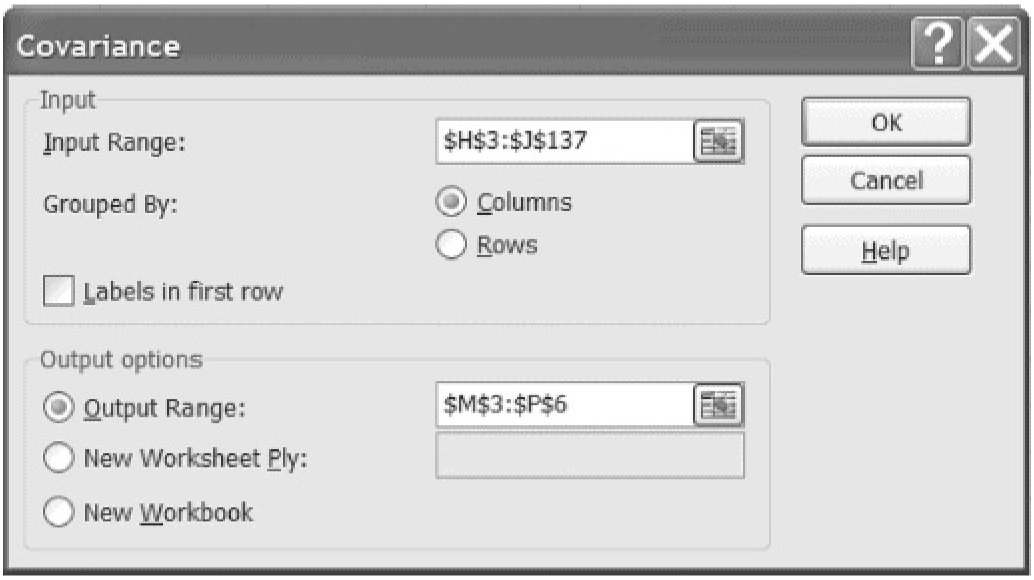
The spreadsheet 'efficient.xls' contains the finished product – the plots of the efficient frontier and capital market line. However, I suggest **starting with a blank spreadsheet, copying across the raw data and starting to reconstruct the formulae again** to get a better of idea of how it is done.

The first step is to construct the returns. The raw prices and T-bill yields are in columns two to six of the sheet. We are going to assume a three-asset portfolio. However, all of the principles outlined below could be very easily and intuitively extended to situations where there were more assets employed.

Since we are dealing with portfolios, it is probably preferable to employ simple rather than continuously compounded returns. So start by **constructing three sets of returns** for the Ford, General Electric and Microsoft share prices in columns H to J, and head these columns 'FORDRET', 'GERET' and 'MSOFTRET', respectively. Column K will comprise the weights on a portfolio containing all three stocks but with varying weights. The way we achieve this is to set up three cells that will contain the weights. To start with, we fix these arbitrarily but later will allow the Solver to choose them optimally. So **write 0.33, 0.33 and 0.34**

in cells N12 to N14, respectively. In cell N15, **calculate the sum of the weights** as a check that this is always one so that the all wealth is invested among the three stocks. We are now in a position to construct the (equally weighted) portfolio returns **(call them 'PORTRET')** in column K. In cell K2, **write =H3*$N$12+I3*$N$13+J3*$N$14** and then **copy this formula down the whole of column K until row 137**.

The next stage is to construct the variance–covariance matrix, which we termed *V* in the description above. So first, **click on Data and Data Analysis** and then select Covariance from the menu. **Complete the Window so that it appears as in** screenshot 2.1 with input range $H$3:$J$137 and output range $M$3:$P$6 and **click OK**.



Screenshot 2.1 Setting up a variance–covariance matrix in Excel

Now **copy the covariances** so that they are also in the upper right triangle of the matrix, and also **replace 'Column 1' etc. with the names of the three stocks in the column and row headers**.

We now want to calculate the average returns for each of the individual stocks (we already have their variances on the leading diagonal of the variance−covariance matrix). To do this, in cells M9 to O9, **write =AVERAGE(H3:H137), =AVERAGE(I3:I137) and =AVERAGE(I3:I137)**.

Next, we can construct summary statistics for the portfolio returns. There are several ways to do this. One way would be to calculate the mean, variance and standard deviation of the returns directly from the monthly portfolio returns in column K. However, to see how we would do this using matrix algebra in Excel, for calculating the average portfolio return **in cell N18, enter the formula** =**MMULT(M9:O9,N12:N14)** which will multiply the returns vector (what we called $E(r)'$) in M9 to O9 by the weights vector w in N12 to N14.

In cell N19, we want the formula for the portfolio variance, which is given by $w'Vw$ and in Excel this is calculated using the formula =**MMULT(MMULT(Q13:S13, N4:P6),N12:N14)**.

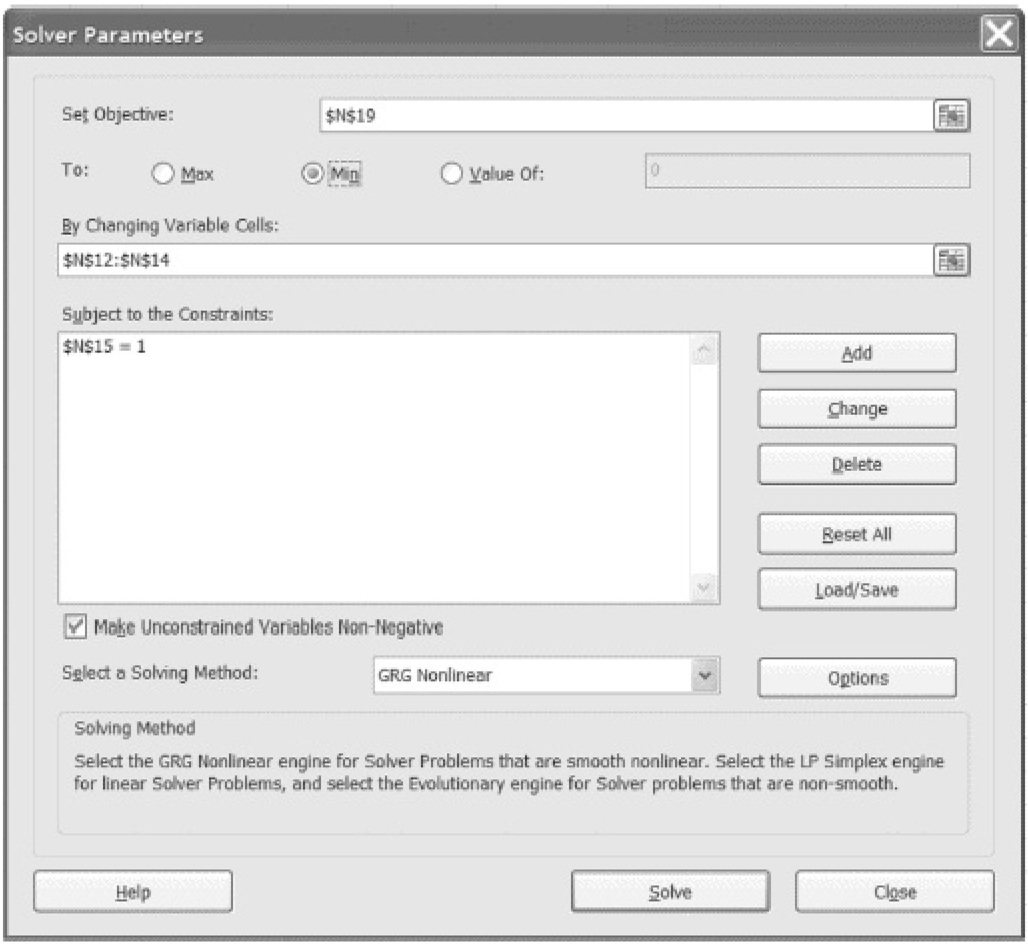
Effectively, we are conducting the multiplication in two stages. First, the internal MMUL is multiplying the transposed weights vector, w' in Q13 to S13 by the variance−covariance matrix V in N4 to P6. We then multiply the resulting product by the weights vector w in N12 to N14. Finally, **calculate the standard deviation of the portfolio returns in N19 as the square root of the variance in N18**.

Take a couple of minutes to examine the summary statistics and the variance−covariance matrix. It is clear that Ford is by far the most volatile stock with an annual variance of 239, while Microsoft is the least at 50. The equally weighted portfolio has a variance of 73.8. Ford also has the highest average return. We now have all of the components needed to construct the mean–variance efficient frontier and the right-hand side of your spreadsheet should appear as in Screenshot 2.2.

M	N	O	P	Q	R	S	T
Variance-Covariance matrix, V							
	FORD	GE	MSOFT				
FORD	293.02	61.55	42.90				
GE	61.55	66.90	25.79				
MSOFT	42.90	25.79	50.05				
Stock Returns							
1.31	0.24	0.39					
Portfolio Weights, w				*Portfolio weights transposed, w'*			
FORD	0.33			FORD	GE	MSOFT	
GE	0.33			0.33	0.33	0.34	
MSOFT	0.34						
	1.00	<<< sum of weights					
Portfolio Statistics							
Mean	0.64						
Variance	73.80						
Std Dev.	8.59						

Screenshot 2.2 The spreadsheet for constructing the efficient frontier

First, let us calculate the minimum variance portfolio. To do this, **click on cell N19**, which is the one containing the portfolio variance formula. Then **click on the Data tab and then on Solver**.[9] A window will appear which should be **completed as in Screenshot 2.3**. So we want to minimise cell $N$19 by changing the weights $N$12:$N$14 subject to the constraint that the weights sum to one ($N$15 = 1). Then **click Solve**. Solver will tell you it has found a solution, so **click OK** again.



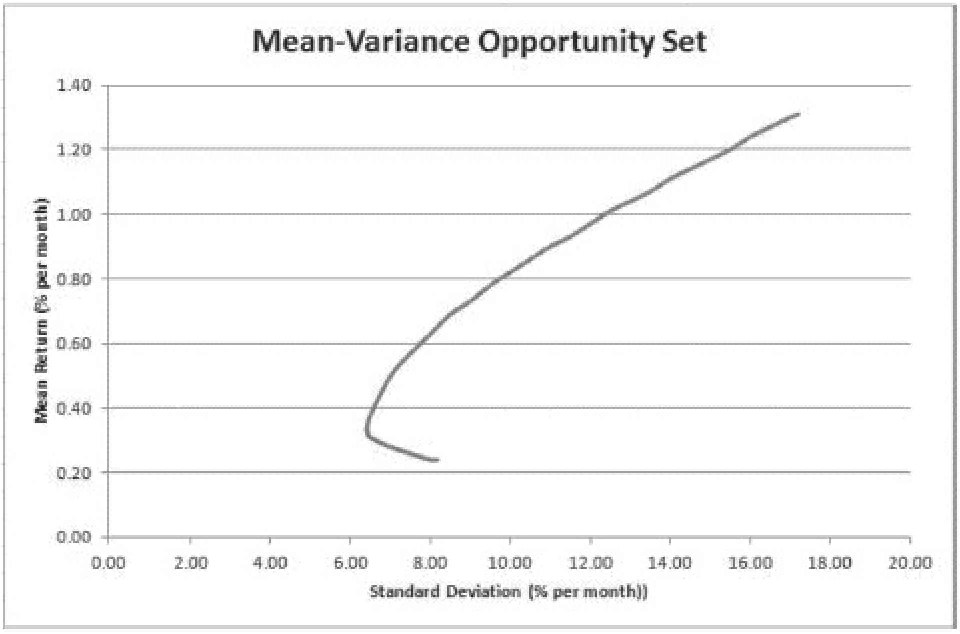
Screenshot 2.3 Completing the Solver window

Note that strictly it is not necessary to use Solver to evaluate this problem when no additional constraints are placed, but if we want to incorporate non-negativity or other constraints on the weights, we could not calculate the weights analytically and Solver would have to be used. The weights in cells N12 to N14 automatically update, as do the portfolio summary statistics in N18 to N20. So the weights that minimise the portfolio variance are with no allocation to Ford, 37% in General Electric and 63% in Microsoft. This achieves a variance of 41 (standard deviation of 6.41%) per month and an average return of 0.33% per month.

So we now have one point on the efficient frontier (the one on the far left), and we repeat this procedure to obtain other points on the frontier. We set a target variance and find the weights that maximise the return subject to this variance. **In cells N25 to N40, we specify the target standard deviations from 6.5 to 17, increasing in units of 0.5**. These figures are somewhat arbitrary, but as a rule of thumb, to get a nice looking frontier, we should have the maximum standard deviation (17) about three times the minimum (6.5). We know not to set any number less than 6.41 since this was the minimum possible standard deviation with these three stocks.

We **click on the cell N18** and then **select Solver** again from the Data tab. Then we use all of the entries as before, except that we want to **choose Max** (to maximise the return subject to a standard deviation constraint) and then **add an additional constraint that $N$20** = **$N$25**, so that the portfolio standard deviation will be equal to the value we want, which is 6.5 in cell N25. **Click Solve** and the new solution will be found. The weights are now 4% in Ford, 30% in GE, and 66% in Microsoft, giving a mean return of 0.38% and a standard deviation of 6.5(%). **Repeat this again for the other standard deviation values from 6.5 through to 17**, each time noting the corresponding mean value (and if you wish, also noting the weights). You will see that if you try to find a portfolio with a standard deviation of 17.5, Solver will not be able to find a solution because there are no combinations of the three stocks that will give such a high value. In fact, the upper left point on the efficient frontier will be the maximum return portfolio which will always be 100% invested in the stock with the highest return (in this case Ford).

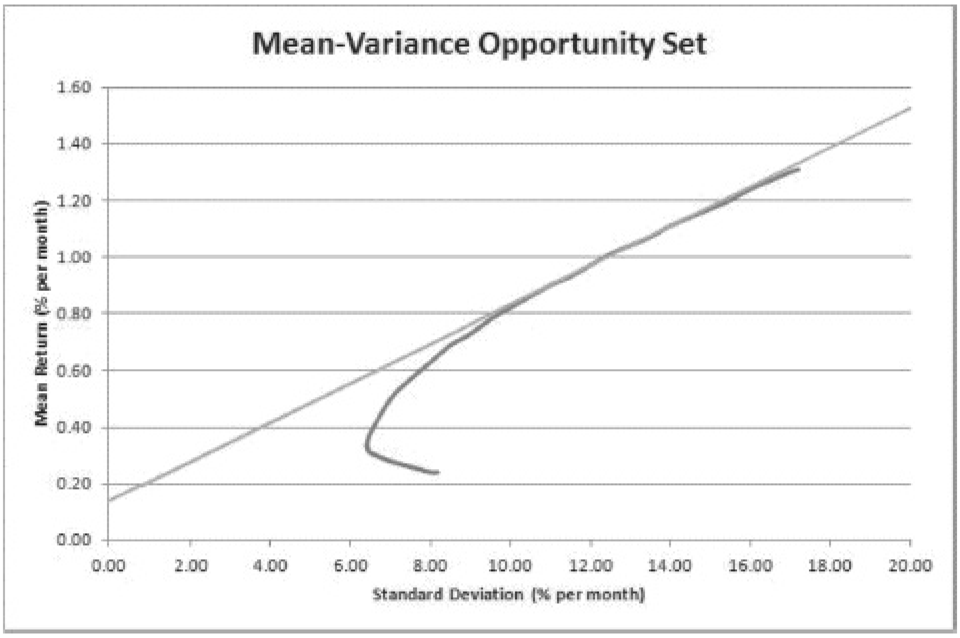
We can now plot the efficient frontier – i.e., the mean return on the *y*-axis against the standard deviation on the *x*-axis. If we also want the lower part of the mean–variance opportunity set (the part where the curve folds back on itself at the bottom), we **repeat the procedure above** – i.e., targeting the standard deviation of 6.5, 7, …, but this time we **minimise the return rather than maximising it**. The minimum return is 0.24 when the portfolio is 100% invested in GE. The plot will appear as in screenshot 2.4. The line is somewhat wiggly, but this arises because the points are insufficiently close together. If we had used standard deviations from 6.5 to 17 in increments of 0.2, say, rather than 0.5 then the plot would have been much smoother.



Screenshot 2.4 A plot of the completed efficient frontier

The final step in the process is to superimpose the capital market line (CML) onto the plot. To do this, we need to find the tangency point, which will be the point at which the Sharpe ratio of the portfolio is maximised. So first we need to **calculate the average of the T-bill series** (dividing it by twelve to get the monthly rate for comparability with the stock returns, which are monthly), putting this in cell N55. We then **calculate the risk premium in N56**, which is the risky portfolio return from N18 less the risk-free rate in N56. Finally, **the Sharpe ratio in N57** is the risk premium from N56 divided by the portfolio standard deviation (N20). We then get Solver to **maximise the value of N57 subject to the weights adding to one** (no other constraints are needed).

The tangency point has a mean return of exactly 1% per month (by coincidence), standard deviation 12.41% and weights of 66%, 0% and 34% in Ford, GE and Microsoft, respectively. We then need **a set of points on the CML to plot** − one will be the point on the *y*-axis where the risk is zero and the return is the average risk-free rate (0.14% per month). Another will be the tangency point we just derived. To get the others, recall that the CML is a straight line with equation $return = R_f + Sharpe\ ratio \times std\ dev$. So all we need to do is to **use a run of standard deviations and then calculate the corresponding returns** − we know that $R_f = 0.14$ and the Sharpe ratio = 0.0694. The minimum variance opportunity set and the CML on the same graph will appear as in screenshot 2.5.



Screenshot 2.5 The capital market line and efficient frontier

KEY CONCEPTS

The key terms to be able to define and explain from this chapter are

- cardinal, ordinal and nominal numbers
- financial econometrics
- time-series data
- panel data
- continuous data
- real and nominal series
- quantiles
- arithmetic progression
- mean
- skewness
- covariance
- population
- present value
- internal rate of return
- geometric mean
- continuously compounded returns
- cross-sectional data
- pooled data
- discrete data
- deflator

- coefficient of variation
- geometric progression
- variance
- kurtosis
- correlation
- sample
- future value

SELF-STUDY QUESTIONS

1. Expand the parentheses as far as possible for the following expressions
   (a) E($ax + by$) for x,y variables and a,b scalars
   (b) E(axy) for x,y independent variables and a a scalar
   (c) E(axy) for x,y correlated variables and a a scalar

2. (a) Explain the difference between a pdf and a cdf
   (b) What shapes are the pdf and cdf for a normally distributed random variable?

3. What is the central limit theorem and why is it important in statistics?

4. Explain the differences between the mean, mode and median. Which is the most useful measure of an average and why?

5. Which is a more useful measure of central tendency for stock returns − the arithmetic mean or the geometric mean? Explain your answer.

6. The covariance between two variables is 0.99. Are they strongly related? Explain your answer.

7. Explain the differences between the following pairs of terms
   (a) Continuous and discrete data
   (b) Ordinal and nominal data
   (c) Time-series and panel data
   (d) Noisy and clean data
   (e) Simple and continuously compounded returns
   (f) Nominal and real series

(g) Bayesian and classical statistics

8. Present and explain a problem that can be approached using a time-series regression, another one using cross-sectional regression, and another using panel data.

9. What are the key features of asset return time-series?

10. The following table gives annual, end of year prices of a bond and of the consumer prices index

Year	Bond value	CPI value
2011	36.9	108.0
2012	39.8	110.3
2013	42.4	113.6
2014	38.1	116.1
2015	36.4	118.4
2016	39.2	120.9
2017	44.6	123.2
2018	45.1	125.4

(a) Calculate the simple returns

(b) Calculate the continuously compounded returns

(c) Calculate the prices of the bond each year in 2018 terms

(d) Calculate the real returns

11. Start with part of the formula for calculating an effective interest rate from a nominal one

$$X = \left[1 + \frac{r}{T}\right]^T$$

Assume that the interest rate, r is 10%, and use T =1, 2, 5, 10, 20, 50, 100, 200, 500, 1000, 5000 to calculate the corresponding value of X in each case. Produce a graph where you plot X on the y-axis and T on the x-axis. What do you notice happening as X is increasing and can you see what would happen as X rises to infinity?

12. Suppose that I place £1000 today in a savings account that pays 3% interest once per year and I make no additions to the account.

(a) How many years would it take for me to double my money?

(b) If instead I switch to a better account that pays 5% per year, how many years would it take?

(c) If I want to triple the money with a 5% interest rate, how long would it take?

(d) If the 5% was paid continuously, how many years would it take to double the money?

13. A saver has a choice of two accounts: one paying 12%, compounded annually, and one paying 11%, compounded monthly. Which should he or she choose and why?

14. (a) Starting now, if I save £200 per month, is it possible that I will become a millionaire in my lifetime assuming a 5% annual growth rate in my investments, compounded annually?

(b) Suppose that I manage instead to save £500 per month, is it possible that I will become a millionaire?

(c) Suppose now that I can save £500 per month but the effects of inflation imply that the real rate of growth in the value of my investments is only 2%. Will I become a millionaire in today's money terms?

1 A more precise and complete definition of the median is surprisingly complex, but is not necessary for our purposes.

2 N is used here to denote the number of observations – i.e., the sample size, which is denoted as T in later chapters for consistency with the standard approach in the time-series literature.

3 Note that there are several slightly different formulae that can be used for calculating quartiles, each of which may provide slightly different answers.

4 Of course, we could also define the positive semi-variance and positive semi-standard deviation where only observations such that $y_i > \bar{y}$ are included in the sum.

5 There are a number of ways to calculate skewness (and kurtosis); the one given in the formula is sometimes known as the moment coefficient of skewness, but it could also be measured using the standardised difference between the mean and the median, or by using the quartiles of the data. Unfortunately, this implies that different software packages will give slightly different values for the skewness and kurtosis coefficients. Also, some packages make a 'degrees of freedom correction' as we do in the equations here, while others do not, so that the divisor in such cases would be N rather than $N - 1$ in the equations.

[6] Strictly, the variance is the second moment, not the standard deviation.

[7] Discretely measured data do not necessarily have to be integers. For example, until they became ‘decimalised’, many financial asset prices were quoted to the nearest 1/16 or 1/32 of a dollar.

[8] Note that $w' \cdot 1_N$ will be 1×1 – i.e., a scalar.

[9] Note that you may have to load the Solver Add-In: see the online Microsoft support for how to do this for your version of Excel and platform.

3

A Brief Overview of the Classical Linear Regression Model

LEARNING OUTCOMES

In this chapter, you will learn how to

- Derive the OLS formulae for estimating parameters and their standard errors
- Explain the desirable properties that a good estimator should have
- Discuss the factors that affect the sizes of standard errors
- Test hypotheses using the test of significance and confidence interval approaches
- Interpret *p*-values

3.1 What is a Regression Model?

Regression analysis is almost certainly the most important tool at the econometrician's disposal. But what is regression analysis? In very general terms, regression is concerned with describing and evaluating the *relationship between a given variable and one or more other variables*. More specifically, regression is an attempt to explain movements in a variable by reference to movements in one or more other variables.

To make this more concrete, denote the variable whose movements the regression seeks to explain by y and the variables which are used to explain those variations by $x_1, x_2, \ldots, x_k$. Hence, in this relatively simple setup, it would be said that variations in k variables (the xs) cause changes in some other variable, y. This chapter will be limited to the case where the model seeks to explain changes in only one variable y (although this restriction will be removed in Chapter 7).

There are various completely interchangeable names for y and the xs, and all of these terms will be used synonymously in this book (see Box 3.1).

BOX 3.1 Names for y and xs in regression models

Names for y	*Names for the xs*
Dependent variable	Independent variables
Regressand	Regressors
Effect variable	Causal variables
Explained variable	Explanatory variables

3.2 Regression versus Correlation

As discussed in Chapter 2, the correlation between two variables measures the *degree of linear association* between them. If it is stated that y and x are correlated, it means that y and x are being treated in a completely symmetrical way. Thus, it is not implied that changes in x cause changes in y, or indeed that changes in y cause changes in x. Rather, it is simply stated that there is evidence for a linear relationship between the two variables, and that movements in the two are on average related to an extent given by the correlation coefficient.

In regression, the dependent variable (y) and the independent variable(s) (xs) are treated very differently. The y variable is assumed to be random or 'stochastic' in some way, i.e., to have a *probability distribution*. The x variables are, however, assumed to have fixed ('non-stochastic') values in repeated samples.[1] Regression as a tool is more flexible and more powerful than correlation.

3.3 Simple Regression

For simplicity, suppose for now that it is believed that y depends on only one x variable. Again, this is of course a severely restricted case, but the case of more explanatory variables will be considered in Chapter 4. Three examples of the kind of relationship that may be of interest include

- How asset returns vary with their level of market risk
- Measuring the long-term relationship between stock prices and dividends
- Constructing an optimal hedge ratio

Suppose that a researcher has some idea that there should be a relationship between two variables y and x, and that financial theory suggests that an increase in x will lead to an increase in y. A sensible first stage to testing whether there is indeed an association between the variables would be to form a scatter plot of them. Suppose that the outcome of this plot is Figure 3.1.

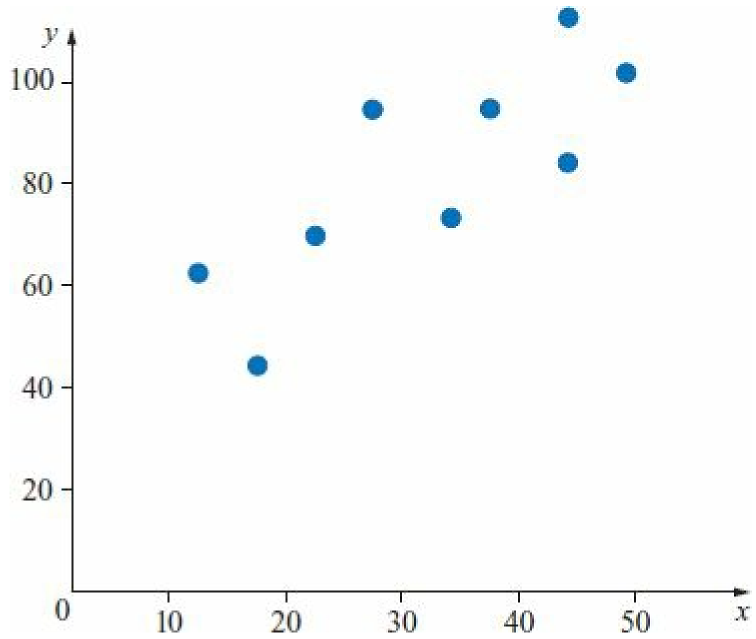


Figure 3.1 Scatter plot of two variables, y and x

In this case, it appears that there is an approximate positive linear relationship between x and y which means that increases in x are usually accompanied by increases in y, and that the relationship between them can be described approximately by a straight line. It would be possible to draw by hand onto the graph a line that appears to fit the data. The intercept and slope of the line fitted by eye could then be measured from the graph. However, in practice such a method is likely to be laborious and inaccurate.

It would therefore be of interest to determine to what extent this relationship can be described by an equation that can be estimated using a defined procedure. It is possible to use the general equation for a straight line

$$y = \alpha + \beta x \tag{3.1}$$

to get the line that best 'fits' the data. The researcher would then be seeking to find the values of the parameters or coefficients, α and β, which would place the line as close as possible to all of the data points taken together.

However, this equation ($y = \alpha + \beta x$) is an exact one. Assuming that this equation is appropriate, if the values of α and β had been calculated, then given a value of x, it would be possible to determine with certainty what the value of y would be. Imagine – a model which says with complete certainty what the value of one variable will be given any value of the other!

Clearly this model is not realistic. Statistically, it would correspond to the case where the model fitted the data perfectly – that is, all of the data points lay exactly on a straight line. To make the model more realistic, a random disturbance term, denoted by u, is added to the equation, thus

$$y_t = \alpha + \beta x_t + u_t \tag{3.2}$$

where the subscript t (= 1, 2, 3, …) denotes the observation number. The disturbance term can capture a number of features (see Box 3.2).

BOX 3.2 Reasons for the inclusion of the disturbance term

- Even in the general case where there is more than one explanatory variable, some determinants of y_t will always in practice be omitted from the model. This might, for example, arise because the number of influences on y is too large to place in a single model, or because some determinants of y may be unobservable or not measurable.
- There may be errors in the way that y is measured which cannot be modelled.
- There are bound to be random outside influences on y that again cannot be modelled. For example, a terrorist attack, a hurricane or a computer failure could all affect financial asset returns in a way that cannot be captured in a model and cannot be forecast reliably. Similarly, many researchers would argue that human behaviour

has an inherent randomness and unpredictability!

So how are the appropriate values of α and β determined? α and β are chosen so that the (vertical) distances from the data points to the fitted lines are minimised (so that the line fits the data as closely as possible). The parameters are thus chosen to minimise collectively the (vertical) distances from the data points to the fitted line. This could be done by 'eye-balling' the data and, for each set of variables y and x, one could form a scatter plot and draw on a line that looks as if it fits the data well by hand, as in Figure 3.2.

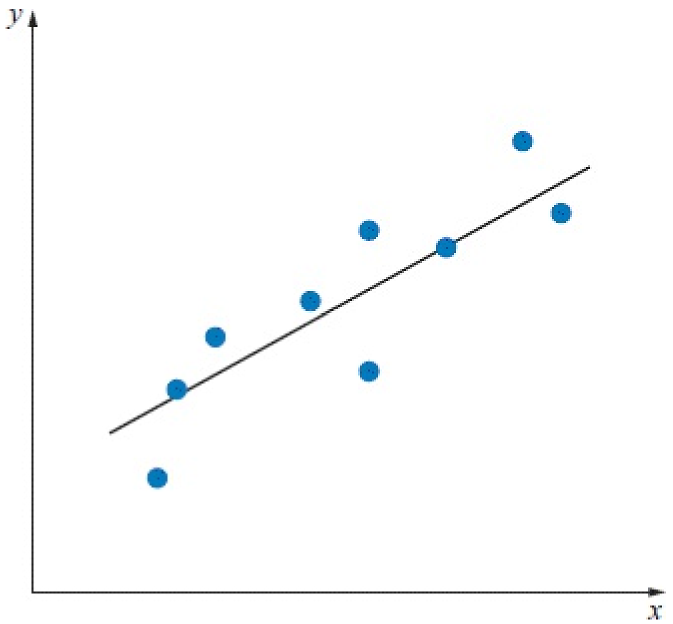


Figure 3.2 Scatter plot of two variables with a line of best fit chosen by eye

Note that the *vertical distances* are usually minimised rather than the horizontal distances or those taken perpendicular to the line. This arises as a result of the assumption that x is fixed in repeated samples, so that the problem becomes one of determining the appropriate model for y given (or conditional upon) the observed values of x.

This 'eye-balling' procedure may be acceptable if only indicative results are required, but of course this method, as well as being tedious, is likely to be imprecise. The most common method used to fit a line to the data is known as ordinary least squares (OLS). This approach forms the workhorse of econometric model estimation, and will be discussed in detail in this and subsequent chapters.

Two alternative estimation methods (for determining the appropriate values of the coefficients α and β) are the method of moments and the method of maximum likelihood. A generalised version of the method of moments, due to Hansen (1982), is popular, but beyond the scope of this book. The method of maximum likelihood is also widely employed, and will be discussed in detail in Chapter 9.

Suppose now, for ease of exposition, that the sample of data contains only five observations. The method of OLS entails taking each vertical distance from the point to the line, squaring it and then minimising the total sum of the areas of squares (hence 'least squares'), as shown in Figure 3.3. This can be viewed as equivalent to minimising the sum of the areas of the squares drawn from the points to the line.

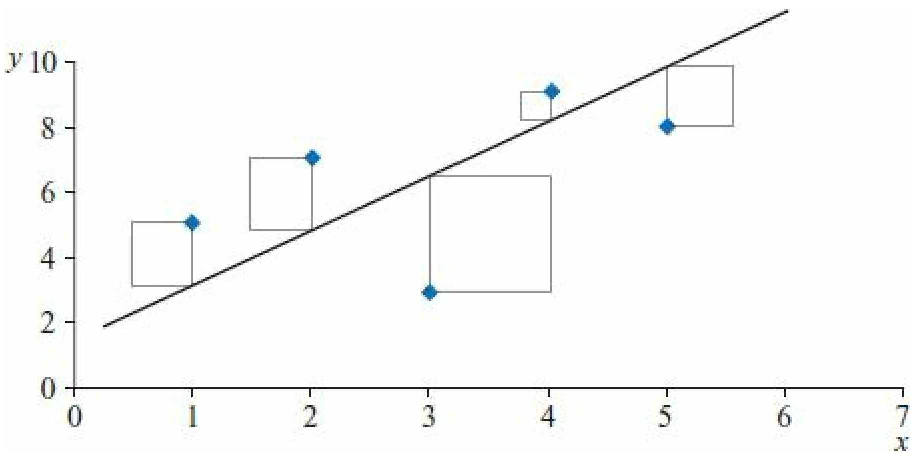


Figure 3.3 Method of OLS fitting a line to the data by minimising the sum of squared residuals

Tightening up the notation, let y_t denote the actual data point for observation t and let $\hat{y}_t$ denote the fitted value from the regression line – in other words, for the given value of x of this observation t, $\hat{y}_t$ is the value for y which the model would have predicted. Note that a hat (ˆ) over a variable or parameter is used to denote a value estimated by a model. Finally, let $\hat{u}_t$ denote the residual, which is the difference between the actual value of y and the value fitted by the model for this data point – i.e., $(y_t - \hat{y}_t)$. This is shown for just one observation t in Figure 3.4.

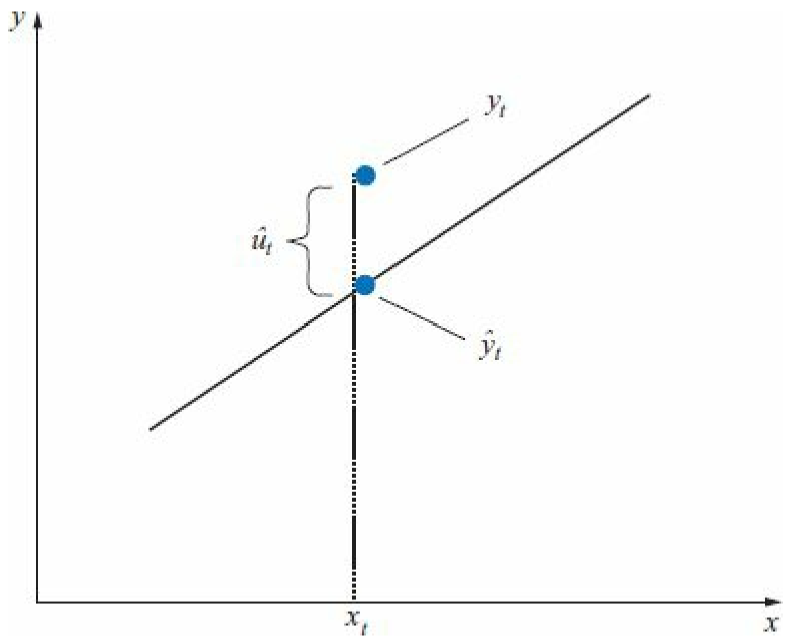


Figure 3.4 Plot of a single observation, together with the line of best fit, the residual and the fitted value

What is done is to minimise the sum of the $\hat{u}_t^2$. The reason that the sum of the squared distances is minimised rather than, for example, finding the sum of $\hat{u}_t$ that is as close to zero as possible, is that in the latter case some points will lie above the line while others lie below it. Then, when the sum to be made as close to zero as possible is formed, the points above the line would count as positive values, while those below would count as negatives. So these distances will in large part cancel each other out, which would mean that one could fit virtually any line to the data, so long as the sum of the distances of the points above the line and the sum of the distances of the points below the line were the same. In that case, there would not be a unique solution for the estimated coefficients. In fact, any fitted line that goes through the mean of the observations (i.e., $\bar{x}, \bar{y}$) would set the sum of the $\hat{u}_t$ to zero. However, taking the squared distances ensures that all deviations that enter the calculation are positive and therefore do not cancel out.

So minimising the sum of squared distances is given by minimising $(\hat{u}_1^2 + \hat{u}_2^2 + \hat{u}_3^2 + \hat{u}_4^2 + \hat{u}_5^2)$, or minimising

$$\left(\sum_{t=1}^{5} \hat{u}_t^2\right)$$

This sum is known as the *residual sum of squares* (RSS) or the sum of squared residuals. But what is $\hat{u}_t$? Again, it is the difference between the actual point and the line, $y_t - \hat{y}_t$. So minimising $\sum_t \hat{u}_t^2$ is equivalent to minimising $\sum_t (y_t - \hat{y}_t)^2$.

Letting $\hat{\alpha}$ and $\hat{\beta}$ denote the values of α and β selected by minimising the RSS, respectively, the equation for the fitted line is given by $\hat{y}_t = \hat{\alpha} + \hat{\beta}x_t$. Now let L denote the RSS, which is also known as a loss function. Take the summation over all of the observations, i.e., from $t = 1$ to T, where T is the number of observations

$$L = \sum_{t=1}^{T} (y_t - \hat{y}_t)^2 = \sum_{t=1}^{T} (y_t - \hat{\alpha} - \hat{\beta}x_t)^2 \tag{3.3}$$

L is minimised with respect to (w.r.t.) $\hat{\alpha}$ and $\hat{\beta}$, to find the values of α and β which minimise the residual sum of squares to give the line that is closest to the data. So L is differentiated w.r.t. $\hat{\alpha}$ and $\hat{\beta}$, setting the first derivatives to zero. A derivation of the OLS estimator is given in Appendix 3.1 to this chapter. The coefficient estimators for the slope and the intercept are given by

$$\hat{\beta} = \frac{\sum x_t y_t - T\bar{x}\bar{y}}{\sum x_t^2 - T\bar{x}^2} \tag{3.4}$$

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} \tag{3.5}$$

Equations (3.4) and (3.5) state that, given only the sets of observations x_t and y_t, it is always possible to calculate the values of the two parameters, $\hat{\alpha}$ and $\hat{\beta}$, that best fit the set of data. Equation (3.4) is the easiest formula to use to calculate the slope estimate, but the formula can also be written, more intuitively, as

$$\hat{\beta} = \frac{\sum (x_t - \bar{x})(y_t - \bar{y})}{\sum (x_t - \bar{x})^2} \tag{3.6}$$

which is equivalent to the sample covariance between x and y divided by the sample variance of x.

To reiterate, this method of finding the optimum is known as OLS. By construction it finds the parameter values that best fit the sample data: any other parameter values would lead to a worse fit and a higher RSS. This is illustrated in Figure 3.5, which shows how the RSS varies with different values of β and $\hat{\beta}$ gives the lowest value. As an exercise, you could set up a spreadsheet that calculates the RSS from equation (3.5) and the sample data in the following Example 3.3, produce a plot similar to Figure 3.5 and demonstrate that any other values of β will give a higher RSS.

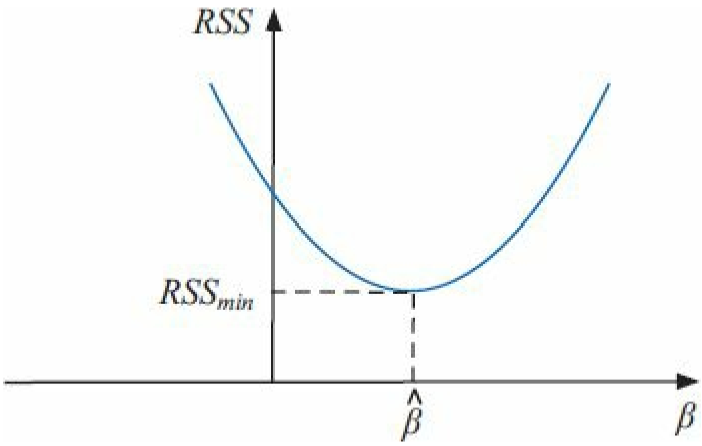


Figure 3.5 How RSS varies with different values of β

It is also worth noting that it is obvious from the equation for $\hat{\alpha}$ that the regression line will go through the mean of the observations – i.e., that the point $(\bar{x}, \bar{y})$ lies on the regression line.

EXAMPLE 3.1

Suppose that some data have been collected on the excess returns on a fund manager's portfolio ('fund XXX') together with the excess returns on a market index, as shown in Table 3.1.

Table 3.1 Sample data on fund XXX to motivate OLS estimation

Year, t	Excess return on fund XXX $= r_{XXX,t} - rf_t$	Excess return on market index $= rm_t - rf_t$
1	17.8	13.7
2	39.0	23.2
3	12.8	6.9
4	24.2	16.8
5	17.2	12.3

The fund manager has some intuition that the beta (in the CAPM framework) on this fund is positive, and she therefore wants to find whether there appears to be a relationship between x and y given the data. Again, the first stage could be to form a scatter plot of the two variables (Figure 3.6).

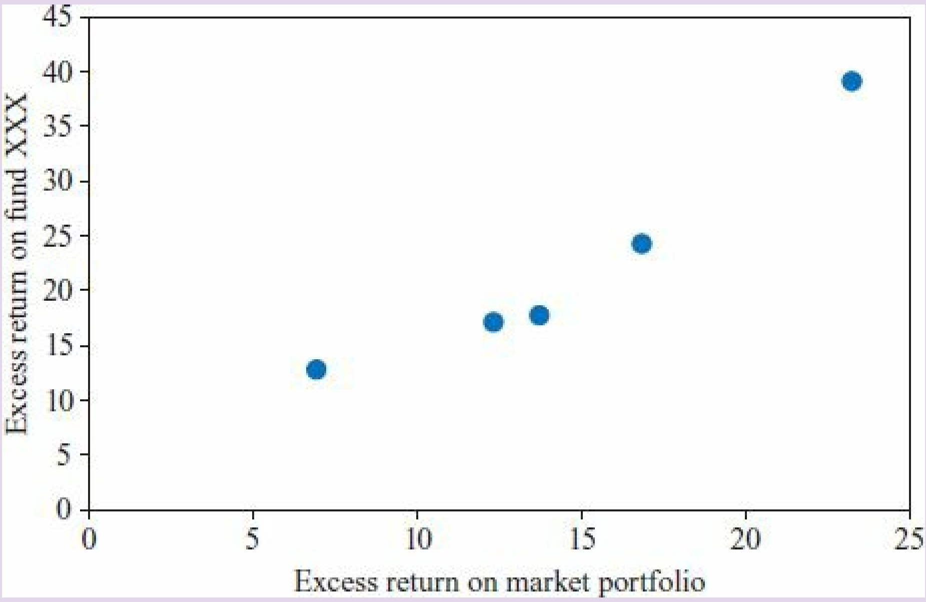


Figure 3.6 Scatter plot of excess returns on fund XXX versus excess returns on the market portfolio

Clearly, there appears to be a positive, approximately linear relationship between x and y, although there is not much data on which to base this conclusion! Plugging the five observations in to make up the formulae given in equations (3.5) and (3.4) would lead to the estimates $\hat{\alpha} = -1.74$ and $\hat{\beta} = 1.64$. The fitted line would be written as

$$\hat{y}_t = -1.74 + 1.64x_t \quad \textbf{(3.7)}$$

where x_t is the excess return of the market portfolio over the risk-free rate (i.e., *rm – rf*), also known as the *market risk premium*.

3.3.1 What are $\hat{\alpha}$ and $\hat{\beta}$ Used For?

This question is probably best answered by posing another question. If an analyst tells you that she expects the market to yield a return 20% higher than the risk-free rate next year, what would you expect the return on fund XXX to be?

The expected value of y = '–1.74 + 1.64 × value of x', so plug x = 20 into (3.7)

$$\hat{y}_t = -1.74 + 1.64 \times 20 = 31.06$$

(3.8)

Thus, for a given expected market risk premium of 20%, and given its riskiness, fund XXX would be expected to earn an excess over the risk-free rate of approximately 31%. In this setup, the regression beta is also the CAPM beta, so that fund XXX has an estimated beta of 1.64, suggesting that the fund is rather risky. In this case, the residual sum of squares reaches its minimum value of 30.33 with these OLS coefficient values.

Although it may be obvious, it is worth stating that it is not advisable to conduct a regression analysis using only five observations! Thus the results presented here can be considered indicative and for illustration of the technique only. Some further discussions on appropriate sample sizes for regression analysis are given in Chapter 5.

The coefficient estimate of 1.64 for β is interpreted as saying that, 'if x increases by 1 unit, y will be expected, everything else being equal, to increase by 1.64 units'. Of course, if $\hat{\beta}$ had been negative, a rise in x would on average cause a fall in y. $\hat{\alpha}$, the intercept coefficient estimate, is interpreted as the value that would be taken by the dependent variable y if the independent variable x took a value of zero. 'Units' here refer to the units of measurement of x_t and y_t. So, for example, suppose that $\hat{\beta} = 1.64$, x is measured in per cent and y is measured in thousands of US dollars. Then it would be said that if x rises by 1%, y will be expected to rise on average by \$1.64 thousand (or \$1,640).

Mathematically, we can interpret the slope coefficient in a regression model as being the derivative of the dependent variable with respect to the independent variable, i.e., $\hat{\beta} = \frac{dy}{dx}$. In cases where there is more than one independent variable (as we will meet in Chapter 4), then the coefficients can be interpreted as partial derivatives of the dependent variable with respect to each independent variable, $\frac{\partial y}{\partial x}$.

Note that changing the scale of y or x will make no difference to the overall results since the coefficient estimates will change by an off-setting factor to leave the overall relationship between y and x unchanged (see Gujarati, 2003, pp. 169–73 for a proof). Thus, if the units of measurement of y were hundreds of dollars instead of thousands, and everything else remains unchanged, the slope coefficient estimate would be 16.4, so that a 1% increase in x would lead to an increase in y of \$16.4 hundreds (or \$1,640) as before. All other properties of the OLS estimator discussed below are also invariant to changes in the scaling of the data.

A word of caution is, however, in order concerning the reliability of estimates of the constant term. Although the strict interpretation of the intercept is indeed as stated above, in practice, it is often the case that there are no values of *x* close to zero in the sample. In such instances, estimates of the value of the intercept will be unreliable. For example, consider Figure 3.7, which demonstrates a situation where no points are close to the *y*-axis.

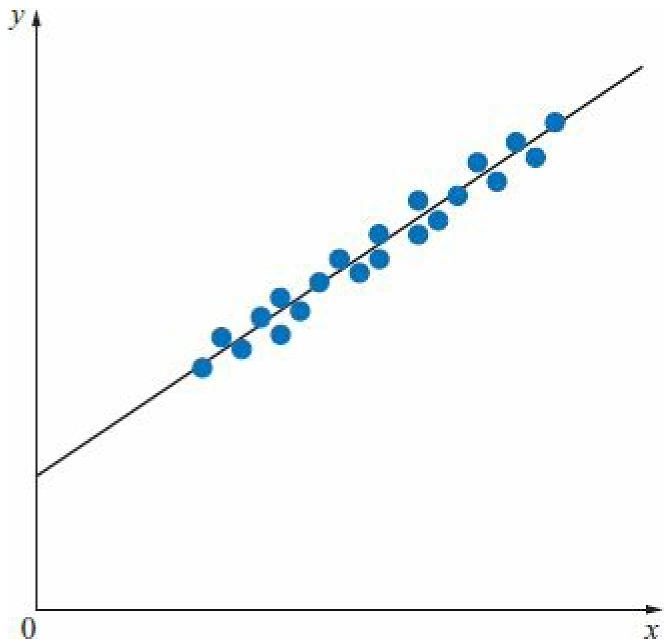


Figure 3.7 No observations close to the *y*-axis

In such cases, one could not expect to obtain robust estimates of the value of *y* when *x* is zero as all of the information in the sample pertains to the case where *x* is considerably larger than zero.

A similar caution should be exercised when producing predictions for *y* using values of *x* that are a long way outside the range of values in the sample. In Example 3.1, *x* takes values between 7% and 23% in the available data. So, it would not be advisable to use this model to determine the expected excess return on the fund if the expected excess return on the market were, say 1% or 30%, or –5% (i.e., the market was expected to fall).

3.4 Some Further Terminology

3.4.1 The Data Generating Process, the Population

Regression Function and the Sample Regression Function

The population regression function (PRF) is a description of the model that is thought to be generating the actual data and it represents the *true relationship between the variables*. The population regression function is also known as the data generating process (DGP). The PRF embodies the true values of α and β, and is expressed as

$$y_t = \alpha + \beta x_t + u_t \tag{3.9}$$

Note that there is a disturbance term in this equation, so that even if one had at one's disposal the entire population of observations on x and y, it would still in general not be possible to obtain a perfect fit of the line to the data. In some textbooks, a distinction is drawn between the PRF (the underlying true relationship between y and x) and the DGP (the process describing the way that the actual observations on y come about), although in this book, the two terms will be used synonymously.

The sample regression function (SRF) is the relationship that has been estimated using the sample observations, and is often written as

$$\hat{y}_t = \hat{\alpha} + \hat{\beta} x_t \tag{3.10}$$

Notice that there is no error or residual term in equation (3.10); all this equation states is that given a particular value of x, multiplying it by $\hat{\beta}$ and adding $\hat{\alpha}$ will give the model fitted or expected value for y, denoted $\hat{y}$. It is also possible to write

$$y_t = \hat{\alpha} + \hat{\beta} x_t + \hat{u}_t \tag{3.11}$$

Equation (3.11) splits the observed value of y into two components: the fitted value from the model, and a residual term.

The SRF is used to infer likely values of the PRF. That is, the estimates $\hat{\alpha}$ and $\hat{\beta}$ are constructed, for the sample of data at hand, but what is really of interest is the true relationship between x and y – in other words, the PRF is what is really wanted, but all that is ever available is the SRF. However, what can be said is how likely it is, given the figures calculated for $\hat{\alpha}$ and $\hat{\beta}$, that the corresponding population parameters take on certain values.

3.4.2 Linearity and Possible Forms for the Regression Function

In order to use OLS, a model that is *linear* is required. This means that, in the simple bivariate case, the relationship between x and y must be capable of being expressed diagramatically using a straight line. More specifically, the model must be linear in the parameters (α and β), but it does not necessarily have to be linear in the variables (y and x). By 'linear in the parameters', it is meant that the parameters are not multiplied together, divided, squared or cubed, etc.

Models that are not linear in the variables can often be made to take a linear form by applying a suitable transformation or manipulation. For example, consider the following exponential regression model

$$Y_t = AX_t^{\beta} e^{u_t} \qquad \textbf{(3.12)}$$

Taking logarithms of both sides, applying the laws of logs and rearranging the RHS

$$\ln Y_t = \ln(A) + \beta \ln X_t + u_t \qquad \textbf{(3.13)}$$

where A and β are parameters to be estimated. Now let $\alpha = \ln(A)$, $y_t = \ln Y_t$ and $x_t = \ln X_t$

$$y_t = \alpha + \beta x_t + u_t \qquad \textbf{(3.14)}$$

This is known as an *exponential regression model* since Y varies according to some exponent (power) function of X. In fact, when a regression equation is expressed in 'double logarithmic form', which means that both the dependent and the independent variables are natural logarithms, the coefficient estimates are interpreted as elasticities (strictly, they are unit changes on a logarithmic scale).

Elasticities are useful since they are unit-free – that is, they are not a function of the units of measurement of either the dependent or the independent variable. Mathematically, as stated above, the slope parameter estimate in a linear regression (not in logarithmic form) can be interpreted as a derivative of y with respect to x. This is sometimes known as a *marginal propensity*. Elasticities can also be calculated in this context by taking

$$\frac{dy}{dx} \times \frac{x_0}{y_0} = \frac{\frac{dy}{y_0}}{\frac{dx}{x_0}}$$

– in other words, multiplying the derivative (i.e., the slope in the regression) by the value of x at some point (call this x_0) and dividing by the corresponding value of y, y_0. We can see from the left-hand side of the expression why this is unit for both x and y have effectively been cancelled out. From the right-hand side, we can see that the elasticity is measuring the proportional change ratio, which is the amount that y changes, dy, as a proportion of its actual value, y_0, divided by the amount that x changes, dx, as a proportion of its actual value, x_0.

Thus a coefficient estimate of 1.2 for $\hat{\beta}$ in equation (3.13) or (3.14) is interpreted as stating that 'a rise in X of 1% will lead on average, everything else being equal, to a rise in Y of 1.2%'. Conversely, for y and x in levels (e.g., equation (3.9)) rather than logarithmic form, the coefficients denote unit changes as described above.

Similarly, if theory suggests that x should be inversely related to y according to a model of the form

$$y_t = \alpha + \frac{\beta}{x_t} + u_t \qquad \textbf{(3.15)}$$

the regression can be estimated using OLS by setting

$$z_t = \frac{1}{x_t}$$

and regressing y on a constant and z. Clearly, then, a surprisingly varied array of models can be estimated using OLS by making suitable transformations to the variables. On the other hand, some models are *intrinsically non-linear*, e.g.

$$y_t = \alpha + \beta x_t^{\gamma} + u_t \qquad \textbf{(3.16)}$$

Such models cannot be estimated using OLS, but might be estimable using a nonlinear estimation method (see Chapter 9).

3.4.3 Estimator or Estimate?

Estimators are the formulae used to *calculate the coefficients* – for example, the expressions given in equation (3.4) and (3.5) above, while the estimates, on the other hand, are the *actual numerical values for the coefficients* that are obtained from the sample.

3.5 The Assumptions Underlying the Classical Linear Regression Model

The model $y_t = \alpha + \beta x_t + u_t$ that has been derived above, together with the assumptions listed below, is known as the *classical linear regression model* (CLRM). Data for x_t are observable, but since y_t also depends on u_t, it is necessary to be specific about how the u_t are generated. The set of assumptions shown in Box 3.3 are usually made concerning the u_t s, the unobservable error or disturbance terms. Note that no assumptions are made concerning their observable counterparts, the estimated model's residuals.

As long as assumption (1) holds, assumption (4) can be equivalently written $E(x_t\ u_t) = 0$. Both formulations imply that the regressor is *orthogonal* to (i.e., unrelated to) the error term. An alternative assumption to (4), which is slightly stronger, is that the x_t are *non-stochastic* or fixed in repeated samples. This means that there is no sampling variation in x_t, and that its value is determined outside the model.

BOX 3.3 Assumptions concerning disturbance terms and their interpretation

Technical notation	*Interpretation*
(1) $E(u_t) = 0$	The errors have zero mean
(2) $\text{var}(u_t) = \sigma^2 < \infty$	The variance of the errors is constant and finite over all values of x_t
(3) $\text{cov}(u_i, u_j) = 0$	The errors are linearly independent of one another
(4) $\text{cov}(u_t, x_t) = 0$	There is no relationship between the error and corresponding x variate
(5) $u_t \sim N(0, \sigma^2)$	– i.e., that u_t is normally distributed.

A fifth assumption is required to make valid inferences about the population parameters (the actual α and β) from the sample parameters ($\hat{\alpha}$

and $\hat{\beta}$) estimated using a finite amount of data, namely that the disturbances follow a normal distribution.

3.6 Properties of the OLS Estimator

If assumptions (1)–(4) hold, then the estimators $\hat{\alpha}$ and $\hat{\beta}$ determined by OLS will have a number of desirable properties, and are known as best linear unbiased estimators (BLUE). What does this acronym stand for?

- 'Estimator' – $\hat{\alpha}$ and $\hat{\beta}$ are estimators of the true value of α and β
- 'Linear' – $\hat{\alpha}$ and $\hat{\beta}$ are linear estimators – that means that the formulae for $\hat{\alpha}$ and $\hat{\beta}$ are linear combinations of the random variables (in this case, *y*)
- 'Unbiased' – on average, the actual values of $\hat{\alpha}$ and $\hat{\beta}$ will be equal to their true values
- 'Best' – means that the OLS estimator $\hat{\beta}$ has minimum variance among the class of linear unbiased estimators; the Gauss–Markov theorem proves that the OLS estimator is best by examining an arbitrary alternative linear unbiased estimator and showing in all cases that it must have a variance no smaller than the OLS estimator

Under assumptions (1)–(4) listed above, the OLS estimator can be shown to have the desirable properties that it is consistent, unbiased and efficient. Unbiasedness and efficiency have already been discussed above, and consistency is an additional desirable property. These three characteristics will now be discussed in turn.

3.6.1 Consistency

The least squares estimators $\hat{\alpha}$ and $\hat{\beta}$ are consistent. One way to state this algebraically for $\hat{\beta}$ (with the obvious modifications made for $\hat{\alpha}$) is

$$\lim_{T \to \infty} \Pr\,[|\hat{\beta} - \beta| > \delta] = 0 \quad \forall\, \delta > 0 \qquad \textbf{(3.17)}$$

This is a technical way of stating that the probability (Pr) that $\hat{\beta}$ is more than some arbitrary fixed distance δ away from its true value tends to zero as the sample size tends to infinity, for all positive values of δ. Thus β is the probability limit of $\hat{\beta}$. In the limit (i.e., for an infinite number of observations), the probability of the estimator being different from the true value is zero. That is, the estimates will converge to their true values as the sample size increases to infinity. Consistency is thus a large sample, or asymptotic property. If an estimator is inconsistent, then even if we had an infinite amount of data, we could not be sure that the estimated value of a parameter will be close to its true value. So consistency is sometimes argued to be the most important property of an estimator. The assumptions that $E(x_t u_t) = 0$ and $E(u_t) = 0$ are sufficient to derive the consistency of the OLS estimator.

3.6.2 Unbiasedness

The least squares estimates of $\hat{\alpha}$ and $\hat{\beta}$ are unbiased. That is

$$E(\hat{\alpha}) = \alpha \tag{3.18}$$

and

$$E(\hat{\beta}) = \beta \tag{3.19}$$

Thus, on average, the estimated values for the coefficients will be equal to their true values. That is, there is no systematic overestimation or underestimation of the true coefficients. To prove this also requires the assumption that $\mathrm{cov}(u_t, x_t) = 0$. Clearly, unbiasedness is a stronger condition than consistency, since it holds for small as well as large samples (i.e., for all sample sizes). An estimator that is consistent may still be biased for small samples, but are all unbiased estimators also consistent? The answer is in fact ‘no’. An unbiased estimator will also be consistent if its variance falls as the sample size increases.

3.6.3 Efficiency

An estimator $\hat{\beta}$ of a parameter β is said to be efficient if no other estimator has a smaller variance. Broadly, if the estimator is efficient, it will be minimising the probability that it is a long way off from the true value of β or, in simpler terms, the variation in the parameter estimate from one sample within the population to another would be minimised. In other words, if the estimator is ‘best’, the uncertainty associated with estimation will be minimised for the class of linear unbiased estimators. A technical way to state this would be to say that an efficient estimator would have a probability distribution that is narrowly dispersed around the true value.

3.6.4 More on Unbiasedness and Efficiency

As stated above, the Gauss–Markov theorem shows that the OLS estimator has the least variance *among the class of linear unbiased estimators*. It is worth exploring these concepts in a little more detail to try to get a better grip on their meaning and implication. It would be possible to find an estimator with a lower variance than the OLS estimator, but it would not be linear and unbiased. As obvious example would be a fixed estimator, e.g., $\hat{\beta} = 2,$ so in other words, whatever the data say, we fix the slope estimate at 2. This estimator would clearly have a lower variance than the OLS estimator – in fact, it would have a variance of zero since there would be no change in the parameter estimate from one sample to another since it is always 2 – but it would clearly be biased and inconsistent, as we increase the sample size, there would be no convergence upon the true population value of β, and the error between our estimated value of 2 and the true value would always be in the same direction and thus biased.

More generally, other data-dependent (non-OLS) estimators could be used but these would either be non-linear or biased or have a higher variance than the OLS estimator. Thus, there is often a trade-off between bias and variance, so that to improve one means worsening the other. The situation is illustrated in Figure 3.8, which plots the distributions of two different estimators. These both display the ranges of estimates for the slope parameter that might arise when selecting different samples from within the population. The distribution which has $\hat{\beta}$ at its centre represents the OLS estimator – this has the true value (β) as the most commonly estimated value in the centre, and is thus unbiased, but it also has a bigger variance than the other estimator, as it is flatter and with more of the distribution in the tails and less in the centre. On the other hand, the alternative estimator, which is represented by the distribution to the right of Figure 3.8, is much more focused on its mean value (which I have called $\tilde{\beta}$) – this estimator clearly has a lower variance but is biased since its centre is not on the true value of β. Up to a certain point, bias is usually considered a more serious problem than variance and hence the widespread use of the OLS estimator as the core of econometric model–building.

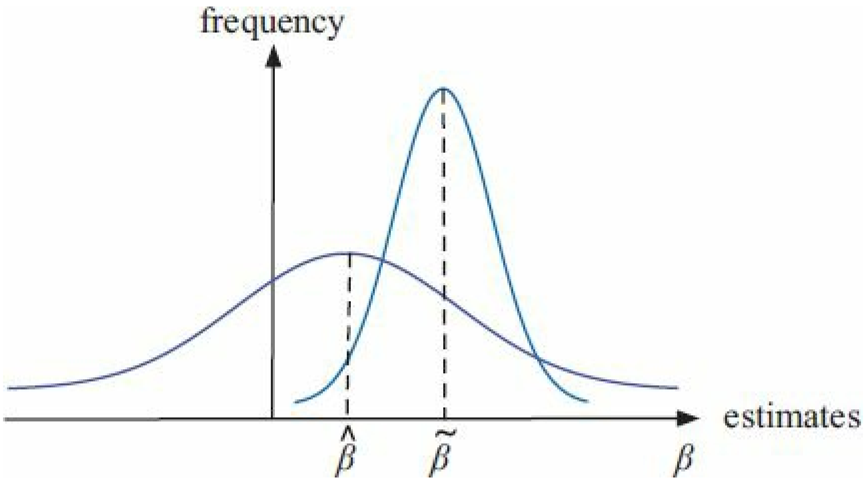


Figure 3.8 The bias versus variance trade-off when selecting between estimators

3.7 Precision and Standard Errors

Any set of regression estimates $\hat{\alpha}$ and $\hat{\beta}$ are specific to the sample used in their estimation. In other words, if a different sample of data was selected from within the population, the data points (the x_t and y_t) will be different, leading to different values of the OLS estimates.

Recall that the OLS estimators ($\hat{\alpha}$ and $\hat{\beta}$) are given by equation (3.4) and (3.5). It would be desirable to have an idea of how 'good' these estimates of α and β are in the sense of having some measure of the reliability or precision of the estimators ($\hat{\alpha}$ and $\hat{\beta}$). It is thus useful to know whether one can have confidence in the estimates, and whether they are likely to vary much from one sample to another sample within the given population. An idea of the sampling variability and hence of the precision of the estimates can be calculated using only the sample of data available. This estimate of precision is given by its standard error. Given assumptions (1)–(4) above, valid estimators of the standard errors can be shown to be given by

$$SE(\hat{\alpha}) = s\sqrt{\frac{\sum x_t^2}{T\sum (x_t - \bar{x})^2}} = s\sqrt{\frac{\sum x_t^2}{T\left(\left(\sum x_t^2\right) - T\bar{x}^2\right)}} \tag{3.20}$$

$$SE(\hat{\beta}) = s\sqrt{\frac{1}{\sum (x_t - \bar{x})^2}} = s\sqrt{\frac{1}{\sum x_t^2 - T\bar{x}^2}} \tag{3.21}$$

where s is the estimated standard deviation of the residuals (see below). These formulae are derived in Appendix 3.1 to this chapter.

It is worth noting that the standard errors give only a general indication of the likely accuracy of the regression parameters. They do not show how accurate a particular set of coefficient estimates is. If the standard errors are small, it shows that the coefficients are likely to be precise on average, not how precise they are for this particular sample. Thus standard errors give a measure of the *degree of uncertainty* in the estimated values for the coefficients. It can be seen that they are a function of the actual observations on the explanatory variable, x, the sample size, T, and another term, s. The last of these is an estimate of the variance of the disturbance term. The actual variance of the disturbance term is usually denoted by σ^2. How can an estimate of σ^2 be obtained?

3.7.1 Estimating the Variance of the Error Term (σ^2)

From elementary statistics, the variance of a random variable u_t is given by

$$\text{var}(u_t) = \text{E}[(u_t) - \text{E}(u_t)]^2 \tag{3.22}$$

Assumption (1) of the CLRM was that the expected or average value of the errors is zero. Under this assumption, equation (3.22) above reduces to

$$\text{var}(u_t) = \text{E}\left[u_t^2\right] \tag{3.23}$$

So what is required is an estimate of the average value of u_t^2, which could be calculated as

$$s^2 = \frac{1}{T}\sum u_t^2 \tag{3.24}$$

Unfortunately equation (3.24) is not workable since u_t is a series of population disturbances, which is not observable. Thus the sample counterpart to u_t, which is $\hat{u}_t$, is used

$$s^2 = \frac{1}{T}\sum \hat{u}_t^2 \tag{3.25}$$

But this estimator is a biased estimator of σ^2. An unbiased estimator, s^2, would be given by equation (3.26) instead of equation (3.25)

$$s^2 = \frac{\sum \hat{u}_t^2}{T-2} \tag{3.26}$$

where $\sum \hat{u}_t^2$ is the residual sum of squares, so that the quantity of relevance for the standard error formulae is the square root of equation (3.26)

$$s = \sqrt{\frac{\sum \hat{u}_t^2}{T-2}} \quad \textbf{(3.27)}$$

s is also known as the *standard error of the regression* or the standard error of the estimate. It is sometimes used as a broad measure of the fit of the regression equation. Everything else being equal, the smaller this quantity is, the closer is the fit of the line to the actual data.

3.7.2 Some Comments on the Standard Error Estimators

It is possible, of course, to derive the formulae for the standard errors of the coefficient estimates from first principles using some algebra, and this is left to Appendix 3.1 to this chapter. Some general intuition is now given as to why the formulae for the standard errors given by equations (3.20) and (3.21) contain the terms that they do and in the form that they do. The presentation offered in Box 3.4 loosely follows that of Hill, Griffiths and Judge (1997), which is the clearest that this author has seen.

BOX 3.4 Standard error estimators

(1) The larger the sample size, T, the smaller will be the coefficient standard errors. T appears explicitly in $SE(\hat{\alpha})$ and implicitly in $SE(\hat{\beta})$. T appears implicitly since the sum $\sum (x_t - \bar{x})^2$ is from $t = 1$ to T. The reason for this is simply that, at least for now, it is assumed that every observation on a series represents a piece of useful information which can be used to help determine the coefficient estimates. So the larger the size of the sample, the more information will have been used in estimation of the parameters, and hence the more confidence will be placed in those estimates.

(2) Both $SE(\hat{\alpha})$ and $SE(\hat{\beta})$ depend on s^2 (or s). Recall from above that s^2 is the estimate of the error variance. The larger this quantity is, the more dispersed are the residuals, and so the greater is the uncertainty in the model. If s^2 is large, the data points are collectively a long way away from the line.

(3) The sum of the squares of the x_t about their mean appears in both formulae – since $\sum (x_t - \bar{x})^2$ appears in the denominators. The larger the sum of squares, the smaller the coefficient variances. Consider what happens if $\sum (x_t - \bar{x})^2$ is small or large, as shown in Figures 3.9 and 3.10, respectively.

In Figure 3.9, the data are close together so that $\sum (x_t - \bar{x})^2$ is small. In this first case, it is more difficult to determine with any degree of certainty exactly where the line should be. On the other hand, in Figure 3.10, the points are widely dispersed across a long section of the line, so that one could hold more confidence in the estimates in this case.

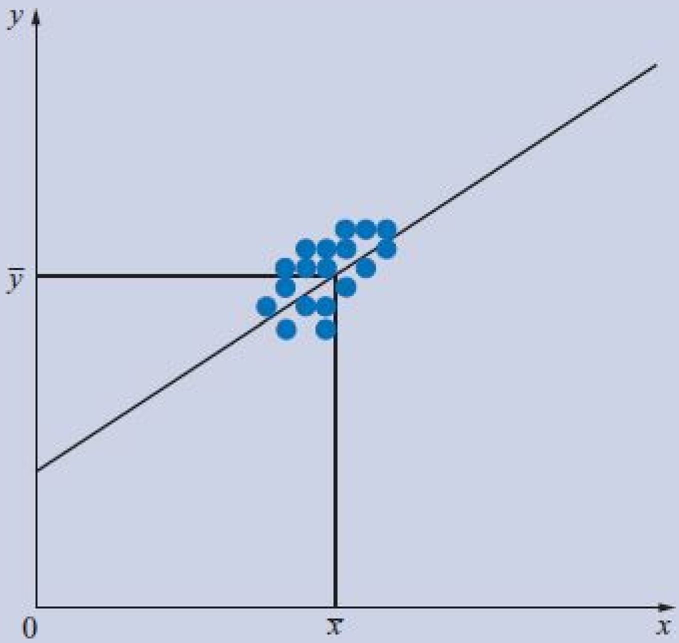


Figure 3.9 Effect on the standard errors of the coefficient estimates when $(x_t - \bar{x})$ are narrowly dispersed

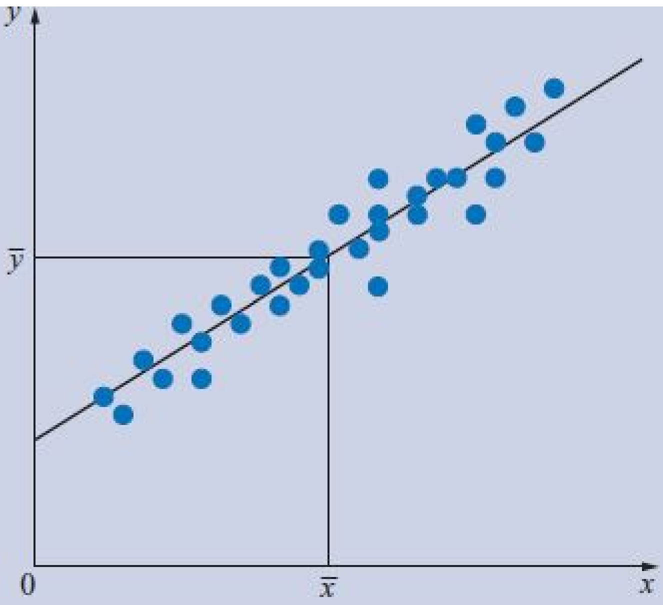


Figure 3.10 Effect on the standard errors of the coefficient estimates when $(x_t - \bar{x})$ are widely dispersed

(4) The term $\sum x_t^2$ affects only the intercept standard error and not the slope standard error. The reason is that $\sum x_t^2$ measures how far the points are away from the y-axis. Consider Figures 3.11 and 3.12.

In Figure 3.11, all of the points are bunched a long way from the y-axis, which makes it more difficult to accurately estimate the point at which the estimated line crosses the y-axis (the intercept). In Figure 3.12, the points collectively are closer to the y-axis and hence it will be easier to determine where the line actually crosses the axis. Note that this intuition will work only in the case where all of the x_t are positive!

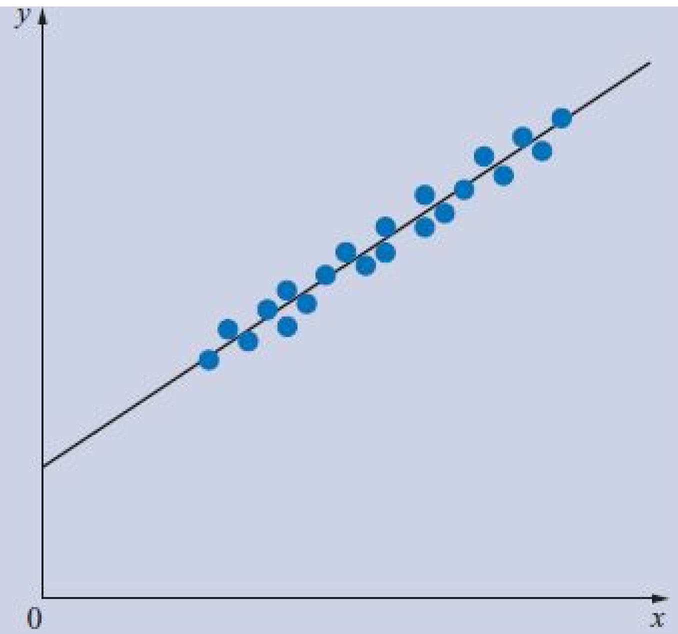


Figure 3.11 Effect on the standard errors of x_t^2 large

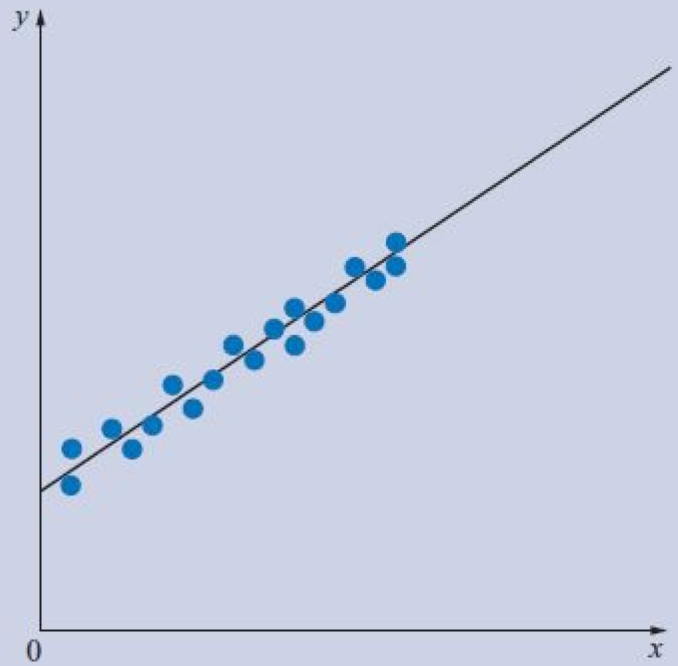


Figure 3.12 Effect on the standard errors of x_t^2 small

EXAMPLE 3.2

Assume that the following data have been calculated from a regression of *y* on a single variable *x* and a constant over twenty-two observations

$$\sum x_t y_t = 830102, \ \ T = 22, \ \ \bar{x} = 416.5, \ \ \bar{y} = 86.65,$$

$$\sum x_t^2 = 3919654, \ \ RSS = 130.6$$

Determine the appropriate values of the coefficient estimates and their standard errors.

This question can simply be answered by plugging the appropriate numbers into the formulae given above. The calculations are

$$\hat{\beta} = \frac{830102 - (22 \times 416.5 \times 86.65)}{3919654 - 22 \times (416.5)^2} = 0.35$$

$$\hat{\alpha} = 86.65 - 0.35 \times 416.5 = -59.12$$

The sample regression function would be written as

$$\hat{y}_t = \hat{\alpha} + \hat{\beta} x_t$$

$$\hat{y}_t = -59.12 + 0.35 x_t$$

Now, turning to the standard error calculations, it is necessary to obtain an estimate, s, of the error variance

$$SE(\text{regression}), \ s = \sqrt{\frac{\sum \hat{u}_t^2}{T-2}} = \sqrt{\frac{130.6}{20}} = 2.55$$

$$SE(\hat{\alpha}) = 2.55 \times \sqrt{\frac{3919654}{22 \times (3919654 - 22 \times 416.5^2)}} = 3.35$$

$$SE(\hat{\beta}) = 2.55 \times \sqrt{\frac{1}{3919654 - 22 \times 416.5^2}} = 0.0079$$

With the standard errors calculated, the results are written as

$$\underset{(3.35)}{\hat{y}_t = -59.12} + \underset{(0.0079)}{0.35 x_t} \quad \textbf{(3.28)}$$

The standard error estimates are usually placed in parentheses under the relevant coefficient estimates.

3.8 An Introduction to Statistical Inference

Often, financial theory will suggest that certain coefficients should take on particular values, or values within a given range. It is thus of interest to determine whether the relationships expected from financial theory are upheld by the data to hand or not. Estimates of α and β have been obtained from the sample, but these values are not of any particular interest; the population values that describe the true relationship between the variables would be of more interest, but are never available. Instead, inferences are made concerning the likely population values from the regression parameters that have been estimated from the sample of data to hand. In doing this, the aim is to determine whether the differences between the coefficient estimates that are actually obtained, and expectations arising from financial theory, are a long way from one another in a statistical sense.

EXAMPLE 3.3

Suppose the following regression results have been calculated:

$$\hat{y}_t = \underset{(14.38)}{20.3} + \underset{(0.2561)}{0.5091}x_t \qquad \textbf{(3.29)}$$

$\hat{\beta} = 0.5091$ is a single (point) estimate of the unknown population parameter, β. As stated above, the reliability of the point estimate is measured by the coefficient's standard error. The information from one or more of the sample coefficients and their standard errors can be used to make inferences about the population parameters. So the estimate of the slope coefficient is $\hat{\beta} = 0.5091$, but it is obvious that this number is likely to vary to some degree from one sample to the next. It might be of interest to answer the question, 'Is it plausible, given this estimate, that the true population parameter, β, could be 0.5? Is it plausible that β could be 1?', etc. Answers to these questions can be obtained through *hypothesis testing*.

3.8.1 Hypothesis Testing: Some Concepts

In the hypothesis testing framework, there are always two hypotheses that go together, known as the *null hypothesis* (denoted H_0 or occasionally H_N) and the *alternative hypothesis* (denoted H_1 or occasionally H_A). The null hypothesis is the statement or the statistical hypothesis that is actually being tested. The alternative hypothesis represents the remaining outcomes of interest.

For example, suppose that given the regression results above, it is of interest to test the hypothesis that the true value of β is in fact 0.5. The following notation would be used.

$$H_0 : \beta = 0.5$$
$$H_1 : \beta \neq 0.5$$

This states that the hypothesis that the true but unknown value of β could be 0.5 is being tested against an alternative hypothesis where β is not 0.5. This would be known as a two-sided test, since the outcomes of both $\beta < 0.5$ and $\beta > 0.5$ are subsumed under the alternative hypothesis.

Sometimes, some prior information may be available, suggesting for example that $\beta > 0.5$ would be expected rather than $\beta < 0.5$. In this case, $\beta < 0.5$ is no longer of interest to us, and hence a one-sided test would be conducted:

$$H_0 : \beta = 0.5$$
$$H_1 : \beta > 0.5$$

Here the null hypothesis that the true value of β is 0.5 is being tested against a one-sided alternative that β is more than 0.5.

On the other hand, one could envisage a situation where there is prior information that $\beta < 0.5$ is expected. For example, suppose that an investment bank bought a piece of new risk management software that is intended to better track the riskiness inherent in its traders' books and that β is some measure of the risk that previously took the value 0.5. Clearly, it would not make sense to expect the risk to have risen, and so $\beta > 0.5$, corresponding to an increase in risk, is not of interest. In this case, the null and alternative hypotheses would be specified as

$$H_0 : \beta = 0.5$$
$$H_1 : \beta < 0.5$$

This prior information should come from the financial theory of the problem under consideration, and not from an examination of the estimated value of the coefficient. Note that there is always an equality under the null hypothesis. So, for example, $\beta < 0.5$ would not be specified under the null hypothesis.

There are two ways to conduct a hypothesis test: via the *test of significance* approach or via the *confidence interval* approach. Both methods centre on a statistical comparison of the estimated value of the coefficient, and its value under the null hypothesis. In very general terms, if the estimated value is a long way away from the hypothesised value, the null hypothesis is likely to be rejected; if the value under the null hypothesis and the estimated value are close to one another, the null hypothesis is less likely to be rejected. For example, consider $\hat{\beta} = 0.5091$ as above. A hypothesis that the true value of β is 5 is more likely to be rejected than a null hypothesis that the true value of β is 0.5. What is required now is a *statistical decision rule* that will permit the formal testing of such hypotheses.

3.8.2 The Probability Distribution of the Least Squares Estimators

In order to test hypotheses, assumption (5) of the CLRM must be used, namely that $u_t \sim N(0, \sigma^2)$ – i.e., that the error term is normally distributed. The normal distribution is a convenient one to use for it involves only two parameters (its mean and variance). This makes the algebra involved in statistical inference considerably simpler than it otherwise would have been. Since y_t depends partially on u_t, it can be stated that if u_t is normally distributed, y_t will also be normally distributed.

Further, since the least squares estimators are linear combinations of the random variables, i.e., $\hat{\beta} = \sum w_t y_t$, where w_t are effectively weights, and since the weighted sum of normal random variables is also normally distributed, it can be said that the coefficient estimates will also be normally distributed. Thus

$$\hat{\alpha} \sim \mathrm{N}(\alpha, \mathrm{var}(\hat{\alpha})) \quad \text{and} \quad \hat{\beta} \sim \mathrm{N}(\beta, \mathrm{var}(\hat{\beta}))$$

Will the coefficient estimates still follow a normal distribution if the errors do not follow a normal distribution? Well, briefly, the answer is usually 'yes' as a result of the central limit theorem, provided that the other assumptions of the CLRM hold, and the sample size is sufficiently large. The issue of non-normality, how to test for it, and its consequences, will be further discussed in Chapter 5.

Standard normal variables can be constructed from $\hat{\alpha}$ and $\hat{\beta}$ by subtracting the mean and dividing by the square root of the variance

$$\frac{\hat{\alpha} - \alpha}{\sqrt{\mathrm{var}(\hat{\alpha})}} \sim \mathrm{N}(0, 1) \quad \text{and} \quad \frac{\hat{\beta} - \beta}{\sqrt{\mathrm{var}(\hat{\beta})}} \sim \mathrm{N}(0, 1)$$

The square roots of the coefficient variances are the standard errors. Unfortunately, the standard errors of the true coefficient values under the PRF are never known – all that is available are their sample counterparts, the calculated standard errors of the coefficient estimates, $SE(\hat{\alpha})$ and $SE(\hat{\beta})$.[2]

Replacing the true values of the standard errors with the sample estimated versions induces another source of uncertainty, and also means that the standardised statistics follow a t-distribution with $T-2$ degrees of freedom (defined below) rather than a normal distribution, so

$$\frac{\hat{\alpha}-\alpha}{SE(\hat{\alpha})} \sim t_{T-2} \quad \text{and} \quad \frac{\hat{\beta}-\beta}{SE(\hat{\beta})} \sim t_{T-2}$$

This result is not formally proved here. For a formal proof, see Hill, Griffiths and Judge (1997, pp. 88–90).

3.8.3 A Note on the *t* and the Normal Distributions

The normal distribution pdf was shown in Figure 3.2 with its characteristic 'bell' shape and its symmetry around the mean (of zero for a standard normal distribution). Any normal variate can be scaled to have zero mean and unit variance by subtracting its mean and dividing by its standard deviation. There is a specific relationship between the t- and the standard normal distribution, and the t-distribution has another parameter, its degrees of freedom.

What does the t-distribution look like? It looks similar to a normal distribution, but with fatter tails, and a smaller peak at the mean, as shown in Figure 3.13.

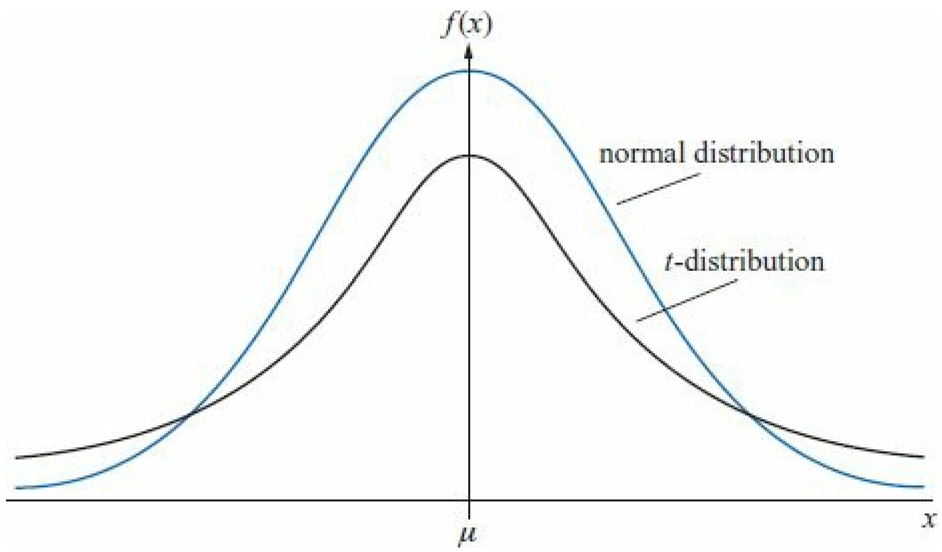


Figure 3.13 The *t*-distribution versus the normal

Some examples of the percentiles from the normal and t-distributions taken from the statistical tables are given in Table 3.2 (more critical values for these two distributions are given in Tables A2.1 and A2.2 of Appendix

2 at the end of this book). When used in the context of a hypothesis test, these percentiles become critical values. The values presented in Table 3.2 would be those critical values appropriate for a one-sided test of the given significance level.

Table 3.2 Critical values from the standard normal versus *t*-distribution

Significance level (%)	**N(0,1)**	t_{40}	t_4
50	0	0	0
5	1.64	1.68	2.13
2.5	1.96	2.02	2.78
0.5	2.57	2.70	4.60

It can be seen that as the number of degrees of freedom for the *t*-distribution increases from 4 to 40, the critical values fall substantially. In Figure 3.13, this is represented by a gradual increase in the height of the distribution at the centre and a reduction in the fatness of the tails as the number of degrees of freedom increases. In the limit, a *t*-distribution with an infinite number of degrees of freedom is a standard normal, i.e., t_∞ = N(0, 1), so the normal distribution can be viewed as a special case of the *t*.

Putting the limit case, t_∞, aside, the critical values for the *t*-distribution are larger in absolute value than those from the standard normal. This arises from the increased uncertainty associated with the situation where the error variance must be estimated. So now the *t*-distribution is used, and for a given statistic to constitute the same amount of reliable evidence against the null, it has to be bigger in absolute value than in circumstances where the normal is applicable.

As stated above, there are broadly two approaches to testing hypotheses under regression analysis: the test of significance approach and the confidence interval approach. Each of these will now be considered in turn.

3.8.4 The Test of Significance Approach (Box 3.5)

Assume the regression equation is given by $y_t = \alpha + \beta x_t + u_t$, $t = 1, 2, \ldots, T$. The steps involved in doing a test of significance are shown in Box 3.5.

BOX 3.5 Conducting a test of significance

(1) Estimate $\hat{\alpha}$, $\hat{\beta}$ and $SE(\hat{\alpha})$, $SE(\hat{\beta})$ in the usual way.

(2) Calculate the test statistic. This is given by the formula

$$test\ statistic = \frac{\hat{\beta} - \beta^*}{SE(\hat{\beta})} \quad \textbf{(3.30)}$$

where β^* is the value of β under the null hypothesis. The null hypothesis is $H_0 : \beta = \beta^*$ and the alternative hypothesis is $H_1 : \beta \neq \beta^*$ (for a two-sided test).

(3) A tabulated distribution with which to compare the estimated test statistics is required. Test statistics derived in this way can be shown to follow a *t*-distribution with $T - 2$ degrees of freedom.

(4) Choose a 'significance level', often denoted α (*not* the same as the regression intercept coefficient). It is conventional to use a significance level of 5%.

(5) Given a significance level, a *rejection region* and *non-rejection region* can be determined. If a 5% significance level is employed, this means that 5% of the total distribution (5% of the area under the curve) will be in the rejection region. That rejection region can either be split in half (for a two-sided test) or it can all fall on one side of the *y*-axis, as is the case for a one-sided test.

For a two-sided test, the 5% rejection region is split equally between the two tails, as shown in Figure 3.14.

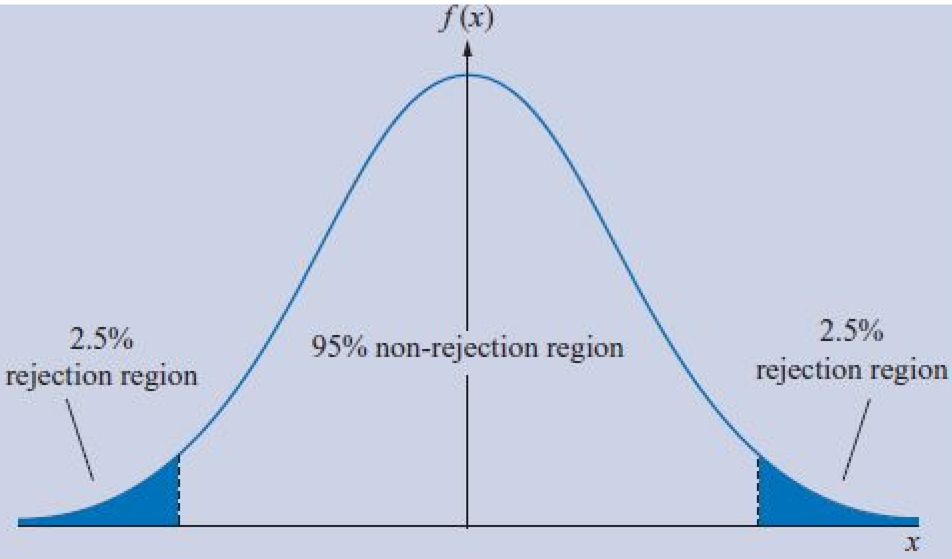


Figure 3.14 Rejection regions for a two-sided 5% hypothesis test

For a one-sided test, the 5% rejection region is located solely in one tail of the distribution, as shown in Figures 3.15 and 3.16, for a test where the alternative is of the 'less than' form, and where the alternative is of the 'greater than' form, respectively.

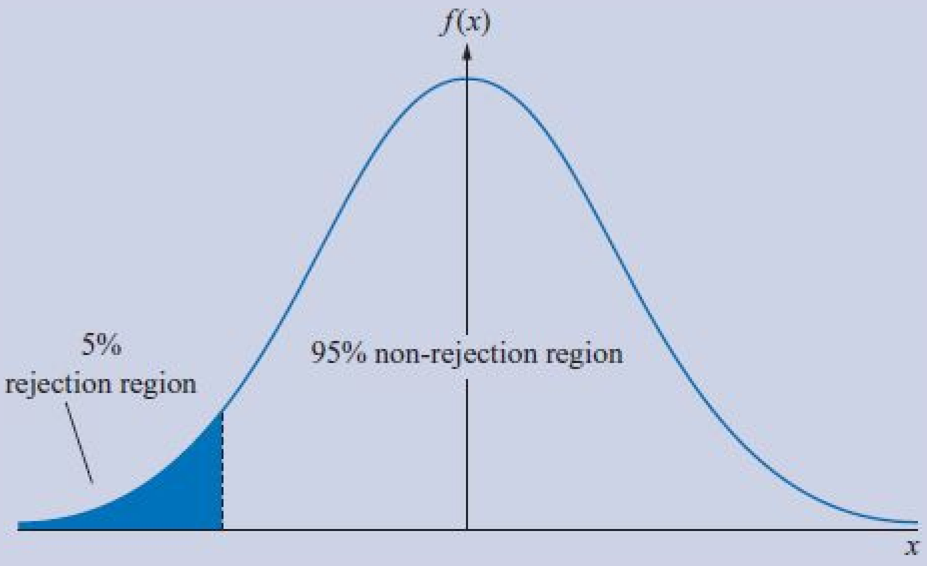


Figure 3.15 Rejection region for a one-sided hypothesis test of the form H_0: $\beta = \beta^*$, H_1: $\beta < \beta^*$

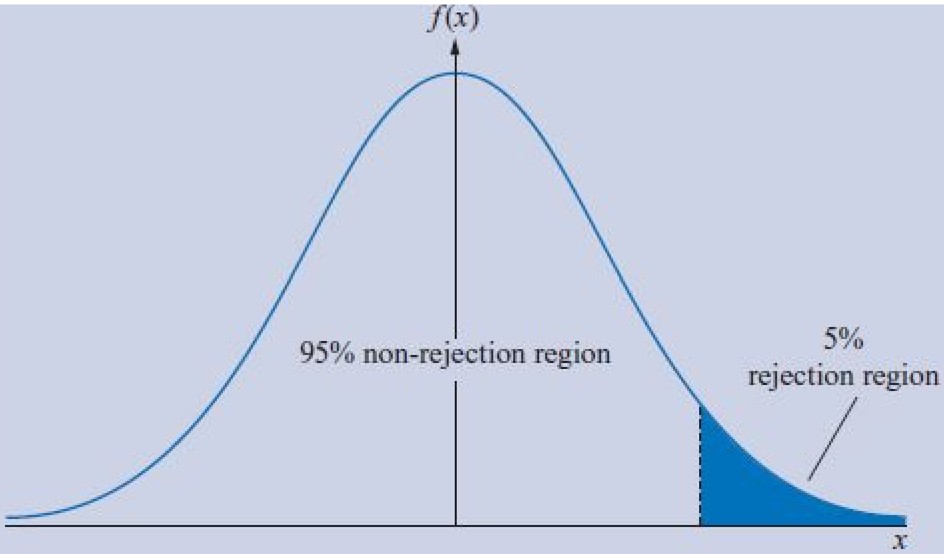


Figure 3.16 Rejection region for a one-sided hypothesis test of the form H_0: $\beta = \beta^*$, H_1: $\beta > \beta^*$

(6) Use the t-tables to obtain a critical value or values with which to compare the test statistic. The critical value will be that value of x that puts 5% into the rejection region.

(7) Finally perform the test. If the test statistic lies in the rejection region then reject the null hypothesis (H_0), else do not reject H_0.

Steps (2)–(7) require further comment. In step (2), the estimated value of β is compared with the value that is subject to test under the null hypothesis, but this difference is 'normalised' or scaled by the standard error of the coefficient estimate. The standard error is a measure of how confident one is in the coefficient estimate obtained in the first stage. If a standard error is small, the value of the test statistic will be large relative to the case where the standard error is large. For a small standard error, it would not require the estimated and hypothesised values to be far away from one another for the null hypothesis to be rejected. Dividing by the standard error also ensures that, under the five CLRM assumptions, the test statistic follows a tabulated distribution.

In this context, the number of degrees of freedom can be interpreted as the number of pieces of additional information beyond the minimum requirement. If two parameters are estimated (α and β – the intercept and the slope of the line, respectively), a minimum of two observations is required to fit this line to the data. As the number of degrees of freedom increases, the critical values in the tables decrease in absolute terms, since less caution is required and one can be more confident that the results are appropriate.

The significance level is also sometimes called the *size of the test* (note that this is completely different from the size of the sample) and it determines the region where the null hypothesis under test will be rejected or not rejected. Remember that the distributions in Figures 3.14–3.16 are for a random variable. Purely by chance, a random variable will take on extreme values (either large and positive values or large and negative values) occasionally. More specifically, a significance level of 5% means that a result as extreme as this or more extreme would be expected only 5% of the time as a consequence of chance alone. To give one illustration, if the 5% critical value for a one-sided test is 1.68, this implies that the test statistic would be expected to be greater than this only 5% of the time by chance alone. There is nothing magical about the test – all that is done is to specify an arbitrary cutoff value for the test statistic that determines whether the null hypothesis would be rejected or not. It is conventional to use a 5% size of test, but 10% and 1% are also commonly used.

However, one potential problem with the use of a fixed (e.g., 5%) size of test is that if the sample size is sufficiently large, any null hypothesis can be rejected. This is particularly worrisome in finance, where tens of thousands of observations or more are often available. What happens is that the standard errors reduce as the sample size increases, thus leading to an increase in the value of all *t*-test statistics. This problem is frequently overlooked in empirical work, but some econometricians have suggested that a lower size of test (e.g., 1%) should be used for large samples (see, for example, Leamer, 1978, for a discussion of these issues).

Note also the use of terminology in connection with hypothesis tests: it is said that the null hypothesis is either *rejected* or *not rejected*. It is incorrect to state that if the null hypothesis is not rejected, it is ‘accepted’ (although this error is frequently made in practice), and it is never said that the alternative hypothesis is accepted or rejected. One reason why it is not sensible to say that the null hypothesis is ‘accepted’ is that it is impossible to know whether the null is actually true or not! In any given situation, many null hypotheses will not be rejected. For example, suppose that $H_0 : \beta = 0.5$ and $H_0 : \beta = 1$ are separately tested against the relevant two-sided alternatives and neither null is rejected. Clearly then it would not make sense to say that ‘$H_0 : \beta = 0.5$ is accepted’ and ‘$H_0 : \beta = 1$ is accepted’, since the true (but unknown) value of β cannot be both 0.5 and 1. So, to summarise, the null hypothesis is either rejected or not rejected on the basis of the available evidence.

3.8.5 The Confidence Interval Approach to Hypothesis Testing (Box 3.6)

To give an example of its usage, one might estimate a parameter, say $\hat{\beta}$, to be 0.93, and a '95% confidence interval' to be (0.77, 1.09). This means that in many repeated samples, 95% of the time, the true value of β will be contained within this interval. Confidence intervals are almost invariably estimated in a two-sided form, although in theory a one-sided interval can be constructed. Constructing a 95% confidence interval is equivalent to using the 5% level in a test of significance.

BOX 3.6 Carrying out a hypothesis test using confidence intervals

(1) Calculate $\hat{\alpha}$, $\hat{\beta}$ and $SE(\hat{\alpha})$, $SE(\hat{\beta})$ as before

(2) Choose a significance level, α (again the convention is 5%). This is equivalent to choosing a $(1 - \alpha)^*100\%$ confidence interval

$$\text{i.e., } 5\% \text{ significance level} = 95\% \text{ confidence interval}$$

(3) Use the t-tables to find the appropriate critical value, which will again have T–2 degrees of freedom

(4) The confidence interval for β is given by

$$(\hat{\beta} - t_{crit} \cdot SE(\hat{\beta}), \hat{\beta} + t_{crit} \cdot SE(\hat{\beta}))$$

Note that a centre dot (·) is sometimes used instead of a cross (×) to denote when two quantities are multiplied together

(5) Perform the test: if the hypothesised value of β (i.e., β^*) lies outside the confidence interval, then reject the null hypothesis that $\beta = \beta^*$, otherwise do not reject the null.

3.8.6 The Test of Significance and Confidence Interval Approaches Always Give the Same Conclusion

Under the test of significance approach, the null hypothesis that $\beta = \beta^*$ will not be rejected if the test statistic lies within the non-rejection region, i.e., if the following condition holds

$$-t_{crit} \leq \frac{\hat{\beta} - \beta^*}{SE(\hat{\beta})} \leq +t_{crit}$$

Rearranging, the null hypothesis would not be rejected if

$$-t_{crit} \cdot SE(\hat{\beta}) \leq \hat{\beta} - \beta^* \leq +t_{crit} \cdot SE(\hat{\beta})$$

i.e., one would not reject if

$$\hat{\beta} - t_{crit} \cdot SE(\hat{\beta}) \leq \beta^* \leq \hat{\beta} + t_{crit} \cdot SE(\hat{\beta})$$

But this is just the rule for non-rejection under the confidence interval approach. So it will always be the case that, for a given significance level, the test of significance and confidence interval approaches will provide the same conclusion by construction. One testing approach is simply an algebraic rearrangement of the other.

EXAMPLE 3.4

Given the regression results above

$$\underset{(14.38)\ (0.2561)}{\hat{y}_t = 20.3 + 0.5091x_t}, \qquad T = 22 \qquad \textbf{(3.31)}$$

Using both the test of significance and confidence interval approaches, test the hypothesis that $\beta = 1$ against a two-sided alternative. This hypothesis might be of interest, for a unit coefficient on the explanatory variable implies a 1:1 relationship between movements in x and movements in y.

The null and alternative hypotheses are, respectively,

$$H_0: \beta = 1$$

$$H_1: \beta \neq 1$$

The results of the test according to each approach are shown in box 3.7.

BOX 3.7 The test of significance and confidence interval approaches compared

Test of significance approach	*Confidence interval approach*

$test\ stat = \dfrac{\hat{\beta} - \beta^*}{SE(\hat{\beta})} = \dfrac{0.5091 - 1}{0.2561} = -1.917$	Find $t_{crit} = t_{20;5\%} = \pm 2.086$
Find $t_{crit} = t_{20;5\%} = \pm 2.086$	$\hat{\beta} \pm t_{crit} \cdot SE(\hat{\beta}) = 0.5091 \pm 2.086 \cdot 0.2561 = (-0.0251, 1.0433)$
Do not reject H_0 since test statistic lies lies within non-rejection region	Do not reject H_0 since 1 within the confidence interval

A couple of comments are in order. First, the critical value from the *t*-distribution that is required is for twenty degrees of freedom and at the 5% level. This means that 5% of the total distribution will be in the rejection region, and since this is a two-sided test, 2.5% of the distribution is required to be contained in each tail. From the symmetry of the *t*-distribution around zero, the critical values in the upper and lower tail will be equal in magnitude, but opposite in sign, as shown in Figure 3.17.

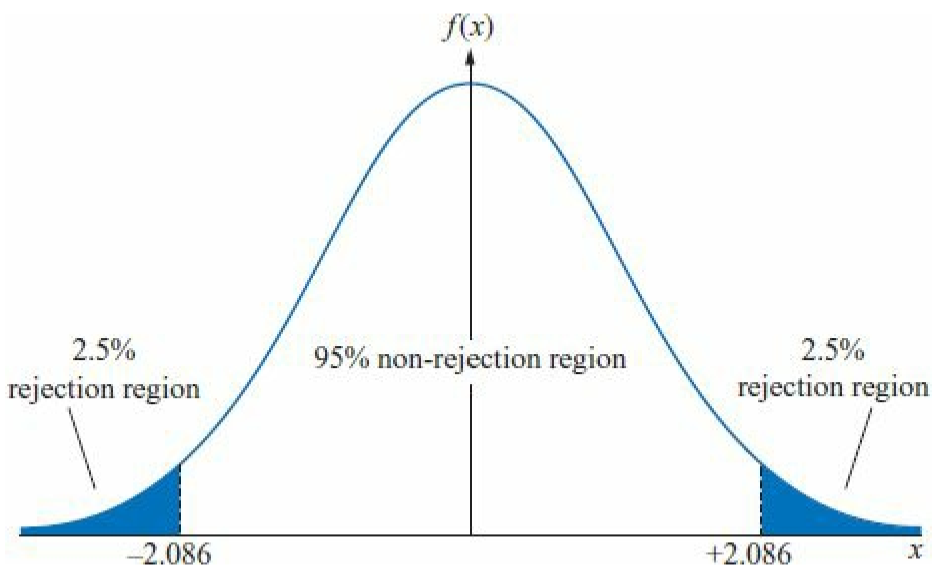


Figure 3.17 Critical values and rejection regions for a $t_{20;5\%}$

What if instead the researcher wanted to test $H_0 : \beta = 0$ or $H_0 : \beta = 2$? In order to test these hypotheses using the test of significance approach, the test statistic would have to be reconstructed in each case, although the critical value would be the same. On the other hand, no additional work would be required if the confidence interval approach had been adopted, since it effectively permits the testing of an infinite number of hypotheses. So for example, suppose that the researcher wanted to test

$$H_0 : \beta = 0$$

versus

$$H_1 : \beta \neq 0$$

and

$$H_0 : \beta = 2$$

versus

$$H_1 : \beta \neq 2$$

In the first case, the null hypothesis (that $\beta = 0$) would not be rejected since 0 lies within the 95% confidence interval. By the same argument, the second null hypothesis (that $\beta = 2$) would be rejected since 2 lies outside the estimated confidence interval.

On the other hand, note that this book has so far considered only the results under a 5% size of test. In marginal cases (e.g., $H_0 : \beta = 1$, where the test statistic and critical value are close together), a completely different answer might arise if a different size of test was used. This is where the test of significance approach is preferable to the construction of a confidence interval.

For example, suppose that now a 10% size of test is used for the null hypothesis given in Example 3.4. Using the test of significance approach,

$$\begin{aligned} \textit{test statistic} &= \frac{\hat{\beta} - \beta^*}{SE(\hat{\beta})} \\ &= \frac{0.5091 - 1}{0.2561} = -1.917 \end{aligned}$$

as above. The only thing that changes is the critical t-value. At the 10% level (so that 5% of the total distribution is placed in each of the tails for this two-sided test), the required critical value is $t_{20;10\%} = \pm 1.725$. So now, as the test statistic lies in the rejection region, H_0 would be rejected. In order to use a 10% test under the confidence interval approach, the interval itself would have to have been re-estimated since the critical value is embedded in the calculation of the confidence interval.

So the test of significance and confidence interval approaches both have their relative merits. The testing of a number of different hypotheses is easier under the confidence interval approach, while a consideration of the effect of the size of the test on the conclusion is easier to address under the test of significance approach.

Caution should therefore be used when placing emphasis on or making decisions in the context of marginal cases (i.e., in cases where the null is only just rejected or not rejected). In this situation, the appropriate conclusion to draw is that the results are marginal and that no strong inference can be made one way or the other. A thorough empirical analysis should involve conducting a sensitivity analysis on the results to determine whether using a different size of test alters the conclusions. It is worth stating again that it is conventional to consider sizes of test of 10%, 5% and 1%. If the conclusion (i.e., 'reject' or 'do not reject') is robust to changes in the size of the test, then one can be more confident that the conclusions are appropriate. If the outcome of the test is qualitatively altered when the size of the test is modified, the conclusion must be that there is no conclusion one way or the other!

It is also worth noting that if a given null hypothesis is rejected using a 1% significance level, it will also automatically be rejected at the 5% level, so that there is no need to actually state the latter. Dougherty (1992, p. 100), gives the analogy of a high jumper. If the high jumper can clear 2 metres, it is obvious that the jumper could also clear 1.5 metres. The 1% significance level is a higher hurdle than the 5% significance level. Similarly, if the null is not rejected at the 5% level of significance, it will automatically not be rejected at any stronger level of significance (e.g., 1%). In this case, if the jumper cannot clear 1.5 metres, there is no way she or he will be able to clear 2 metres.

3.8.7 Some More Terminology

If the null hypothesis is rejected at the 5% level, it would be said that the result of the test is 'statistically significant'. If the null hypothesis is not rejected, it would be said that the result of the test is 'not significant', or that it is 'insignificant'. Finally, if the null hypothesis is rejected at the 1% level, the result is termed 'highly statistically significant'.

Note that a statistically significant result may be of no practical significance. For example, if the estimated beta for a stock under a CAPM regression is 1.05, and a null hypothesis that $\beta = 1$ is rejected, the result will be statistically significant. But it may be the case that a slightly higher beta will make no difference to an investor's choice as to whether to buy the stock or not. In that case, one would say that the result of the test was statistically significant but financially or practically insignificant.

3.8.8 Classifying the Errors That Can be Made Using Hypothesis Tests

H_0 is usually rejected if the test statistic is statistically significant at a chosen significance level. There are two possible errors that could be made:

(1) Rejecting H_0 when it was really true; this is called a *type I error*

(2) Not rejecting H_0 when it was in fact false; this is called a *type II error*

The possible scenarios can be summarised in Table 3.3. The probability of a type I error is just α, the significance level or size of test chosen. To see this, recall what is meant by 'significance' at the 5% level: it is only 5% likely that a result as or more extreme as this could have occurred purely by chance. Or, to put this another way, it is only 5% likely that this null would be rejected when it was in fact true.

Table 3.3 Classifying hypothesis testing errors and correct conclusions

		Reality	
		H_0 is true	**H_0 is false**
	Significant	Type I error $= \alpha$	✓
Result of test	(reject H_0)		
	Insignificant	✓	Type II error $= \beta$
	(do not reject H_0)		

Note that there is no chance for a free lunch (i.e., a cost-less gain) here! What happens if the size of the test is reduced (e.g., from a 5% test to a 1% test)? The chances of making a type I error would be reduced …but so would the probability that the null hypothesis would be rejected at all, so increasing the probability of a type II error. The two competing effects of reducing the size of the test are shown in Box 3.8.

BOX 3.8 Type I and type II errors

Reduce size of test (e.g., 5% to 1%)	→	Stricter criterion for rejection	→	Reject null hypothesis less often	↗	Less likely to falsely reject	→	Lower chance of type I error
					↘	More likely to incorrectly not reject	→	Higher chance of type II error

So there always exists, therefore, a direct trade-off between type I and type II errors when choosing a significance level. The only way to reduce the chances of both is to increase the sample size or to select a sample with more variation, thus increasing the amount of information upon which the results of the hypothesis test are based. In practice, up to a certain level, type I errors are usually considered more serious and hence a small size of test is usually chosen (5% or 1% are the most common).

The probability of a type I error is the probability of incorrectly rejecting a correct null hypothesis, which is also the size of the test. Another important piece of terminology in this area is the *power of a test*. The power of a test is defined as the probability of (appropriately) rejecting an incorrect null hypothesis. The power of the test is also equal to one minus the probability of a type II error.

In addition to the significance level chosen and the sample size, the power of a statistical test also depends on how 'wrong' the value proposed under the null hypothesis is compared with the true value. For example, suppose that the true value of some parameter, β is 3. The power of a test is higher (i.e., we are more likely to reject the null hypothesis) if the value under the null β^* is 1 rather than 2. Finally, it is sometimes possible to test a particular null hypothesis using several different approaches – for example, in Chapter 8 we will see that there are several different tests for unit roots based on different forms of the test statistic – and it is likely that different types of test will have different levels of power.

An optimal test would be one with an actual test size that matched the nominal size and which had as high a power as possible. Such a test would imply, for example, that using a 5% significance level would result in the null being rejected exactly 5% of the time by chance alone, and that an incorrect null hypothesis would be rejected close to 100% of the time.

3.9 A Special Type of Hypothesis Test: The *t*-ratio

Recall that the formula under a test of significance approach to hypothesis testing using a *t*-test for the slope parameter was

$$\text{test statistic} = \frac{\hat{\beta} - \beta^*}{SE(\hat{\beta})} \tag{3.32}$$

with the obvious adjustments to test a hypothesis about the intercept. If the test is

$$H_0 : \beta = 0$$
$$H_1 : \beta \neq 0$$

i.e., a test that the population parameter is zero against a two-sided alternative, this is known as a *t*-ratio test. Since $\beta^* = 0$, the expression in equation (3.32) collapses to

$$\text{test statistic} = \frac{\hat{\beta}}{SE(\hat{\beta})} \tag{3.33}$$

Thus the ratio of the coefficient to its standard error, given by this expression, is known as the *t-ratio* or *t-statistic*.

EXAMPLE 3.5

Suppose that we have calculated the estimates for the intercept and the slope (1.10 and –19.88, respectively) and their corresponding standard errors (1.35 and 1.98, respectively). The *t*-ratios associated with each of the intercept and slope coefficients would be given by

	$\hat{\alpha}$	$\hat{\beta}$
Coefficient	1.10	−19.88
SE	1.35	1.98
t-ratio	0.81	−10.04

Note that if a coefficient is negative, its *t*-ratio will also be negative. In order to test (separately) the null hypotheses that $\alpha = 0$ and $\beta = 0$, the test statistics would be compared with the appropriate critical value from a *t*-distribution. In this case, the number of degrees of freedom, given by $T - k$, is equal to 15 – 2 = 13. The 5% critical value for this two-sided test (remember, 2.5% in each tail for a 5% test) is 2.16, while the 1% two-sided critical value (0.5% in each tail) is 3.01. Given these *t*-ratios and critical values, would the following null hypotheses be rejected?

$$H_0 : \alpha = 0? \qquad (No)$$

$$H_0 : \beta = 0? \qquad (Yes)$$

If H_0 is rejected, it would be said that the test statistic is *significant*. If the variable is not 'significant', it means that while the estimated value of the coefficient is not exactly zero (e.g. 1.10 in the example above), the coefficient is indistinguishable statistically from zero. If a zero were placed in the fitted equation instead of the estimated value, this would mean that whatever happened to the value of that explanatory variable, the dependent variable would be unaffected. This would then be taken to mean that the variable is not helping to explain variations in *y*, and that it could therefore be removed from the regression equation. For example, if the *t*-ratio associated with *x* had been –1.04 rather than –10.04 (assuming that the standard error stayed the same), the variable would be classed as insignificant (i.e., not statistically different from zero). The only insignificant term in the above regression is the intercept. There are good statistical reasons for always retaining the constant, even if it is not significant; see Chapter 5.

It is worth noting that, for degrees of freedom greater than around 25, the 5% two-sided critical value is approximately ±2. So, as a rule of thumb (i.e., a rough guide), the null hypothesis would be rejected if the *t*-statistic exceeds 2 in absolute value.

Some authors place the *t*-ratios in parentheses below the corresponding coefficient estimates rather than the standard errors. One thus needs to check which convention is being used in each particular application, and also to state this clearly when presenting estimation results.

There will now follow two finance case studies that involve only the estimation of bivariate linear regression models and the construction and interpretation of *t*-ratios.

3.10 An Example of a Simple *t*-test of a Theory in Finance: Can US Mutual Funds Beat the Market?

Jensen (1968) was the first to systematically test the performance of mutual funds, and in particular examine whether any ‘beat the market’. He used a sample of annual returns on the portfolios of 115 mutual funds from 1945 to 64. Each of the 115 funds was subjected to a separate OLS time-series regression of the form

$$R_{jt} - R_{ft} = \alpha_j + \beta_j(R_{mt} - R_{ft}) + u_{jt} \quad \textbf{(3.34)}$$

where R_{jt} is the return on portfolio j at time t, R_{ft} is the return on a risk-free proxy (a one-year government bond), R_{mt} is the return on a market portfolio proxy, u_{jt} is an error term, and α_j, β_j are parameters to be estimated. The quantity of interest is the significance of α_j, since this parameter defines whether the fund outperforms or underperforms the market index. Thus the null hypothesis is given by: $H_0 : \alpha_j = 0$. A positive and significant α_j for a given fund would suggest that the fund is able to earn significant abnormal returns in excess of the market-required return for a fund of this given riskiness. This coefficient has become known as ‘Jensen’s alpha’. Some summary statistics across the 115 funds for the estimated regression results for equation (3.34) are given in Table 3.4.

Table 3.4 Summary statistics for the estimated regression results for equation (3.34)

Item	Mean value	Median value	Extremal values	
			Minimum	Maximum
$\hat{\alpha}$	–0.011	–0.009	–0.080	0.058
$\hat{\beta}$	0.840	0.848	0.219	1.405
Sample size	17	19	10	20

Source: Jensen (1968). Reprinted with the permission of Blackwell Publishers.

As Table 3.4 shows, the average (defined as either the mean or the median) fund was unable to ‘beat the market’, recording a negative alpha in both cases. There were, however, some funds that did manage to perform significantly better than expected given their level of risk, with the best fund of all yielding an alpha of 0.058. Interestingly, the average fund had a beta estimate of around 0.85, indicating that, in the CAPM context, most funds were less risky than the market index. This result may be attributable to the funds investing predominantly in (mature) blue chip stocks rather than small caps.

The most visual method of presenting the results was obtained by plotting the number of mutual funds in each t-ratio category for the alpha coefficient, first gross and then net of transactions costs, as in Figure 3.18 and Figure 3.19, respectively.

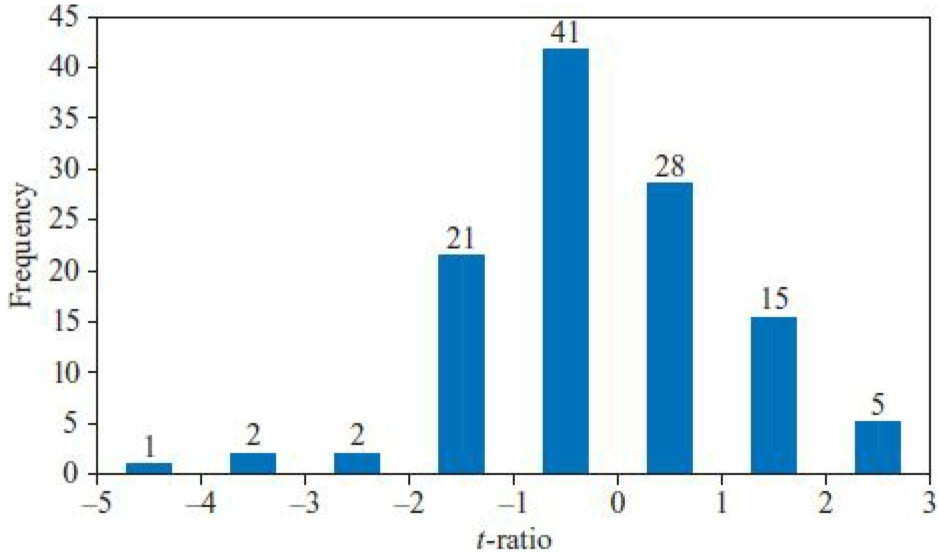


Figure 3.18 Frequency distribution of t-ratios of mutual fund alphas (gross of transactions costs). *Source:* Jensen (1968). Reprinted with the permission of Blackwell Publishers

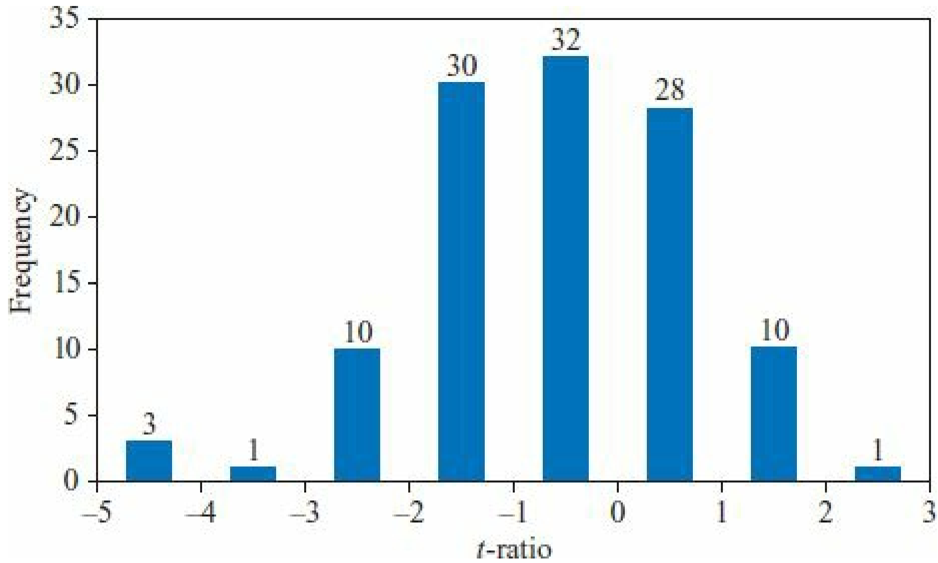


Figure 3.19 Frequency distribution of t-ratios of mutual fund alphas (net of transactions costs). Source: Jensen (1968). Reprinted with the

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The appropriate critical value for a two-sided test of $\alpha_j = 0$ is approximately 2.10 (assuming twenty years of annual data leading to eighteen degrees of freedom). As can be seen, only five funds have estimated *t*-ratios greater than 2 and are therefore implied to have been able to outperform the market before transactions costs are taken into account. Interestingly, five firms have also significantly underperformed the market, with *t*-ratios of −2 or less.

When transactions costs are taken into account (Figure 3.19), only one fund out of 115 is able to significantly outperform the market, while 14 significantly underperform it. Given that a nominal 5% two-sided size of test is being used, one would expect two or three funds to 'significantly beat the market' by chance alone. It would thus be concluded that, during the sample period studied, US fund managers appeared unable to systematically generate positive abnormal returns.

3.11 Can UK Unit Trust Managers Beat the Market?

Jensen's study has proved pivotal in suggesting a method for conducting empirical tests of the performance of fund managers. However, it has been criticised on several grounds. One of the most important of these in the context of this book is that only between ten and twenty annual observations were used for each regression. Such a small number of observations is really insufficient for the asymptotic theory underlying the testing procedure to be validly invoked.

A variant on Jensen's test is now estimated in the context of the UK market, by considering monthly returns on seventy-six equity unit trusts. The data cover the period January 1979–May 2000 (257 observations for each fund). Some summary statistics for the funds are presented in Table 3.5.

Table 3.5 Summary statistics for unit trust returns, January 1979–May 2000

	Mean (%)	Minimum (%)	Maximum (%)	Median (%)
Average monthly	1.0	0.6	1.4	1.0

return, 1979–2000				
Standard deviation of returns over time	5.1	4.3	6.9	5.0

From these summary statistics, the average continuously compounded return is 1% per month, although the most interesting feature is the wide variation in the performances of the funds. The worst-performing fund yields an average return of 0.6% per month over the twenty-year period, while the best would give 1.4% per month. This variability is further demonstrated in Figure 3.20, which plots over time the value of £100 invested in each of the funds in January 1979.

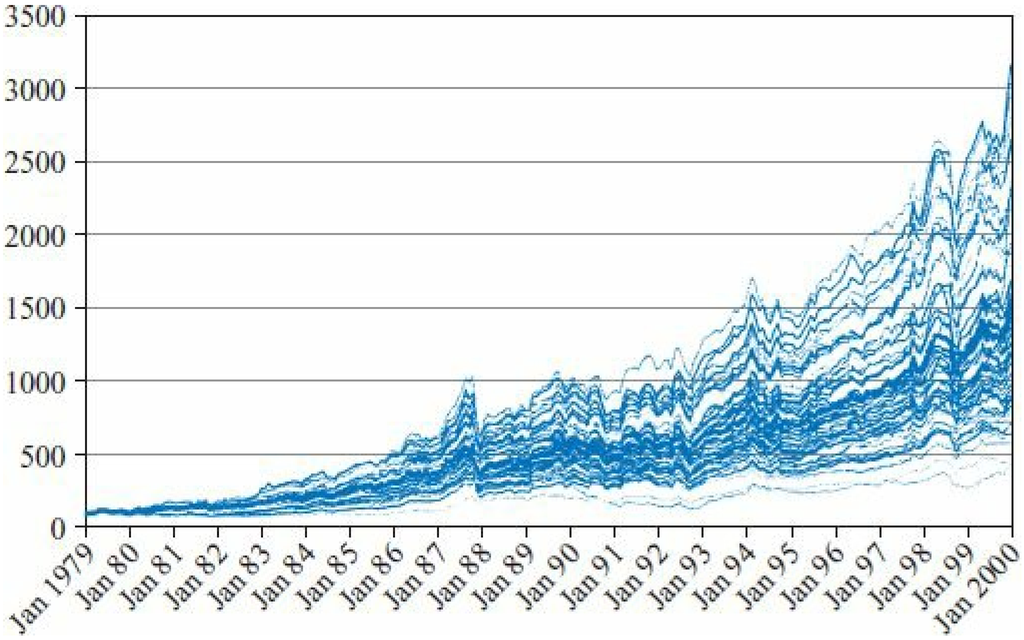


Figure 3.20 Performance of UK unit trusts, 1979–2000

A regression of the form (3.34) is applied to the UK data, and the summary results presented in Table 3.6. A number of features of the regression results are worthy of further comment. First, most of the funds have estimated betas less than one again, perhaps suggesting that the fund managers have historically been risk averse or investing disproportionately in blue chip companies in mature sectors. Second, gross of transactions costs, nine funds of the sample of seventy-six were able to significantly outperform the market by providing a significant positive alpha, while seven funds yielded significant negative alphas. The average fund (where 'average' is measured using either the mean or the median) is not able to earn any excess return over the required rate given its level of risk.

Table 3.6 CAPM regression results for unit trust returns, January 1979–May 2000

Estimates of	Mean	Minimum	Maximum	Median
$\alpha(\%)$	−0.02	−0.54	0.33	−0.03
β	0.91	0.56	1.09	0.91
t-ratio on α	−0.07	−2.44	3.11	−0.25

3.12 The Overreaction Hypothesis and the UK Stock Market

3.12.1 Motivation

Two studies by DeBondt and Thaler (1985, 1987) showed that stocks experiencing a poor performance over a three to five-year period subsequently tend to outperform stocks that had previously performed relatively well. This implies that, on average, stocks which are 'losers' in terms of their returns subsequently become 'winners', and vice versa. This chapter now examines a paper by Clare and Thomas (1995) that conducts a similar study using monthly UK stock returns from January 1955 to 1990 (thirty-six years) on all firms traded on the London Stock Exchange (LSE).

This phenomenon seems at first blush to be inconsistent with the efficient markets hypothesis, and Clare and Thomas (1995) propose two explanations (see Box 3.9). Zarowin (1990) also finds that 80% of the extra return available from holding the losers accrues to investors in January, so that almost all of the 'overreaction effect' seems to occur at the start of the calendar year.

BOX 3.9 Reasons for stock market overreactions

(1) *That the 'overreaction effect' is just another manifestation of the 'size effect'*. The size effect is the tendency of small firms to generate, on average, superior returns to large firms. The argument would follow that the losers were small firms and that these small firms would subsequently outperform the large firms. DeBondt and Thaler did not believe this to be a sufficient explanation, but Zarowin (1990) found that allowing for firm size did reduce the subsequent return on the losers.

(2) *That the reversals of fortune reflect changes in equilibrium required returns*. The losers are argued to be likely to have considerably higher CAPM betas, reflecting investors' perceptions that they are more risky. Of course, betas can change over time, and a substantial fall in the firms' share prices (for the losers) would lead to a rise in their leverage ratios, leading in all likelihood to an increase in their perceived riskiness. Therefore, the required rate of return on the losers will be larger, and their *ex post* performance better. Ball and Kothari (1989) find the CAPM betas of losers to be considerably higher than those of winners.

3.12.2 Methodology

Clare and Thomas (1995) take a random sample of 1000 firms and, for each, they calculate the monthly excess return of the stock for the market over a twelve-, twentyfour- or thirty-six-month period for each stock *i*

$$U_{it} = R_{it} - R_{mt} \quad t = 1, \ldots, n; \quad i = 1, \ldots, 1000; \quad n = 12, 24 \text{ or } 36 \tag{3.35}$$

Then the average monthly return over each stock *i* for the first twelve-, twenty-four-, or thirty-six-month period is calculated

$$\bar{R}_i = \frac{1}{n} \sum_{t=1}^{n} U_{it} \tag{3.36}$$

The stocks are then ranked from highest average return to lowest and from these five portfolios are formed and returns are calculated assuming an equal weighting of stocks in each portfolio (Box 3.10).

BOX 3.10 Ranking stocks and forming portfolios

Portfolio	*Ranking*
Portfolio 1	Best performing 20% of firms
Portfolio 2	Next 20%

Portfolio 3	Next 20%
Portfolio 4	Next 20%
Portfolio 5	Worst performing 20% of firms

The same sample length n is used to monitor the performance of each portfolio. Thus, for example, if the portfolio formation period is one, two or three years, the subsequent portfolio tracking period will also be one, two or three years, respectively. Then another portfolio formation period follows and so on until the sample period has been exhausted. How many samples of length n will there be? n = 1, 2 or 3 years. First, suppose n = 1 year. The procedure adopted would be as shown in Box 3.11.

BOX 3.11 Portfolio monitoring

Estimate $\bar{R}_i$ for year 1
Monitor portfolios for year 2
Estimate $\bar{R}_i$ for year 3
⋮
Monitor portfolios for year 36

So if n = 1, there are eighteen independent (non-overlapping) observation periods and eighteen independent tracking periods. By similar arguments, n = 2 gives nine independent periods and n = 3 gives six independent periods. The mean return for each month over the 18, 9 or 6 periods for the winner and loser portfolios (the top 20% and bottom 20% of firms in the portfolio formation period) are denoted by $\bar{R}^W_{pt}$ and $\bar{R}^L_{pt}$, respectively. Define the difference between these as $\bar{R}_{Dt} = \bar{R}^L_{pt} - \bar{R}^W_{pt}$.

The first regression to be performed is of the excess return of the losers over the winners on a constant only

$$\bar{R}_{Dt} = \alpha_1 + \eta_t \tag{3.37}$$

where η_t is an error term. The test is of whether α_1 is significant and positive. However, a significant and positive α_1 is not a sufficient condition for the over-reaction effect to be confirmed, because it could be owing to higher returns being required on loser stocks owing to loser stocks being more risky. The solution, Clare and Thomas, (1995) argue, is to allow for risk differences by regressing against the market risk premium

$$\bar{R}_{Dt} = \alpha_2 + \beta(R_{mt} - R_{ft}) + \eta_t \quad \textbf{(3.38)}$$

where R_{mt} is the return on the FTA All-Share, and R_{ft} is the return on a UK government three-month Treasury Bill. The results for each of these two regressions are presented in Table 3.7.

Table 3.7 Is there an overreaction effect in the UK stock market?

Panel A: all months			
	n = 12	n = 24	n = 36
Return on loser	0.0033	0.0011	0.0129
Return on winner	0.0036	–0.0003	0.0115
Implied annualised return difference	–0.37%	1.68%	1.56%
Coefficient for (3.37): $\hat{\alpha}_1$	–0.00031	0.0014**	0.0013*
	(–0.29)	(2.01)	(1.55)
Coefficients for (3.38): $\hat{\alpha}_2$	–0.00034	0.00147**	0.0013
	(–0.30)	(2.01)	(1.41)
Coefficients for (3.38): $\hat{\beta}$	–0.022	0.010	–0.0025
	(–0.25)	(0.21)	(–0.06)
Panel B: all months except January			
Coefficient for (3.37): $\hat{\alpha}_1$	–0.0007	0.0012*	0.0009
	(–0.72)	(1.63)	(1.05)

Notes: t-ratios in parentheses; * and ** denote significance at the 10% and 5% levels, respectively.

Source: Clare and Thomas (1995). Reprinted with the permission of Blackwell Publishers.

As can be seen by comparing the returns on the winners and losers in the first two rows of Table 3.7, twelve months is not a sufficiently long time for losers to become winners. By the two-year tracking horizon, however, the losers have become winners, and similarly for the three-year samples. This translates into an average 1.68% higher return on the losers than the winners at the two-year horizon, and 1.56% higher return at the three-year horizon. Recall that the estimated value of the coefficient in a regression of a variable on a constant only is equal to the average value of that variable. It can also be seen that the estimated coefficients on the constant terms for each horizon are exactly equal to the differences between the returns of the losers and the winners. This coefficient is statistically significant at the two-year horizon, and marginally significant at the three-year horizon.

In the second test regression, $\hat{\beta}$ represents the difference between the market betas of the winner and loser portfolios. None of the beta coefficient estimates are even close to being significant, and the inclusion of the risk term makes virtually no difference to the coefficient values or significances of the intercept terms.

Removal of the January returns from the samples reduces the subsequent degree of overperformance of the loser portfolios, and the significances of the $\hat{\alpha}_1$ terms is somewhat reduced. It is concluded, therefore, that only a part of the overreaction phenomenon occurs in January. Clare and Thomas (1995), then proceed to examine whether the overreaction effect is related to firm size, although the results are not presented here.

3.12.3 Conclusions

The main conclusions from Clare and Thomas' study are:

(1) There appears to be evidence of overreactions in UK stock returns, as found in previous US studies

(2) These overreactions are unrelated to the CAPM beta

(3) Losers that subsequently become winners tend to be small, so that most of the overreaction in the UK can be attributed to the size effect

3.13 The Exact Significance Level

The exact significance level is also commonly known as the *p*-value. It gives the *marginal significance level* where one would be indifferent between rejecting and not rejecting the null hypothesis. If the test statistic is 'large' in absolute value, the *p*-value will be small, and vice versa. For example, consider a test statistic that is distributed as a t_{62} and takes a value of 1.47. Would the null hypothesis be rejected? It would depend on the size of the test. Now, suppose that the *p*-value for this test is calculated to be 0.12

- Is the null rejected at the 5% level? *No*
- Is the null rejected at the 10% level? *No*
- Is the null rejected at the 20% level? *Yes*

In fact, the null would have been rejected at the 12% level or higher. To see this, consider conducting a series of tests with size 0.1%, 0.2%, 0.3%, 0.4%, …1%, …, 5%, …10%, …Eventually, the critical value and test statistic will meet and this will be the *p*-value. *p*-values are almost always provided automatically by software packages. Note how useful they are! They provide all of the information required to conduct a hypothesis test without requiring of the researcher the need to calculate a test statistic or to find a critical value from a table – both of these steps have already been taken by the package in producing the *p*-value. The *p*-value is also useful since it avoids the requirement of specifying an arbitrary significance level (α). Sensitivity analysis of the effect of the significance level on the conclusion occurs automatically.

Informally, the *p*-value is also often referred to as the probability of being wrong when the null hypothesis is rejected. Thus, for example, if a *p*-value of 0.05 or less leads the researcher to reject the null (equivalent to a 5% significance level), this is equivalent to saying that if the probability of incorrectly rejecting the null is more than 5%, do not reject it. The *p*-value has also been termed the 'plausibility' of the null hypothesis; so, the smaller is the *p*-value, the less plausible is the null hypothesis.

KEY CONCEPTS

The key terms to be able to define and explain from this chapter are

- regression model
- population
- linear model
- unbiasedness
- standard error

- null hypothesis
- *t*-distribution
- test statistic
- type I error
- size of a test
- *p*-value
- disturbance term
- sample
- consistency
- efficiency
- statistical inference
- alternative hypothesis
- confidence interval
- rejection region
- type II error
- power of a test
- asymptotic

Appendix 3.1 Mathematical Derivations of CLRM Results

3A.1 Derivation of the OLS Coefficient Estimator in the Bivariate Case

$$L = \sum_{t=1}^{T}(y_t - \hat{y}_t)^2 = \sum_{t=1}^{T}(y_t - \hat{\alpha} - \hat{\beta}x_t)^2 \quad \textbf{(3A.1)}$$

It is necessary to minimise L w.r.t. $\hat{\alpha}$ and $\hat{\beta}$, to find the values of α and β that give the line that is closest to the data. So L is differentiated w.r.t. $\hat{\alpha}$ and $\hat{\beta}$, and the first derivatives are set to zero. The first derivatives are given by

$$\frac{\partial L}{\partial \hat{\alpha}} = -2\sum_t (y_t - \hat{\alpha} - \hat{\beta}x_t) = 0 \quad \textbf{(3A.2)}$$

$$\frac{\partial L}{\partial \hat{\beta}} = -2\sum_t x_t(y_t - \hat{\alpha} - \hat{\beta}x_t) = 0 \quad \textbf{(3A.3)}$$

The next step is to rearrange equations (3A.2) and (3A.3) in order to obtain expressions for $\hat{\alpha}$ and $\hat{\beta}$. From equation (3A.2)

$$\sum_t (y_t - \hat{\alpha} - \hat{\beta} x_t) = 0 \quad \textbf{(3A.4)}$$

Expanding the parentheses and recalling that the sum runs from 1 to T so that there will be T terms in $\hat{\alpha}$

$$\sum y_t - T\hat{\alpha} - \hat{\beta} \sum x_t = 0 \quad \textbf{(3A.5)}$$

But $\sum y_t = T\bar{y}$ and $\sum x_t = T\bar{x}$, so it is possible to write equation (3A.5) as

$$T\bar{y} - T\hat{\alpha} - T\hat{\beta}\bar{x} = 0 \quad \textbf{(3A.6)}$$

or

$$\bar{y} - \hat{\alpha} - \hat{\beta}\bar{x} = 0 \quad \textbf{(3A.7)}$$

From equation (3A.3)

$$\sum_t x_t (y_t - \hat{\alpha} - \hat{\beta} x_t) = 0 \quad \textbf{(3A.8)}$$

equation From (3A.7)

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} \quad \textbf{(3A.9)}$$

Substituting into equation (3A.8) for $\hat{\alpha}$ from equation (3A.9)

$$\sum_t x_t (y_t - \bar{y} + \hat{\beta}\bar{x} - \hat{\beta} x_t) = 0 \quad \textbf{(3A.10)}$$

$$\sum_t x_t y_t - \bar{y} \sum x_t + \hat{\beta}\bar{x} \sum x_t - \hat{\beta} \sum x_t^2 = 0 \quad \textbf{(3A.11)}$$

$$\sum_t x_t y_t - T\bar{x}\bar{y} + \hat{\beta} T\bar{x}^2 - \hat{\beta} \sum x_t^2 = 0 \quad \textbf{(3A.12)}$$

Rearranging for $\hat{\beta}$,

$$\hat{\beta}\left(T\bar{x}^2 - \sum x_t^2\right) = T\bar{x}\bar{y} - \sum x_t y_t \quad \textbf{(3A.13)}$$

Dividing both sides of equation (3A.13) by $(T\bar{x}^2 - \sum x_t^2)$ gives

$$\hat{\beta} = \frac{\sum x_t y_t - T\bar{x}\bar{y}}{\sum x_t^2 - T\bar{x}^2} \quad \text{and} \quad \hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} \quad \textbf{(3A.14)}$$

3A.2 Derivation of the OLS Standard Error Estimators for the Intercept and Slope in the Bivariate Case

Recall that the variance of the random variable $\hat{\alpha}$ can be written as

$$\text{var}(\hat{\alpha}) = E(\hat{\alpha} - E(\hat{\alpha}))^2 \tag{3A.15}$$

and since the OLS estimator is unbiased

$$\text{var}(\hat{\alpha}) = E(\hat{\alpha} - \alpha)^2 \tag{3A.16}$$

By similar arguments, the variance of the slope estimator can be written as

$$\text{var}(\hat{\beta}) = E(\hat{\beta} - \beta)^2 \tag{3A.17}$$

Working first with equation (3A.17), replacing $\hat{\beta}$ with the formula for it given by the OLS estimator

$$\text{var}(\hat{\beta}) = E\left(\frac{\sum (x_t - \bar{x})(y_t - \bar{y})}{\sum (x_t - \bar{x})^2} - \beta\right)^2 \tag{3A.18}$$

Replacing y_t with $\alpha + \beta x_t + u_t$, and replacing $\bar{y}$ with $\alpha + \beta\bar{x}$ in equation (3A.18)

$$\text{var}(\hat{\beta}) = E\left(\frac{\sum (x_t - \bar{x})(\alpha + \beta x_t + u_t - \alpha - \beta\bar{x})}{\sum (x_t - \bar{x})^2} - \beta\right)^2 \tag{3A.19}$$

Cancelling α and multiplying the last β term in equation (3A.19) by $\frac{\sum (x_t - \bar{x})^2}{\sum (x_t - \bar{x})^2}$

$$\text{var}(\hat{\beta}) = E\left(\frac{\sum (x_t - \bar{x})(\beta x_t + u_t - \beta\bar{x}) - \beta\sum (x_t - \bar{x})^2}{\sum (x_t - \bar{x})^2}\right)^2 \tag{3A.20}$$

Rearranging

$$\text{var}(\hat{\beta}) = E\left(\frac{\sum (x_t - \bar{x})\beta(x_t - \bar{x}) + \sum u_t(x_t - \bar{x}) - \beta\sum (x_t - \bar{x})^2}{\sum (x_t - \bar{x})^2}\right)^2 \tag{3A.21}$$

$$\text{var}(\hat{\beta}) = E\left(\frac{\beta\sum(x_t-\bar{x})^2 + \sum u_t(x_t-\bar{x}) - \beta\sum(x_t-\bar{x})^2}{\sum(x_t-\bar{x})^2}\right)^2 \tag{3A.22}$$

Now the β terms equation in (3A.22) will cancel to give

$$\text{var}(\hat{\beta}) = E\left(\frac{\sum u_t(x_t-\bar{x})}{\sum(x_t-\bar{x})^2}\right)^2 \tag{3A.23}$$

Now let x_t^* denote the mean-adjusted observation for x_t, i.e. $(x_t - \bar{x})$. Equation (3A.23) can be written

$$\text{var}(\hat{\beta}) = E\left(\frac{\sum u_t x_t^*}{\sum x_t^{*2}}\right)^2 \tag{3A.24}$$

The denominator of equation (3A.24) can be taken through the expectations operator under the assumption that x is fixed or non-stochastic

$$\text{var}(\hat{\beta}) = \frac{1}{\left(\sum x_t^{*2}\right)^2} E\left(\sum u_t x_t^*\right)^2 \tag{3A.25}$$

Writing the terms out in the last summation of equation (3A.25)

$$\text{var}(\hat{\beta}) = \frac{1}{\left(\sum x_t^{*2}\right)^2} E\left(u_1 x_1^* + u_2 x_2^* + \cdots + u_T x_T^*\right)^2 \tag{3A.26}$$

Now expanding the brackets of the squared term in the expectations operator of equation (3A.26)

$$\text{var}(\hat{\beta}) = \frac{1}{\left(\sum x_t^{*2}\right)^2} E\left(u_1^2 x_1^{*2} + u_2^2 x_2^{*2} + \cdots + u_T^2 x_T^{*2} + \textit{cross-products}\right) \tag{3A.27}$$

where '*cross-products*' in equation (3A.27) denotes all of the terms $u_i x_i^* u_j x_j^*$ $(i \neq j)$. These cross-products can be written as $u_i u_j x_i^* x_j^*$ $(i \neq j)$ and their expectation will be zero under the assumption that the error terms are uncorrelated with one another. Thus, the '*cross-products*' term in equation (3A.27) will drop out. Recall also from the chapter text that $E(u_t^2)$ is the error variance, which is estimated using s^2

$$\text{var}(\hat{\beta}) = \frac{1}{\left(\sum x_t^{*2}\right)^2}\left(s^2 x_1^{*2} + s^2 x_2^{*2} + \cdots + s^2 x_T^{*2}\right) \quad \textbf{(3A.28)}$$

which can also be written

$$\text{var}(\hat{\beta}) = \frac{s^2}{\left(\sum x_t^{*2}\right)^2}\left(x_1^{*2} + x_2^{*2} + \cdots + x_T^{*2}\right) = \frac{s^2 \sum x_t^{*2}}{\left(\sum x_t^{*2}\right)^2} \quad \textbf{(3A.29)}$$

A term in $\sum x_t^{*2}$ can be cancelled from the numerator and denominator of equation (3A.29), and recalling that $x_t^* = (x_t - \bar{x})$, this gives the variance of the slope coefficient as

$$\text{var}(\hat{\beta}) = \frac{s^2}{\sum (x_t - \bar{x})^2} \quad \textbf{(3A.30)}$$

so that the standard error can be obtained by taking the square root of (3A.30)

$$SE(\hat{\beta}) = s\sqrt{\frac{1}{\sum (x_t - \bar{x})^2}} \quad \textbf{(3A.31)}$$

Turning now to the derivation of the intercept standard error, this is much more difficult than that of the slope standard error. In fact, both are very much easier using matrix algebra as shown below. Therefore, this derivation will be offered in summary form. It is possible to express $\hat{\alpha}$ as a function of the true α and of the disturbances, u_t

$$\hat{\alpha} = \alpha + \frac{\sum u_t \left[\sum x_t^2 - x_t \sum x_t\right]}{\left[T \sum x_t^2 - \left(\sum x_t\right)^2\right]} \quad \textbf{(3A.32)}$$

Denoting all of the elements in square brackets as g_t, equation (3A.32) can be written

$$\hat{\alpha} - \alpha = \sum u_t g_t \quad \textbf{(3A.33)}$$

From equation (3A.15), the intercept variance would be written

$$\text{var}(\hat{\alpha}) = E\left(\sum u_t g_t\right)^2 = \sum g_t^2 E\left(u_t^2\right) = s^2 \sum g_t^2 \quad \textbf{(3A.34)}$$

Writing equation (3A.34) out in full for g_t^2 and expanding the brackets

$$\text{var}(\hat{\alpha}) = \frac{s^2\left[T\left(\sum x_t^2\right)^2 - 2\sum x_t\left(\sum x_t^2\right)\sum x_t + \left(\sum x_t^2\right)\left(\sum x_t\right)^2\right]}{\left[T\sum x_t^2 - \left(\sum x_t\right)^2\right]^2} \quad \textbf{(3A.35)}$$

This looks rather complex, but fortunately, if we take $\sum x_t^2$ outside the square brackets in the numerator, the remaining numerator cancels with a term in the denominator to leave the required result

$$SE(\hat{\alpha}) = s\sqrt{\frac{\sum x_t^2}{T\sum(x_t - \bar{x})^2}} \quad \textbf{(3A.36)}$$

SELF-STUDY QUESTIONS

1. (a) Why does OLS estimation involve taking vertical deviations of the points to the line rather than horizontal distances?
   (b) Why are the vertical distances squared before being added together?
   (c) Why are the squares of the vertical distances taken rather than the absolute values?
2. Explain, with the use of equations, the difference between the sample regression function and the population regression function.
3. What is an estimator? Is the OLS estimator superior to all other estimators? Why or why not?
4. What five assumptions are usually made about the unobservable error terms in the classical linear regression model (CLRM)? Briefly explain the meaning of each. Why are these assumptions made?
5. Which of the following models can be estimated (following a suitable rearrangement if necessary) using ordinary least squares (OLS), where X, y, Z are variables and α, β, γ are parameters to be estimated? (*Hint*: the models need to be linear in the parameters.)

$$y_t = \alpha + \beta x_t + u_t \tag{3.39}$$

$$y_t = e^{\alpha} x_t^{\beta} e^{u_t} \tag{3.40}$$

$$y_t = \alpha + \beta \gamma x_t + u_t \tag{3.41}$$

$$\ln(y_t) = \alpha + \beta \ln(x_t) + u_t \tag{3.42}$$

$$y_t = \alpha + \beta x_t z_t + u_t \tag{3.43}$$

6. The capital asset pricing model (CAPM) can be written as

$$E(R_i) = R_f + \beta_i [E(R_m) - R_f] \tag{3.44}$$

using the standard notation.

The first step in using the CAPM is to estimate the stock's beta using the market model. The market model can be written as

$$R_{it} = \alpha_i + \beta_i R_{mt} + u_{it} \tag{3.45}$$

where R_{it} is the excess return for security i at time t, R_{mt} is the excess return on a proxy for the market portfolio at time t, and u_t is an iid random disturbance term. The cofficient beta in this case is also the CAPM beta for security i.

Suppose that you had estimated equation (3.45) and found that the estimated value of beta for a stock, $\hat{\beta}$ was 1.147. The standard error associated with this coefficient $SE(\hat{\beta})$ is estimated to be 0.0548.

A city analyst has told you that this security closely follows the market, but that it is no more risky, on average, than the market. This can be tested by the null hypotheses that the value of beta is one. The model is estimated over sixty-two daily observations. Test this hypothesis against a one-sided alternative that the security is more risky than the market, at the 5% level. Write down the null and alternative hypothesis. What do you conclude? Are the analyst's claims empirically verified?

7. The analyst also tells you that shares in Chris Mining plc have no systematic risk, in other words that the returns on its shares are completely unrelated to movements in the market. The value of beta and its standard error are calculated to be 0.214 and 0.186, respectively. The model is estimated over thirty-eight quarterly observations. Write down the null and alternative hypotheses. Test this null hypothesis against a two-sided alternative.

8. Form and interpret a 95% and a 99% confidence interval for beta using the figures given in Question 7.

9. Are hypotheses tested concerning the actual values of the coefficients (i.e., β) or their estimated values (i.e., $\hat{\beta}$) and why?

[1] Strictly, the assumption that the *x*s are non-stochastic is stronger than required, an issue that will be discussed in more detail in Chapter 5.

[2] Strictly, these are the estimated standard errors conditional on the parameter estimates, and so should be denoted $\hat{SE}(\hat{\alpha})$ and $\hat{SE}(\hat{\beta})$, but the additional layer of hats will be omitted here since the meaning should be obvious from the context.

4

Further Development and Analysis of the Classical Linear Regression Model

LEARNING OUTCOMES

In this chapter, you will learn how to

- Construct models with more than one explanatory variable
- Test multiple hypotheses using an *F*-test
- Determine how well a model fits the data
- Form a restricted regression
- Derive the OLS parameter and standard error estimators using matrix algebra
- Construct and interpret quantile regression models

4.1 Generalising the Simple Model to Multiple Linear Regression

Previously, a model of the following form has been used

$$y_t = \alpha + \beta x_t + u_t \quad t = 1, 2, \ldots, T \tag{4.1}$$

Equation (4.1) is a simple bivariate regression model. That is, changes in the dependent variable are explained by reference to changes in one single explanatory variable x. But what if the financial theory or idea that is sought to be tested suggests that the dependent variable is influenced by more than one independent variable? For example, simple estimation and tests of the capital asset pricing model (CAPM) can be conducted using an equation of the form of (4.1), but arbitrage pricing theory does not presuppose that there is only a single factor affecting stock returns. So, to give one illustration, stock returns might be purported to depend on their sensitivity to unexpected changes in:

(1) inflation

(2) differences in returns on short- and long-dated bonds

(3) industrial production

(4) default risks

Having just one independent variable would be no good in this case. It would of course be possible to use each of the four proposed explanatory factors in separate regressions. But it is of greater interest and it is more valid to have more than one explanatory variable in the regression equation at the same time, and therefore to examine the effect of all of the explanatory variables together on the explained variable.

It is very easy to generalise the simple model to one with *k* regressors (independent variables). Equation (4.1) becomes

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \cdots + \beta_k x_{kt} + u_t, \quad t = 1, 2, \ldots, T \tag{4.2}$$

So the variables x_{2t}, x_{3t}, …, x_{kt} are a set of $k - 1$ explanatory variables which are thought to influence *y*, and the coefficient estimates β_1, β_2, …, β_k are the parameters which quantify the effect of each of these explanatory variables on *y*. The coefficient interpretations are slightly altered in the multiple regression context. Each coefficient is now known as a partial regression coefficient, interpreted as representing the partial effect of the given explanatory variable on the explained variable, after holding constant, or eliminating the effect of, all other explanatory variables. For example, $\hat{\beta}_2$ measures the effect of x_2 on *y* after eliminating the effects of x_3, x_4, …, x_k. Stating this in other words, each coefficient measures the average change in the dependent variable per unit change in a given independent variable, holding all other independent variables constant at their average values.

4.2 The Constant Term

In equation (4.2) above, astute readers will have noticed that the explanatory variables are numbered x_2, x_3, … i.e., the list starts with x_2 and not x_1. So, where is x_1? In fact, it is the constant term, usually represented by a column of ones of length T:

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \tag{4.3}$$

Thus there is a variable implicitly hiding next to β_1, which is a column vector of ones, the length of which is the number of observations in the sample. The x_1 in the regression equation is not usually written, in the same way that one unit of p and two units of q would be written as 'p + 2q' and not '1p + 2q'. β_1 is the coefficient attached to the constant term (which was called α in Chapter 3). This coefficient can still be referred to as the *intercept*, which can be interpreted as the average value which y would take if all of the explanatory variables took a value of zero.

A tighter definition of k, the number of explanatory variables, is probably now necessary. Throughout this book, k is defined as the number of 'explanatory variables' or 'regressors' including the constant term. This is equivalent to the number of parameters that are estimated in the regression equation. Strictly speaking, it is not sensible to call the constant an explanatory variable, since it does not explain anything and it always takes the same values. However, this definition of k will be employed for notational convenience.

Equation (4.2) can be expressed even more compactly by writing it in matrix form

$$y = X\beta + u \tag{4.4}$$

where: y is of dimension $T \times 1$
X is of dimension $T \times k$
β is of dimension $k \times 1$
u is of dimension $T \times 1$

The difference between (4.2) and (4.4) is that all of the time observations have been stacked up in a vector, and also that all of the different explanatory variables have been squashed together so that there is a column for each in the X matrix. Such a notation may seem unnecessarily complex, but in fact, the matrix notation is usually more compact and convenient. So, for example, if k is 2, i.e., there are two regressors, one of which is the constant term (equivalent to a simple bivariate regression $y_t = \alpha + \beta x_t + u_t$), it is possible to write

$$\underset{T \times 1}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix}} = \underset{T \times 2}{\begin{bmatrix} 1 & x_{21} \\ 1 & x_{22} \\ \vdots & \vdots \\ 1 & x_{2T} \end{bmatrix}} \underset{2 \times 1}{\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}} + \underset{T \times 1}{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_T \end{bmatrix}} \tag{4.5}$$

so that the x_{ij} element of the matrix X represents the jth time observation on the ith variable. Notice that the matrices written in this way are *conformable* – in other words, there is a valid matrix multiplication and addition on the RHS.

The above presentation is the standard way to express matrices in the time-series econometrics literature, although the ordering of the indices is different to that used in the mathematics of matrix algebra (as presented in Chapter 1 of this book). In the latter case, x_{ij} would represent the element in row i and column j, although in the notation used in the body of this book it is the other way around.

4.3 How are the Parameters (the Elements of the β Vector) Calculated in the Generalised Case?

Previously, the residual sum of squares, $\sum \hat{u}_i^2$ was minimised with respect to α and β. In the multiple regression context, in order to obtain estimates of the parameters, $\beta_1, \beta_2, \ldots, \beta_k$, the *RSS* would be minimised with respect to all the elements of β. Now, the residuals can be stacked in a vector:

$$\hat{u} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_T \end{bmatrix} \tag{4.6}$$

The *RSS* is still the relevant loss function, and would be given in a matrix notation by

$$L = \hat{u}'\hat{u} = [\hat{u}_1 \hat{u}_2 \cdots \hat{u}_T] \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_T \end{bmatrix} = \hat{u}_1^2 + \hat{u}_2^2 + \cdots + \hat{u}_T^2 = \sum \hat{u}_t^2 \qquad \textbf{(4.7)}$$

Using a similar procedure to that employed in the bivariate regression case, i.e., substituting into equation (4.7), and denoting the vector of estimated parameters as $\hat{\beta}$, it can be shown (see Appendix 4.1 to this chapter) that the coefficient estimates will be given by the elements of the expression

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_k \end{bmatrix} = (X'X)^{-1}X'y \qquad \textbf{(4.8)}$$

If one were to check the dimensions of the RHS of equation (4.8), it would be observed to be $k \times 1$. This is as required since there are k parameters to be estimated by the formula for $\hat{\beta}$.

But how are the standard errors of the coefficient estimates calculated? Previously, to estimate the variance of the errors, σ^2, an estimator denoted by s^2 was used

$$s^2 = \frac{\sum \hat{u}_t^2}{T-2} \qquad \textbf{(4.9)}$$

The denominator of equation (4.9) is given by $T - 2$, which is the number of degrees of freedom for the bivariate regression model (i.e., the number of observations minus two). This essentially applies since two observations are effectively 'lost' in estimating the two model parameters (i.e., in deriving estimates for α and β). In the case where there is more than one explanatory variable plus a constant, and using the matrix notation, equation (4.9) would be modified to

$$s^2 = \frac{\hat{u}'\hat{u}}{T-k} \qquad \textbf{(4.10)}$$

where k = number of regressors including a constant. In this case, k observations are 'lost' as k parameters are estimated, leaving $T - k$ degrees of freedom. It can also be shown (see Appendix 4.1 to this chapter) that the parameter variance–covariance matrix is given by

$$\text{var}(\hat{\beta}) = s^2(X'X)^{-1} \tag{4.11}$$

The leading diagonal terms give the coefficient variances while the off-diagonal terms give the covariances between the parameter estimates, so that the variance of $\hat{\beta}_1$ is the first diagonal element, the variance of $\hat{\beta}_2$ is the second element on the leading diagonal, and the variance of $\hat{\beta}_k$ is the *k*th diagonal element. The coefficient standard errors are thus simply given by taking the square roots of each of the terms on the leading diagonal.

4.4 Testing Multiple Hypotheses: The *F*-test

The *t*-test was used to test single hypotheses, i.e., hypotheses involving only one coefficient. But what if it is of interest to test more than one coefficient simultaneously? For example, what if a researcher wanted to determine whether a restriction that the coefficient values for β_2 and β_3 are both unity could be imposed, so that an increase in either one of the two variables x_2 or x_3 would cause y to rise by one unit? The *t*-testing framework is not sufficiently general to cope with this sort of hypothesis test. Instead, a more general framework is employed, centring on an *F*-test. Under the *F*-test framework, two regressions are required, known as the unrestricted and the restricted regressions. The unrestricted regression is the one in which the coefficients are freely determined by the data, as has been constructed previously. The restricted regression is the one in which the coefficients are restricted, i.e., the restrictions are imposed on some βs. Thus the *F*-test approach to hypothesis testing is also termed restricted least squares, for obvious reasons.

The residual sums of squares from each regression are determined, and the two residual sums of squares are 'compared' in the test statistic. The *F*-test statistic for testing multiple hypotheses about the coefficient estimates is given by

$$\textit{test statistic} = \frac{RRSS - URSS}{URSS} \times \frac{T-k}{m} \tag{4.12}$$

where the following notation applies:

EXAMPLE 4.1

The following model with three regressors (including the constant, so k = 3) is estimated over fifteen observations

$$y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + u \quad \textbf{(4.13)}$$

and the following data have been calculated from the original xs

$$(X'X)^{-1} = \begin{bmatrix} 2.0 & 3.5 & -1.0 \\ 3.5 & 1.0 & 6.5 \\ -1.0 & 6.5 & 4.3 \end{bmatrix}, \quad (X'y) = \begin{bmatrix} -3.0 \\ 2.2 \\ 0.6 \end{bmatrix}, \quad \hat{u}'\hat{u} = 10.96$$

Calculate the coefficient estimates and their standard errors.

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_k \end{bmatrix} = (X'X)^{-1}X'y = \begin{bmatrix} 2.0 & 3.5 & -1.0 \\ 3.5 & 1.0 & 6.5 \\ -1.0 & 6.5 & 4.3 \end{bmatrix} \times \begin{bmatrix} -3.0 \\ 2.2 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 1.10 \\ -4.40 \\ 19.88 \end{bmatrix} \quad \textbf{(4.14)}$$

To calculate the standard errors, an estimate of σ^2 is required

$$s^2 = \frac{RSS}{T-k} = \frac{10.96}{15-3} = 0.91 \quad \textbf{(4.15)}$$

The variance–covariance matrix of $\hat{\beta}$ is given by

$$s^2(X'X)^{-1} = 0.91(X'X)^{-1} = \begin{bmatrix} 1.82 & 3.19 & -0.91 \\ 3.19 & 0.91 & 5.92 \\ -0.91 & 5.92 & 3.91 \end{bmatrix} \quad \textbf{(4.16)}$$

The coefficient variances are on the diagonals, and the standard errors are found by taking the square roots of each of the coefficient variances

$$\text{var}(\hat{\beta}_1) = 1.82 \qquad SE(\hat{\beta}_1) = 1.35 \quad \textbf{(4.17)}$$

$$\text{var}(\hat{\beta}_2) = 0.91 \Leftrightarrow SE(\hat{\beta}_2) = 0.95 \quad \textbf{(4.18)}$$

$$\text{var}(\hat{\beta}_3) = 3.91 \qquad SE(\hat{\beta}_3) = 1.98 \quad \textbf{(4.19)}$$

The estimated equation would be written

$$\hat{y} = 1.10 - 4.40x_2 + 19.88x_3 \quad (4.20)$$
$$(1.35) \ (0.95) \qquad (1.98)$$

Fortunately, in practice all econometrics software packages will estimate the coefficient values and their standard errors. Clearly, though, it is still useful to understand where these estimates came from.

$URSS$ = residual sum of squares from unrestricted regression
$RRSS$ = residual sum of squares from restricted regression
m = number of restrictions
T = number of observations
k = number of regressors in unrestricted regression including the constant

The most important part of the test statistic to understand is the numerator expression *RRSS* – *URSS*. To see why the test centres around a comparison of the residual sums of squares from the restricted and unrestricted regressions, recall that OLS estimation involved choosing the model that minimised the residual sum of squares, with no constraints imposed. Now if, after imposing constraints on the model, a residual sum of squares results that is not much higher than the unconstrained model's residual sum of squares, it would be concluded that the restrictions were supported by the data. On the other hand, if the residual sum of squares increased considerably after the restrictions were imposed, it would be concluded that the restrictions were not supported by the data and therefore that the hypothesis should be rejected.

It can be further stated that $RRSS \geq URSS$. Only under a particular set of very extreme circumstances will the residual sums of squares for the restricted and unrestricted models be exactly equal. This would be the case when the restriction was already present in the data, so that it is not really a restriction at all (it would be said that the restriction is 'not binding', i.e., it does not make any difference to the parameter estimates). So, for example, if the null hypothesis is H_0: $\beta_2 = 1$ and $\beta_3 = 1$, then $RRSS = URSS$ only in the case where the coefficient estimates for the unrestricted regression had been $\hat{\beta}_2 = 1$ and $\hat{\beta}_3 = 1$. Of course, such an event is extremely unlikely to occur in practice.

It is worth noting that the F-test statistic is sometimes expressed in a slightly different way, which is simply a rearrangement of (4.12) as

$$test\ statistic = \frac{(RRSS - URSS)/m}{URSS/(T-k)}$$

Sometimes, m is termed the *degrees of freedom for the numerator* while $T - k$ are the *degrees of freedom for the denominator* and writing the formula for the F-test statistic in this way makes it easy to see why.

EXAMPLE 4.2

Dropping the time subscripts for simplicity, suppose that the general regression is

$$y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + u \qquad \textbf{(4.21)}$$

and that the restriction $\beta_3 + \beta_4 = 1$ is under test (there exists some hypothesis from theory which suggests that this would be an interesting hypothesis to study). The unrestricted regression is equation (4.21) above, but what is the restricted regression? It could be expressed as

$$y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + u \quad \text{s.t. (subject to) } \beta_3 + \beta_4 = 1 \qquad \textbf{(4.22)}$$

The restriction ($\beta_3 + \beta_4 = 1$) is substituted into the regression so that it is automatically imposed on the data. The way that this would be achieved would be to make either β_3 or β_4 the subject of equation (4.22), e.g.

$$\beta_3 + \beta_4 = 1 \Rightarrow \beta_4 = 1 - \beta_3 \qquad \textbf{(4.23)}$$

and then substitute into equation (4.21) for β_4

$$y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + (1 - \beta_3) x_4 + u \qquad \textbf{(4.24)}$$

Equation (4.24) is already a restricted form of the regression, but it is not yet in the form that is required to estimate it using a computer package. In order to be able to estimate a model using OLS, software packages usually require each RHS variable to be multiplied by one coefficient only. Therefore, a little more algebraic manipulation is required. First, expanding the brackets around $(1 - \beta_3)$

$$y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + x_4 - \beta_3 x_4 + u \tag{4.25}$$

Then, gathering all of the terms in each β_i together and rearranging

$$(y - x_4) = \beta_1 + \beta_2 x_2 + \beta_3 (x_3 - x_4) + u \tag{4.26}$$

Note that any variables without coefficients attached (e.g. x_4 in equation (4.25)) are taken over to the LHS and are then combined with y. Equation (4.26) is the restricted regression. It is actually estimated by creating two new variables – call them, say, P and Q, where $P = y - x_4$ and $Q = x_3 - x_4$ – so the regression that is actually estimated is

$$P = \beta_1 + \beta_2 x_2 + \beta_3 Q + u \tag{4.27}$$

What would have happened if instead β_3 had been made the subject of equation (4.23) and β_3 had therefore been removed from the equation? Although the equation that would have been estimated would have been different from equation (4.27), the value of the residual sum of squares for these two models (both of which have imposed upon them the same restriction) would be the same.

The test statistic follows the *F*-distribution under the null hypothesis. The *F*-distribution has two degrees of freedom parameters (recall that the *t*-distribution had only one degree of freedom parameter, equal to $T - k$). The value of the degrees of freedom parameters for the *F*-test are m, the number of restrictions imposed on the model, and $(T - k)$, the number of observations less the number of regressors for the unrestricted regression, respectively. Note that the order of the degree of freedom parameters is important. The appropriate critical value will be in column m, row $(T - k)$ of the *F*-distribution tables.

4.4.1 The Relationship Between the *t*- and the *F*-Distributions

Any hypothesis that could be tested with a *t*-test could also have been tested using an *F*-test, but not the other way around. So, single hypotheses involving one coefficient can be tested using a *t*- or an *F*-test, but multiple hypotheses can be tested only using an *F*-test. For example, consider the hypothesis

$$H_0 : \beta_2 = 0.5$$
$$H_1 : \beta_2 \neq 0.5$$

This hypothesis could have been tested using the usual *t*-test

$$test\ stat = \frac{\hat{\beta}_2 - 0.5}{SE(\hat{\beta}_2)} \qquad \textbf{(4.28)}$$

or it could be tested in the framework above for the *F*-test. Note that the two tests always give the same conclusion since the *t*-distribution is just a special case of the *F*-distribution. For example, consider any random variable Z that follows a *t*-distribution with $T – k$ degrees of freedom, and square it. The square of the t is equivalent to a particular form of the *F*-distribution

$$Z^2 \sim t^2\ (T - k) \text{ then also } Z^2 \sim F(1, T - k)$$

Thus the square of a *t*-distributed random variable with $T – k$ degrees of freedom also follows an *F*-distribution with 1 and $T – k$ degrees of freedom. This relationship between the t and the *F*-distributions will always hold – take some examples from the statistical tables and try it!

The *F*-distribution has only positive values and is not symmetrical. Therefore, the null is rejected only if the test statistic exceeds the critical *F*-value, although the test is a two-sided one in the sense that rejection will occur if $\hat{\beta}_2$ is significantly bigger or significantly smaller than 0.5.

4.4.2 Determining the Number of Restrictions, *m*

How is the appropriate value of m decided in each case? Informally, the number of restrictions can be seen as 'the number of equality signs under the null hypothesis'. To give some examples

H_0 : *hypothesis*	*No. of restrictions, m*
$\beta_1 + \beta_2 = 2$	1
$\beta_2 = 1$ and $\beta_3 = -1$	2
$\beta_2 = 0, \beta_3 = 0$ and $\beta_4 = 0$	3

At first glance, you may have thought that in the first of these cases, the number of restrictions was two. In fact, there is only one restriction that involves two coefficients. The number of restrictions in the second two examples is obvious, as they involve two and three separate component restrictions, respectively.

The last of these three examples is particularly important. If the model is

$$y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + u \qquad \textbf{(4.29)}$$

then the null hypothesis of

$$H_0 : \beta_2 = 0 \quad \text{and} \quad \beta_3 = 0 \quad \text{and} \quad \beta_4 = 0$$

is tested by 'THE' regression *F*-statistic. It tests the null hypothesis that all of the coefficients except the intercept coefficient are zero. This test is sometimes called a test for 'junk regressions', since if this null hypothesis cannot be rejected, it would imply that none of the independent variables in the model was able to explain variations in *y*.

Note the form of the alternative hypothesis for all tests when more than one restriction is involved

$$H_1 : \beta_2 \neq 0 \quad \text{or} \quad \beta_3 \neq 0 \quad \text{or} \quad \beta_4 \neq 0$$

In other words, 'and' occurs under the null hypothesis and 'or' under the alternative, so that it takes only one part of a joint null hypothesis to be wrong for the null hypothesis as a whole to be rejected.

4.4.3 Hypotheses that Cannot be Tested with Either an *F*- or a *t*-Test

It is not possible to test hypotheses that are not linear or that are multiplicative using this framework – for example, $H_0 : \beta_2\beta_3 = 2$, or H_0 : $H_0 : \beta_2^2 = 1$ cannot be tested.

EXAMPLE 4.3

Suppose that a researcher wants to test whether the returns on a company stock (*y*) show unit sensitivity to two factors (factor x_2 and factor x_3) among three considered. The regression is carried out on 144 monthly observations. The regression is

$$y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + u \qquad \textbf{(4.30)}$$

(1) What are the restricted and unrestricted regressions?

(2) If the two *RSS* are 436.1 and 397.2, respectively, perform the test.

Unit sensitivity to factors x_2 and x_3 implies the restriction that the coefficients on these two variables should be unity, so H_0: $\beta_2 = 1$ and

β_3 = 1. The unrestricted regression will be the one given by equation (4.30) above. To derive the restricted regression, first impose the restriction:

$$y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + u \quad \text{s.t.} \quad \beta_2 = 1 \quad \text{and} \quad \beta_3 = 1 \tag{4.31}$$

Replacing β_2 and β_3 by their values under the null hypothesis

$$y = \beta_1 + x_2 + x_3 + \beta_4 x_4 + u \tag{4.32}$$

Rearranging

$$y - x_2 - x_3 = \beta_1 + \beta_4 x_4 + u \tag{4.33}$$

Defining $z = y - x_2 - x_3$, the restricted regression is one of z on a constant and x_4

$$z = \beta_1 + \beta_4 x_4 + u \tag{4.34}$$

The formula for the F-test statistic is given in equation (4.12) above. For this application, the following inputs to the formula are available: T = 144, k = 4, m = 2, $RRSS$ = 436.1, $URSS$ = 397.2. Plugging these into the formula gives an F-test statistic value of 6.86. This statistic should be compared with an $F(m, T - k)$, which in this case is an $F(2, 140)$. The critical values are 3.07 at the 5% level and 4.79 at the 1% level. Note that the table does not include a row for 140, so we use the closest, which is 120 rather than ∞. The test statistic clearly exceeds the critical values at both the 5% and 1% levels, and hence the null hypothesis is rejected. It would thus be concluded that the restriction is not supported by the data.

4.4.4 A Note on Sample Sizes and Asymptotic Theory

A question that is often asked by those new to econometrics is ‘what is an appropriate sample size for model estimation?’ While there is no definitive answer to this question, it should be noted that most testing procedures in econometrics rely on asymptotic theory. That is, the results in theory hold only if there is an *infinite number of observations*. In practice, an infinite number of observations will never be available and fortunately, an infinite number of observations are not usually required to invoke the asymptotic theory. An approximation to the asymptotic behaviour of the test statistics can be obtained using finite samples, provided that they are large enough. In general, as many observations as possible should be used (although there are important caveats to this statement relating to 'structural stability', discussed in Chapter 5). The reason is that all the researcher has at his disposal is a sample of data from which to estimate parameter values and to infer their likely population counterparts. A sample may fail to deliver something close to the exact population values owing to sampling error. Even if the sample is randomly drawn from the population, some samples will be more representative of the behaviour of the population than others, purely owing to 'luck of the draw'. Sampling error is minimised by increasing the size of the sample, since the larger the sample, the less likely it is that all of the data drawn will be unrepresentative of the population.

4.5 Data Mining and the True Size of the Test

Recall that the probability of rejecting a correct null hypothesis is equal to the size of the test, denoted α. The possibility of rejecting a correct null hypothesis arises from the fact that test statistics are assumed to follow a random distribution and hence they will take on extreme values that fall in the rejection region some of the time by chance alone. A consequence of this is that it will almost always be possible to find significant relationships between variables if enough variables are examined. For example, suppose that a dependent variable y_t and twenty explanatory variables $x_{2t}, \ldots, x_{21t}$ (excluding a constant term) are generated separately as independent normally distributed random variables. Then y is regressed separately on each of the twenty explanatory variables plus a constant, and the significance of each explanatory variable in the regressions is examined. If this experiment is repeated many times, on average one of the twenty regressions will have a slope coefficient that is significant at the 5% level for each experiment. The implication is that for any regression, if enough explanatory variables are employed in a regression, often one or more will be significant by chance alone. More concretely, it could be stated that if an α% size of test is used, on average one in every $(100/\alpha)$ regressions will have a significant slope coefficient by chance alone.

Trying many variables in a regression without basing the selection of the candidate variables on a financial or economic theory is known as 'data mining' or 'data snooping'. The result in such cases is that the true significance level will be considerably greater than the nominal significance level assumed. For example, suppose that twenty separate regressions are conducted, of which three contain a significant regressor, and a 5% nominal significance level is assumed, then the true significance level would be much higher (e.g., 25%). Therefore, if the researcher then shows only the results for the regression containing the final three equations and states that they are significant at the 5% level, inappropriate conclusions concerning the significance of the variables would result.

As well as ensuring that the selection of candidate regressors for inclusion in a model is made on the basis of financial or economic theory, another way to avoid data mining is by examining the forecast performance of the model in an 'out-of-sample' data set (see Chapter 6). The idea is essentially that a proportion of the data is not used in model estimation, but is retained for model testing. A relationship observed in the estimation period that is purely the result of data mining, and is therefore spurious, is very unlikely to be repeated for the out-of-sample period. Therefore, models that are the product of data mining are likely to fit very poorly and to give very inaccurate forecasts for the out-of-sample period.

4.6 Qualitative Variables

There are many situations when building an econometric model where we would like to capture the effect of qualitative information. For example, we might be interested in modelling credit ratings, or comparing the performance of men versus women traders to determine who takes more risk on average. In both these situations (credit ratings and sex of traders), the data do not have numbers associated with them initially. The way that we would turn qualitative information into quantitative variables that can be incorporated into the model is via the construction of one or more *dummy variables*. Dummy variables are usually specified to take on one of a narrow range of integer values, and in most instances only zero and one are used.

Dummy variables can be used in the context of cross-sectional or time-series regressions. In each case, the dummy variables are used in the same way as other explanatory variables and the coefficients on the dummy variables can be interpreted as the average differences in the values of the dependent variable for each category, given all of the other factors in the model. For example, suppose we have data and estimate the following regression model

$$salary_i = \beta_1 + \beta_2 age_i + \beta_3 sex_i + \beta_4 location_i + \beta_5 edu_i + u_i \quad \textbf{(4.35)}$$

where $salary_i$ is the annual salary in US dollars of trader *i*, *age* is his or her age, *sex* is his or her sex (with 1 = male; 0 = female), *location* = 0 if the trader is based in New York, 1 if he or she is based in London, 2 if he or she is based in Paris, and *edu* = 1 if the trader has a first degree or higher and 0 otherwise.

In this case, all four of the explanatory variables in the model would be dummies and three of them take only values 0 and 1. For the latter, the coefficients on the dummies would be easy to interpret as the average difference in salary between a trader with this characteristic and an otherwise identical one without. For example, suppose that $\hat{\beta}_3 = 2850$, this would suggest that on average male traders (dummy value 1) earn \$2850 per year more than otherwise equivalent female traders. Similarly, if the estimated value of β_5 is $\hat{\beta}_5 = 8500$, this would show that traders having at least a degree earn on average \$8500 per year more than those without.

But what about the *location* dummy? This is more tricky to interpret and in fact it would probably be inappropriate to set up a dummy in this way. The dummy has three values, and thus implies an ordinal ranking of numbers which was probably not intended. Thus instead we would need to set up two or three separate 0–1 dummy variables, assuming that all traders in our sample are based in one of these locations and no other places. So, for example, we could have a NY_i dummy variable that took the value 1 if the trader is based in New York and 0 otherwise; similarly we would set up a London dummy and a Paris dummy. We could either include all three dummy variables together (and not include an intercept in the regression equation) or only include two of the dummy variables and retain the intercept. If we include all three dummy variables and the intercept at the same time, the regression model could not be estimated and this is known as the *dummy variable trap* – see Section 10.3 in Chapter 10 for details.

All of the above variables are known as *intercept dummies* since, effectively, they modify the intercept in each case (e.g., allowing for a different intercept or average salary for men versus women) but they do not alter the relationship between the dependent variable and the other independent variables – the latter would be known as *slope dummy variables*, which are discussed in Section 10.3 of Chapter 10.

It should be evident given this brief introduction that dummy variables are extremely useful and will be used extensively in other parts of the book, but most notably to allow for outliers in Chapter 5, to account for seasonality in Chapter 10 and when investigating limited dependent variables, which is the subject of the whole of Chapter 12.

4.7 Goodness of Fit Statistics

4.7.1 R^2

It is desirable to have some measure of how well the regression model actually fits the data. In other words, it is desirable to have an answer to the question, 'how well does the model containing the explanatory variables that was proposed actually explain variations in the dependent variable?' Quantities known as *goodness of fit statistics* are available to test how well the sample regression function (SRF) fits the data – that is, how 'close' the fitted regression line is to all of the data points taken together. Note that it is not possible to say how well the sample regression function fits the population regression function – i.e., how the estimated model compares with the true relationship between the variables, since the latter is never known.

But what measures might make plausible candidates to be goodness of fit statistics? A first response to this might be to look at the residual sum of squares (*RSS*). Recall that OLS selected the coefficient estimates that minimised this quantity, so the lower was the minimised value of the *RSS*, the better the model fitted the data. Consideration of the *RSS* is certainly one possibility, but *RSS* is unbounded from above (strictly, *RSS* is bounded from above by the total sum of squares – see below) – i.e., it can take any (non-negative) value. So, for example, if the value of the *RSS* under OLS estimation was 136.4, what does this actually mean? It would therefore be very difficult, by looking at this number alone, to tell whether the regression line fitted the data closely or not. The value of *RSS* depends to a great extent on the scale of the dependent variable. Thus, one way to pointlessly reduce the *RSS* would be to divide all of the observations on y by 10!

In fact, a *scaled version* of the residual sum of squares is usually employed. The most common goodness of fit statistic is known as R^2. One way to define R^2 is to say that it is the square of the correlation coefficient between y and $\hat{y}$ – that is, the square of the correlation between the values of the dependent variable and the corresponding fitted values from the model. A correlation coefficient must lie between –1 and +1 by definition. Since R^2 defined in this way is the square of a correlation coefficient, it must lie between 0 and 1. If this correlation is high, the model fits the data well, while if the correlation is low (close to zero), the model is not providing a good fit to the data.

Another definition of R^2 requires a consideration of what the model is attempting to explain. What the model is trying to do in effect is to explain variability of *y* about its mean value, $\bar{y}$. This quantity, $\bar{y}$, which is more specifically known as the unconditional mean of *y*, acts like a benchmark since, if the researcher had no model for *y*, he could do no worse than to regress *y* on a constant only. In fact, the coefficient estimate for this regression would be the mean of *y*. So, from the regression

$$y_t = \beta_1 + u_t \tag{4.36}$$

the coefficient estimate $\hat{\beta}_1$, will be the mean of *y*, i.e., $\bar{y}$. The total variation across all observations of the dependent variable about its mean value is known as the total sum of squares, *TSS*, which is given by:

$$TSS = \sum_t (y_t - \bar{y})^2 \tag{4.37}$$

The *TSS* can be split into two parts: the part that has been explained by the model (known as the explained sum of squares, *ESS*) and the part that the model was not able to explain (the *RSS*). That is

$$TSS = ESS + RSS \tag{4.38}$$

$$\sum_t (y_t - \bar{y})^2 = \sum_t (\hat{y}_t - \bar{y})^2 + \sum_t \hat{u}_t^2 \tag{4.39}$$

Recall also that the residual sum of squares can also be expressed as

$$\sum_t (y_t - \hat{y}_t)^2$$

since a residual for observation *t* is defined as the difference between the actual and fitted values for that observation. The goodness of fit statistic is given by the ratio of the explained sum of squares to the total sum of squares:

$$R^2 = \frac{ESS}{TSS} \tag{4.40}$$

but since *TSS* = *ESS* + *RSS*, it is also possible to write

$$R^2 = \frac{ESS}{TSS} = \frac{TSS - RSS}{TSS} = 1 - \frac{RSS}{TSS} \quad \textbf{(4.41)}$$

R^2 must always lie between zero and one (provided that there is a constant term in the regression). This is intuitive from the correlation interpretation of R^2 given above, but for another explanation, consider two extreme cases

$$RSS = TSS \quad \text{i.e.,} \quad ESS = 0 \quad \text{so} \quad R^2 = ESS/TSS = 0$$
$$ESS = TSS \quad \text{i.e.,} \quad RSS = 0 \quad \text{so} \quad R^2 = ESS/TSS = 1$$

In the first case, the model has not succeeded in explaining any of the variability of y about its mean value, and hence the residual and total sums of squares are equal. This would happen only where the estimated values of all of the coefficients were exactly zero. In the second case, the model has explained all of the variability of y about its mean value, which implies that the residual sum of squares will be zero. This would happen only in the case where all of the observation points lie exactly on the fitted line. Neither of these two extremes is likely in practice, of course, but they do show that R^2 is bounded to lie between zero and one, with a higher R^2 implying, everything else being equal, that the model fits the data better.

To sum up, a simple way (but crude, as explained next) to tell whether the regression line fits the data well is to look at the value of R^2. A value of R^2 close to 1 indicates that the model explains nearly all of the variability of the dependent variable about its mean value, while a value close to zero indicates that the model fits the data poorly. The two extreme cases, where $R^2 = 0$ and $R^2 = 1$, are indicated in Figures 4.1 and 4.2 in the context of a simple bivariate regression.

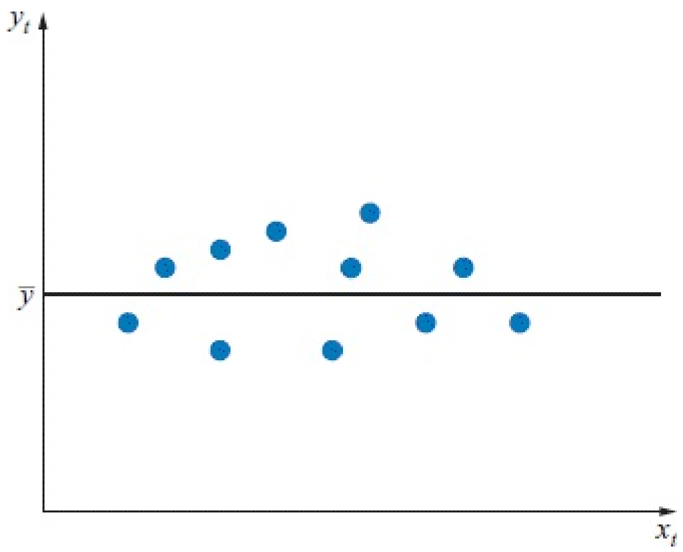


Figure 4.1 $R^2 = 0$ demonstrated by a flat estimated line, i.e., a zero slope coefficient

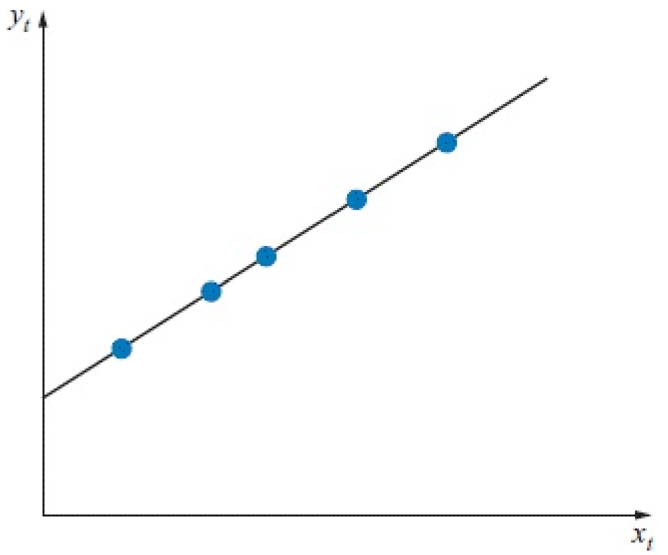


Figure 4.2 $R^2 = 1$ when all data points lie exactly on the estimated line

4.7.2 Problems with R^2 as a Goodness of Fit Measure

R^2 is simple to calculate, intuitive to understand, and provides a broad indication of the fit of the model to the data. However, there are a number of problems with R^2 as a goodness of fit measure:

(1) R^2 is defined in terms of variation about the mean of y so that if a model is reparameterised (rearranged) and the dependent variable changes, R^2 will change, even if the second model was a simple rearrangement of the first, with identical *RSS*. Thus it is not sensible to compare the value of R^2 across models with different dependent variables.

(2) R^2 never falls if more regressors are added to the regression. For example, consider the following two models:

$$\text{Regression 1: } y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + u \tag{4.42}$$

$$\text{Regression 2: } y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + u \tag{4.43}$$

R^2 will always be at least as high for regression 2 relative to regression 1. The R^2 from regression 2 would be exactly the same as that for regression 1 only if the estimated value of the coefficient on the new variable were exactly zero, i.e., $\hat{\beta}_4 = 0$. In practice, $\hat{\beta}_4$ will always be non-zero, even if not significantly so, and thus in practice R^2 always rises as more variables are added to a model. This feature of R^2 essentially makes it impossible to use as a determinant of whether a given variable should be present in the model or not.

(3) R^2 can take values of 0.9 or higher for time-series regressions, and hence it is not good at discriminating between models, since a wide array of models will frequently have broadly similar (and high) values of R^2.

4.7.3 Adjusted R^2

In order to get around the second of these three problems, a modification to R^2 is often made which takes into account the loss of degrees of freedom associated with adding extra variables. This is known as $\bar{R}^2$, or adjusted R^2, which is defined as

$$\bar{R}^2 = 1 - \left[\frac{T-1}{T-k}(1 - R^2)\right] \tag{4.44}$$

So if an extra regressor (variable) is added to the model, k increases and unless R^2 increases by a more than off-setting amount, $\bar{R}^2$ will actually fall. Hence $\bar{R}^2$ can be used as a decision-making tool for determining whether a given variable should be included in a regression model or not, with the rule being: include the variable if $\bar{R}^2$ rises and do not include it if $\bar{R}^2$ falls.

However, there are still problems with the maximisation of $\bar{R}^2$ as criterion for model selection, and principal among these is that it is a ‘soft’ rule, implying that by following it, the researcher will typically end up with a large model, containing a lot of marginally significant or insignificant variables. Also, while R^2 must be at least zero if an intercept is included in the regression, its adjusted counterpart may take negative values, even with an intercept in the regression, if the model fits the data very poorly.

To provide a couple of illustrations, if we consider a hedging example regression of spot returns on futures returns giving an R^2 value of 0.99, this would indicate that almost all of the variation in spot returns is explained by the futures returns. However, regressions in the context of the CAPM time-series regression of excess stock returns on excess market returns often fit less well – suppose, for example, the value is $R^2 = 0.35$ (i.e., about 35%), the conclusion would be that for that stock and sample period, around a third of the monthly movement in the excess returns can be attributed to movements in the market as a whole, as measured by the S&P500.

There now follows another case study of the application of the OLS method of regression estimation, including interpretation of t-ratios and R^2.

4.8 Hedonic Pricing Models

An application of econometric techniques where the coefficients have a particularly intuitively appealing interpretation is in the area of hedonic pricing models. *Hedonic models* are used to value real assets, especially housing, and view the asset as representing a bundle of characteristics, each of which gives either utility or disutility to its consumer. Hedonic models are often used to produce appraisals or valuations of properties, given their characteristics (e.g., size of dwelling, number of bedrooms, location, number of bathrooms, etc.). In these models, the coefficient estimates represent ‘prices of the characteristics’.

One such application of a hedonic pricing model is given by Des Des Rosiers and Thériault (1996), who consider the effect of various amenities on rental values for buildings and apartments in five sub-markets in the Quebec area of Canada. After accounting for the effect of ‘contract-specific’ features which will affect rental values (such as whether furnishings, lighting, or hot water are included in the rental price), they arrive at a model where the rental value in Canadian dollars per month (the dependent variable) is a function of nine to fourteen variables (depending on the area under consideration). The paper employs 1990 data for the Quebec City region, and there are 13,378 observations. The twelve explanatory variables are:

LnAGE	log of the apparent age of the property
NBROOMS	number of bedrooms
AREABYRM	area per room (in square met
ELEVATOR	a dummy variable = 1 if the building has an elevator; 0 otherwise
BASEMENT	a dummy variable = 1 if the u is located in a basement; 0 otherwise
OUTPARK	number of outdoor parking spaces
INDPARK	number of indoor parking sp
NOLEASE	a dummy variable = 1 if the u has no lease attached to it; 0 otherwise
LnDISTCBD	log of the distance in kilomet to the central business distric (CBD)
SINGLPAR	percentage of single-parent families in the area where the building stands
DSHOPCNTR	distance in kilometres to the nearest shopping centre
VACDIFF1	vacancy difference between t building and the census figur

This list includes several dummy variables, including ELEVATOR, BASEMENT, OUTPARK, INDPARK, NOLEASE. The interpretation of the coefficients on these dummies will be discussed when considering the output below. Des Rosiers and Thériault (1996) report several specifications for five different regions, and they present results for the model with variables as discussed here in their Exhibit 4, which is adapted and reported here as Table 4.1.

Table 4.1 Hedonic model of rental values in Quebec City, 1990. Dependent variable: Canadian dollars per month

Variable	Coefficient	t-ratio	sign expected *a priori*
Intercept	282.21	56.09	+
LnAGE	−53.10	−59.71	−
NBROOMS	48.47	104.81	+
AREABYRM	3.97	29.99	+
ELEVATOR	88.51	45.04	+
BASEMENT	−15.90	−11.32	−
OUTPARK	7.17	7.07	+
INDPARK	73.76	31.25	+
NOLEASE	−16.99	−7.62	−
LnDISTCBD	5.84	4.60	−
SINGLPAR	−4.27	−38.88	−
DSHOPCNTR	−10.04	−5.97	−
VACDIFF1	0.29	5.98	−

Notes: Adjusted R^2 = 0.651; regression F-statistic = 2082.27.

Source: Des Rosiers and Thériault (1996). Reprinted with the permission of the American Real Estate Society.

The adjusted R^2 value indicates that 65% of the total variability of rental prices about their mean value is explained by the model. For a cross-sectional regression, this is quite high. Also, all variables are significant at the 0.01% level or lower and consequently, the regression F-statistic rejects very strongly the null hypothesis that all coefficient values on explanatory variables are zero. Note that there is a relationship between the regression F-statistic and R^2, as shown in Box 4.1.

BOX 4.1 The relationship between the regression F-statistic and R^2

There is a particular relationship between a regression's R^2 value and the regression F-statistic. Recall that the regression F-statistic tests the null hypothesis that all of the regression slope parameters are simultaneously zero. Let us call the residual sum of squares for the unrestricted regression including all of the explanatory variables RSS, while the restricted regression will simply be one of y_t on a constant

$$y_t = \beta_1 + u_t \tag{4.45}$$

Since there are no slope parameters in this model, none of the variability of y_t about its mean value would have been explained. Thus the residual sum of squares for equation (4.45) will actually be the total sum of squares of y_t, TSS. We could write the usual F-statistic formula for testing this null that all of the slope parameters are jointly zero as

$$F\text{-}stat = \frac{TSS - RSS}{RSS} \times \frac{T - k}{k - 1} \tag{4.46}$$

In this case, the number of restrictions ('m') is equal to the number of slope parameters, $k - 1$. Recall that $TSS - RSS = ESS$ and dividing the numerator and denominator of equation (4.46) by TSS, we obtain

$$F\text{-}stat = \frac{ESS/TSS}{RSS/TSS} \times \frac{T - k}{k - 1} \tag{4.47}$$

Now the numerator of the left-hand part of equation (4.47) is R^2, while the denominator is $1 - R^2$, so that the F-statistic can be written

$$F\text{-}stat = \frac{R^2(T - k)}{(1 - R^2)(k - 1)} \tag{4.48}$$

This relationship between the F-statistic and R^2 holds only for a test of this null hypothesis and not for any others.

As stated above, one way to evaluate an econometric model is to determine whether it is consistent with theory. In this instance, no real theory is available, but instead there is a notion that each variable will affect rental values in a given direction. The actual signs of the coefficients can be compared with their expected values, given in the last column of Table 4.1 (as determined by this author). It can be seen that all coefficients except two (the log of the distance to the CBD and the vacancy differential) have their predicted signs. It is argued by Des Rosiers and Thériault that the 'distance to the CBD' coefficient may be expected to have a positive sign since, while it is usually viewed as desirable to live close to a town centre, everything else being equal, in this instance most of the least desirable neighbourhoods are located towards the centre.

The coefficient estimates themselves show the Canadian dollar rental price per month of each feature of the dwelling. To offer a few illustrations, the NBROOMS value of 48 (rounded) shows that, everything else being equal, one additional bedroom will lead to an average increase in the rental price of the property by $48 per month at 1990 prices. A basement coefficient of –16 suggests that an apartment located in a basement commands a rental $16 less than an identical apartment above ground. Finally the coefficients for parking suggest that on average each outdoor parking space adds $7 to the rent while each indoor parking space adds $74, and so on. The intercept shows, in theory, the rental that would be required of a property that had zero values on all the attributes. This case demonstrates, as stated previously, that the coefficient on the constant term often has little useful interpretation, as it would refer to a dwelling that has just been built, has no bedrooms each of zero size, no parking spaces, no lease, right in the CBD and shopping centre, etc.

One limitation of such studies that is worth mentioning at this stage is their assumption that the implicit price of each characteristic is identical across types of property, and that these characteristics do not become saturated. In other words, it is implicitly assumed that if more and more bedrooms or allocated parking spaces are added to a dwelling indefinitely, the monthly rental price will rise each time by $48 and $7, respectively. This assumption is very unlikely to be upheld in practice, and will result in the estimated model being appropriate for only an 'average' dwelling. For example, an additional indoor parking space is likely to add far more value to a luxury apartment than a basic one. Similarly, the marginal value of an additional bedroom is likely to be bigger if the dwelling currently has one bedroom than if it already has ten. One potential remedy for this would be to use dummy variables with fixed effects in the regressions; see, for example, Chapter 11 for an explanation of these.

4.9 Tests of Non-Nested Hypotheses

All of the hypothesis tests conducted thus far in this book have been in the context of 'nested' models. This means that, in each case, the test involved imposing restrictions on the original model to arrive at a restricted formulation that would be a sub-set of, or nested within, the original specification.

However, it is sometimes of interest to compare between non-nested models. For example, suppose that there are two researchers working independently, each with a separate financial theory for explaining the variation in some variable, y_t. The models selected by the researchers respectively could be

$$y_t = \alpha_1 + \alpha_2 x_{2t} + u_t \tag{4.49}$$

$$y_t = \beta_1 + \beta_2 x_{3t} + v_t \tag{4.50}$$

where u_t and v_t are iid error terms. Model (4.49) includes variable x_2 but not x_3, while model (4.50) includes x_3 but not x_2. In this case, neither model can be viewed as a restriction of the other, so how then can the two models be compared as to which better represents the data, y_t? Given the discussion in Section 4.7, an obvious answer would be to compare the values of R^2 or adjusted R^2 between the models. Either would be equally applicable in this case since the two specifications have the same number of RHS variables. Adjusted R^2 could be used even in cases where the number of variables was different across the two models, since it employs a penalty term that makes an allowance for the number of explanatory variables. However, adjusted R^2 is based upon a particular penalty function (that is, $T - k$ appears in a specific way in the formula). This form of penalty term may not necessarily be optimal. Also, given the statement above that adjusted R^2 is a soft rule, it is likely on balance that use of it to choose between models will imply that models with more explanatory variables are favoured. Several other similar rules are available, each having more or less strict penalty terms; these are collectively known as ‘information criteria’. These are explained in some detail in Chapter 6, but suffice to say for now that a different strictness of the penalty term will in many cases lead to a different preferred model.

An alternative approach to comparing between non-nested models would be to estimate an encompassing or hybrid model. In the case of equations (4.49) and (4.50), the relevant encompassing model would be

$$y_t = \gamma_1 + \gamma_2 x_{2t} + \gamma_3 x_{3t} + w_t \tag{4.51}$$

where w_t is an error term. Formulation (4.51) contains both equations (4.49) and (4.50) as special cases when γ_3 and γ_2 are zero, respectively. Therefore, a test for the best model would be conducted via an examination of the significances of γ_2 and γ_3 in model (4.51). There will be four possible outcomes (Box 4.2).

BOX 4.2 Selecting between models

(1) γ_2 is statistically significant but γ_3 is not. In this case, model (4.51) collapses to model (4.49), and the latter is the preferred model.

(2) γ_3 is statistically significant but γ_2 is not. In this case, model (4.51) collapses to model (4.50), and the latter is the preferred model.

(3) γ_2 and γ_3 are both statistically significant. This would imply that both x_2 and x_3 have incremental explanatory power for y, in which case both variables should be retained. Models (4.49) and (4.50) are both ditched and model (4.51) is the preferred model.

(4) Neither γ_2 nor γ_3 are statistically significant. In this case, none of the models can be dropped, and some other method for choosing between them must be employed.

However, there are several limitations to the use of encompassing regressions to select between non-nested models. Most importantly, even if models (4.49) and (4.50) have a strong theoretical basis for including the RHS variables that they do, the hybrid model may be meaningless. For example, it could be the case that financial theory suggests that y could either follow model (4.49) or model (4.50), but model (4.51) is implausible.

Also, if the competing explanatory variables x_2 and x_3 are highly related (i.e., they are near collinear), it could be the case that if they are both included, neither γ_2 nor γ_3 is statistically significant, while each is significant in their separate regressions (4.49) and (4.50); see the section on multicollinearity in Chapter 5.

An alternative approach is via the J-encompassing test due to Davidson and MacKinnon (1981). Interested readers are referred to their work or to Gujarati (2003, pp. 533–6) for further details.

4.10 Quantile Regression

4.10.1 Background and Motivation

Standard regression approaches effectively model the (conditional) mean of the dependent variable – that is, they capture the average value of *y* given the average values of all of the explanatory variables. We could of course calculate from the fitted regression line the value that *y* would take for any values of the explanatory variables, but this would essentially be an extrapolation of the behaviour of the relationship between *y* and *x* at the mean to the remainder of the data.

As a motivational example of why this approach will often be sub-optimal, suppose that it is of interest to capture the cross-sectional relationship across countries between the degree of regulation of banks and gross domestic product (GDP). Starting from a very low level of regulation (or no regulation), an increase in regulation is likely to encourage a rise in economic activity as the banking system functions better as a result of more trust and stability in the financial environment. However, there is likely to come a point where further increasing the amount of regulation may impede economic growth by stifling innovation and the responsiveness of the banking sector to the needs of the industries it serves. Thus we may think of there being a non-linear (∩-shaped) relationship between regulation and GDP growth, and estimating a standard linear regression model may lead to seriously misleading estimates of this relationship as it will 'average' the positive and negative effects from very low and very high regulation.

Of course, in this situation it would be possible to include non-linear (i.e., polynomial) terms in the regression model (for example, squared, cubic, … terms of regulation in the equation). But *quantile regressions*, developed by Koenker and Bassett (1978), represent a more natural and flexible way to capture the complexities inherent in the relationship by estimating models for the conditional quantile functions. Quantile regressions can be conducted in both time-series and cross-sectional contexts, although the latter are more common. It is usually assumed that the dependent variable, often called the *response variable* in the literature on quantile regressions, is independently distributed and homoscedastic; these assumptions can of course be relaxed but at the cost of additional complexity. Quantile regressions represent a comprehensive way to analyse the relationships between a set of variables, and are far more robust to outliers and non-normality than OLS regressions, in the same fashion that the median is often a better measure of average or 'typical' behaviour than the mean when the distribution is considerably skewed by a few large outliers. Quantile regression is a non-parametric technique since no distributional assumptions are required to optimally estimate the parameters.

The notation and approaches commonly used in quantile regression modelling are different to those that we are familiar with in financial econometrics, and this probably limited the early take up of the technique, which was historically more widely used in other disciplines. Numerous applications in labour economics were developed, for example. However, the more recent availability of the techniques in econometric software packages and increased interest in modelling the 'tail behaviour' of series have spurred applications of quantile regression in finance. The most common use of the technique here is to value at risk modelling. This seems natural given that the models are based on estimating the quantile of a distribution of possible losses – see, for example, the study by Chernozhukov and Umantsev (2001) and the development of the CaViaR model by Engle and Manganelli (2004).[1]

Quantiles, denoted τ, refer to the position where an observation falls within an ordered series for y – for example, the median is the observation in the very middle; the (lower) tenth percentile is the value that places 10% of observations below it (and therefore 90% of observations above), and so on. More precisely, we can define the τ-th quantile, $Q(\tau)$, of a random variable y having cumulative distribution $F(y)$ as

$$Q(\tau) = \inf y : F(y) \geq \tau \qquad \textbf{(4.52)}$$

where inf refers to the infimum, or the 'greatest lower bound' which is the smallest value of y satisfying the inequality. By definition, quantiles must lie between zero and one.

Quantile regressions take the concept of quantiles a stage further and effectively model the entire conditional distribution of y given the explanatory variables (rather than only the mean as is the case for OLS) – thus they examine their impact on not only the location and scale of the distribution of y, but also on the shape of the distribution as well. So we can determine how the explanatory variables affect the fifth or ninetieth percentiles of the distribution of y or its median and so on.

4.10.2 Estimation of Quantile Functions

In the same fashion as the OLS estimator finds the mean value that minimises the sum of the squared residuals, minimising the sum of the absolute values of the residuals will yield the median value. By definition, the absolute value function is symmetrical so that the median always has the same number of data points above it as below it. But if instead the absolute residuals are weighted differently depending on whether they are positive or negative, we can calculate the quantiles of the distribution. To estimate the τ-th quantile, we would set the weight on positive observations to τ, which is the quantile of interest, and that on negative observations to $1 - \tau$. We can select the quantiles of interest (or the software might do this for us), but common choices would be 0.05, 0.1, 0.25, 0.5, 0.75, 0.9, 0.95. The fit is not always good for values of τ too close to its limits of 0 and 1, so it is advisable to avoid such values.

We could write the minimisation problem for a set of quantile regression parameters $\hat{\beta}_\tau$, each element of which is a $k \times 1$ vector, as

$$\hat{\beta}_\tau = \text{argmin}_\beta \left(\sum_{i:y_i > \beta x_i} \tau |y_i - \beta x_i| + \sum_{i:y_i < \beta x_i} (1-\tau)|y_i - \beta x_i| \right) \quad \textbf{(4.53)}$$

This equation makes it clear where the weighting enters into the optimisation. As above, for the median, $\tau = 0.5$ and the weights are symmetric, but for all other quantiles they will be asymmetric. This optimisation problem can be solved using a linear programming representation via the simplex algorithm or it can be cast within the generalised method of moments (GMM) framework.

As an alternative to quantile regression, it would be tempting to think of partitioning the data and running separate regressions on each of them – for example, dropping the top 90% of the observations on y and the corresponding data points for the xs, and running a regression on the remainder. However, this process, tantamount to truncating the dependent variable, would be wholly inappropriate and could lead to potentially severe sample selection biases of the sort discussed in Chapter 12 and highlighted by Heckman (1976). In fact, quantile regression does not partition the data – all observations are used in the estimation of the parameters for every quantile.

It is quite useful to plot each of the estimated parameters, $\hat{\beta}_{i,\tau}$ (for $i = 1, \ldots, k$), against the quantile, τ (from 0 to 1) so that we can see whether the estimates vary across the quantiles or are roughly constant. Sometimes ±2 standard error bars are also included on the plot, and these tend to widen as the limits of τ are approached. Producing these standard errors for the quantile regression parameters is unfortunately more complex conceptually than estimating the parameters themselves and thus a discussion of these is beyond the scope of this book. Under some assumptions, Koenker (2005) demonstrates that the quantile regression parameters are asymptotically normally distributed. A number of approaches have been proposed for estimating the variance-covariance matrix of the parameters, including one based on a bootstrap – see Chapter 13 for a discussion of this.

4.10.3 An Application of Quantile Regression: Evaluating Fund Performance

A study by Bassett and Chen (2001) performs a style attribution analysis for a mutual fund and, for comparison, the S&P500 index. In order to examine how a portfolio's exposure to various styles varies with performance, they use a quantile regression approach.

Effectively evaluating the performance of mutual fund managers is made difficult by the observation that certain investment styles – notably, value and small cap – yield higher returns on average than the equity market as a whole. In response to this, factor models such as those of Fama and French (1993) have been employed to remove the impact of these characteristics – see Chapter 14 for a detailed presentation of these models. The use of such models also ensures that fund manager skill in picking highly performing stocks is not confused with randomly investing within value and small cap styles that will outperform the market in the long run. For example, if a manager invests a relatively high proportion of his portfolio in small firms, we would expect to observe higher returns than average from this manager because of the firm size effect alone.

Bassett and Chen (2001) conduct a style analysis in this spirit by regressing the returns of a fund on the returns of a large growth portfolio, the returns of a large value portfolio, the returns of a small growth portfolio, and the returns of a small value portfolio. These style portfolio returns are based on the Russell style indices. In this way, the parameter estimates on each of these style-mimicking portfolio returns will measure the extent to which the fund is exposed to that style. Thus we can determine the actual investment style of a fund without knowing anything about its holdings purely based on an analysis of its returns *ex post* and their relationships with the returns of style indices. Table 4.2 presents the results from a standard OLS regression and quintile regressions for $\tau = 0.1$, 0.3, 0.5 (i.e., the median), 0.7 and 0.9. The data are observed over the five years to December 1997 and the standard errors are based on a bootstrapping procedure.

Table 4.2 OLS and quantile regression results for the Magellan fund

	OLS	***Q*(0.1)**	***Q*(0.3)**	***Q*(0.5)**	***Q*(0.7)**	***Q*(0.9)**
Large growth	0.14 (0.15)	0.35 (0.31)	0.19 (0.22)	0.01 (0.16)	0.12 (0.20)	0.01 (0.22)
Large value	0.69 (0.20)	0.31 (0.38)	0.75 (0.30)	0.83 (0.25)	0.85 (0.30)	0.82 (0.36)
Small growth	0.21 (0.11)	–0.01 (0.15)	0.10 (0.16)	0.14 (0.17)	0.27 (0.17)	0.53 (0.15)
Small value	–0.03 (0.20)	0.31 (0.31)	0.08 (0.27)	0.07 (0.29)	–0.31 (0.32)	–0.51 (0.35)
Constant	–0.05 (0.25)	–1.90 (0.39)	–1.11 (0.27)	–0.30 (0.38)	0.89 (0.40)	2.31 (0.57)

Notes: Standard errors in parentheses.

Source: Bassett and Chen (2001). Reprinted with the permission of Springer-Verlag.

Notice that the sum of the style parameters for a given regression is always one (except for rounding errors). To conserve space, I only present the results for the Magellan active fund and not those for the S&P – the latter exhibit very little variation in the estimates across the quantiles. The OLS results (column 2) show that the mean return has by far its biggest exposure to large value stocks (and this parameter estimate is also statistically significant), but it also exposed to small growth and, to a lesser extent, large growth stocks. It is of interest to compare the mean (OLS) results with those for the median, $Q(0.5)$. The latter show much higher exposure to large value, less to small growth and none at all to large growth.

It is also of interest to examine the factor tilts as we move through the quantiles from left ($Q(0.1)$) to right ($Q(0.9)$). We can see that the loading on large growth monotonically falls from 0.31 at $Q(0.1)$ to 0.01 at $Q(0.9)$ while the loadings on large value and small growth substantially increase. The loading on small value falls from 0.31 at $Q(0.1)$ to -0.51 at $Q(0.9)$. A way to interpret (those of the current authors rather than those of Bassett and Chen) these results is to say that when the fund has historically performed poorly, this has resulted in equal amounts from its overweight exposure to large value and growth, and small growth. On the other hand, when it has historically performed well, this is a result of its exposure to large value and small growth but it was underweight small value stocks. Finally, it is obvious that the intercept (coefficient on the constant) estimates should be monotonically increasing from left to right since the quantile regression effectively sorts on average performance and the intercept can be interpreted as the performance expected if the fund had zero exposure to all of the styles.

KEY CONCEPTS

The key terms to be able to define and explain from this chapter are

- multiple regression model
- restricted regression
- residual sum of squares
- multiple hypothesis test
- R^2
- hedonic model
- data mining
- dummy variables
- variance-covariance matrix
- F-distribution
- total sum of squares
- non-nested hypotheses
- $\bar{R}^2$
- encompassing regression
- quantile regression
- qualitative data

Appendix 4.1 Mathematical Derivations of CLRM

Results

Derivation of the OLS Coefficient Estimator in the Multiple Regression Context

In the multiple regression context, in order to obtain the parameter estimates for $\beta_1, \beta_2, \ldots, \beta_k$, the *RSS* would be minimised with respect to all the elements of β. Now the residuals are expressed in a vector

$$\hat{u} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_T \end{bmatrix} \quad \textbf{(4A.1)}$$

The *RSS* is still the relevant loss function, and would be given in a matrix notation by expression (4A.2)

$$L = \hat{u}'\hat{u} = [\hat{u}_1 \hat{u}_2 \ldots \hat{u}_T] \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_T \end{bmatrix} = \hat{u}_1^2 + \hat{u}_2^2 + \cdots + \hat{u}_T^2 = \sum \hat{u}_t^2 \quad \textbf{(4A.2)}$$

Denoting the vector of estimated parameters as $\hat{\beta}$, it is also possible to write

$$L = \hat{u}'\hat{u} = (y - X\hat{\beta})'(y - X\hat{\beta}) = y'y - \hat{\beta}'X'y - y'X\hat{\beta} + \hat{\beta}'X'X\hat{\beta} \quad \textbf{(4A.3)}$$

It turns out that $\hat{\beta}'X'y$ is $(1 \times k) \times (k \times T) \times (T \times 1) = 1 \times 1$ and also that $y'X\hat{\beta}$ is $(1 \times T) \times (T \times k) \times (k \times 1) = 1 \times 1$, so in fact $\hat{\beta}'X'y = y'X\hat{\beta}$. Thus equation (4A.3) can be written

$$L = \hat{u}'\hat{u} = (y - X\hat{\beta})'(y - X\hat{\beta}) = y'y - 2\hat{\beta}'X'y + \hat{\beta}'X'X\hat{\beta} \quad \textbf{(4A.4)}$$

Differentiating this expression with respect to $\hat{\beta}$ and setting it to zero in order to find the parameter values that minimise the residual sum of squares would yield

$$\frac{\partial L}{\partial \hat{\beta}} = -2X'y + 2X'X\hat{\beta} = 0 \quad \textbf{(4A.5)}$$

This expression arises since the derivative of $y'y$ is zero with respect to $\hat{\beta}$, and $\hat{\beta}'X'X\hat{\beta}$ acts like a square of $X\hat{\beta}$, which is differentiated to $2X'X\hat{\beta}$. Rearranging expression (4A.5)

$$2X'y = 2X'X\hat{\beta} \quad \textbf{(4A.6)}$$

$$X'y = X'X\hat{\beta} \quad \textbf{(4A.7)}$$

Pre-multiplying both sides of (4A.7) by the inverse of $X'X$

$$\hat{\beta} = (X'X)^{-1}X'y \quad \textbf{(4A.8)}$$

Thus, the vector of OLS coefficient estimates for a set of k parameters is given by

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_k \end{bmatrix} = (X'X)^{-1}X'y \quad \textbf{(4A.9)}$$

Derivation of the OLS Standard Error Estimator in the Multiple Regression Context

The variance of a vector of random variables $\hat{\beta}$ is given by the formula $E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)']$. Since $y = X\beta + u$, it can also be stated, given (4A.9), that

$$\hat{\beta} = (X'X)^{-1}X'(X\beta + u) \quad \textbf{(4A.10)}$$

Expanding the parentheses

$$\hat{\beta} = (X'X)^{-1}X'X\beta + (X'X)^{-1}X'u \quad \textbf{(4A.11)}$$

$$\hat{\beta} = \beta + (X'X)^{-1}X'u \quad \textbf{(4A.12)}$$

Thus, it is possible to express the variance of $\hat{\beta}$ as

$$E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'] = E[(\beta + (X'X)^{-1}X'u - \beta)(\beta + (X'X)^{-1}X'u - \beta)'] \quad \textbf{(4A.13)}$$

Cancelling the β terms in each set of parentheses

$$E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'] = E[((X'X)^{-1}X'u)((X'X)^{-1}X'u)'] \quad \textbf{(4A.14)}$$

Expanding the parentheses on the RHS of (4A.14) gives

$$E[(\hat{\beta}-\beta)(\hat{\beta}-\beta)'] = E[(X'X)^{-1}X'uu'X(X'X)^{-1}] \quad \textbf{(4A.15)}$$

$$E[(\hat{\beta}-\beta)(\hat{\beta}-\beta)'] = (X'X)^{-1}X'E[uu']X(X'X)^{-1} \quad \textbf{(4A.16)}$$

Now $E[uu']$ is estimated by s^2I, so that

$$E[(\hat{\beta}-\beta)(\hat{\beta}-\beta)'] = (X'X)^{-1}X's^2IX(X'X)^{-1} \quad \textbf{(4A.17)}$$

where I is a $k \times k$ identity matrix. Rearranging further

$$E[(\hat{\beta}-\beta)(\hat{\beta}-\beta)'] = s^2(X'X)^{-1}X'X(X'X)^{-1} \quad \textbf{(4A.18)}$$

The $X'X$ and the last $(X'X)^{-1}$ term cancel out to leave

$$\text{var}(\hat{\beta}) = s^2(X'X)^{-1} \quad \textbf{(4A.19)}$$

as the expression for the parameter variance–covariance matrix. This quantity, $s^2(X'X)^{-1}$, is known as the estimated variance–covariance matrix of the coefficients. The leading diagonal terms give the estimated coefficient variances while the off-diagonal terms give the estimated covariances between the parameter estimates. The variance of $\hat{\beta}_1$ is the first diagonal element, the variance of $\hat{\beta}_2$ is the second element on the leading diagonal, …, and the variance of $\hat{\beta}_k$ is the kth diagonal element, etc. as discussed in the body of the chapter.

Appendix 4.2 A Brief Introduction to Factor Models and Principal Components Analysis

Factor models are employed primarily as dimensionality reduction techniques in situations where we have a large number of closely related variables and where we wish to allow for the most important influences from all of these variables at the same time. Factor models decompose the structure of a set of series into factors that are common to all series and a proportion that is specific to each series (idiosyncratic variation). There are broadly two types of such models, which can be loosely characterised as either macroeconomic or mathematical factor models. The key distinction between the two is that the factors are observable for the former but are latent (unobservable) for the latter. Observable factor models include the

APT model of Ross (1976). The most common mathematical factor model is principal components analysis (PCA). PCA is a technique that may be useful where explanatory variables are closely related – for example, in the context of near multicollinearity. Specifically, if there are k explanatory variables in the regression model, PCA will transform them into k uncorrelated new variables. To elucidate, suppose that the original explanatory variables are denoted $x_1, x_2, \ldots, x_k$, and denote the principal components by $p_1, p_2, \ldots, p_k$. These principal components are independent linear combinations of the original data

$$\begin{aligned} p_1 &= \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1k}x_k \\ p_2 &= \alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2k}x_k \\ \ldots & \quad \ldots \quad \ldots \quad \ldots \\ p_k &= \alpha_{k1}x_1 + \alpha_{k2}x_2 + \cdots + \alpha_{kk}x_k \end{aligned} \qquad \textbf{(4A.20)}$$

where α_{ij} are coefficients to be calculated, representing the coefficient on the jth explanatory variable in the ith principal component. These coefficients are also known as factor loadings. Note that there will be T observations on each principal component if there were T observations on each explanatory variable.

It is also required that the sum of the squares of the coefficients for each component is one, i.e.

$$\begin{aligned} \alpha_{11}^2 + \alpha_{12}^2 + \cdots + \alpha_{1k}^2 &= 1 \\ \vdots \qquad \vdots & \\ \alpha_{k1}^2 + \alpha_{k2}^2 + \cdots + \alpha_{kk}^2 &= 1 \end{aligned} \qquad \textbf{(4A.21)}$$

This requirement could also be expressed using sigma notation

$$\sum_{j=1}^{k} \alpha_{ij}^2 = 1 \quad \forall \quad i = 1, \ldots, k \qquad \textbf{(4A.22)}$$

Constructing the components is a purely mathematical exercise in constrained optimisation, and thus no assumption is made concerning the structure, distribution, or other properties of the variables.

The principal components are derived in such a way that they are in descending order of importance. Although there are k principal components, the same as the number of explanatory variables, if there is some collinearity between these original explanatory variables, it is likely that some of the (last few) principal components will account for so little of the variation that they can be discarded. However, if all of the original explanatory variables were already essentially uncorrelated, all of the components would be required, although in such a case there would have been little motivation for using PCA in the first place.

The principal components can also be understood as the eigenvalues of (*X′X*), where *X* is the matrix of observations on the original variables. Thus thenumber of eigenvalues will be equal to the number of variables, *k*. If the ordered eigenvalues are denoted λ_i ($i = 1, \ldots, k$), the ratio

$$\phi_i = \frac{\lambda_i}{\sum_{i=1}^{k} \lambda_i}$$

gives the proportion of the total variation in the original data explained by the principal component *i*. Suppose that only the first *r* ($0 < r < k$) principal components are deemed sufficiently useful in explaining the variation of (*X′X*), and that they are to be retained, with the remaining $k - r$ components being discarded. The regression finally estimated, after the principal components have been formed, would be one of *y* on the *r* principal components

$$y_t = \gamma_0 + \gamma_1 p_{1t} + \cdots + \gamma_r p_{rt} + u_t \qquad \textbf{(4A.23)}$$

In this way, the principal components are argued to keep most of the important information contained in the original explanatory variables, but are orthogonal. This may be particularly useful for independent variables that are very closely related. The principal component estimates $(\hat{\gamma}_i, i = 1, \ldots, r)$ will be biased estimates, although they will be more efficient than the OLS estimators since redundant information has been removed. In fact, if the OLS estimator for the original regression of *y* on *x* is denoted $\hat{\beta}$, it can be shown that

$$\hat{\gamma}_r = P'_r \hat{\beta} \qquad \textbf{(4A.24)}$$

where $\hat{\gamma}_r$ are the coefficient estimates for the principal components, and P_r is a matrix of the first *r* principal components. The principal component coefficient estimates are thus simply linear combinations of the original OLS estimates.

An Application of Principal Components to Interest Rates

Many economic and financial models make use of interest rates in some form or another as independent variables. Researchers may wish to include interest rates on a large number of different assets in order to reflect the variety of investment opportunities open to investors. However, market interest rates could be argued to be not sufficiently independent of one another to make the inclusion of several interest rate series in an econometric model statistically sensible. One approach to examining this issue would be to use PCA on several related interest rate series to determine whether they did move independently of one another over some historical time period or not.

Fase (1973) conducted such a study in the context of monthly Dutch market interest rates from January 1962 until December 1970 (108 months). Fase examined both 'money market' and 'capital market' rates, although only the money market results will be discussed here in the interests of brevity. The money market instruments investigated were

- Call money
- Three-month Treasury paper
- One-year Treasury paper
- Two-year Treasury paper
- Three-year Treasury paper
- Five-year Treasury paper
- Loans to local authorities: three-month
- Loans to local authorities: one-year
- Eurodollar deposits
- Netherlands Bank official discount rate.

Prior to analysis, each series was standardised to have zero mean and unit variance by subtracting the mean and dividing by the standard deviation in each case. The three largest of the ten eigenvalues are given in Table 4A.1.

Table 4A.1 Principal component ordered eigenvalues for Dutch interest rates, 1962–70

	Monthly data			Quarterly data
	Jan 62–Dec 70	Jan 62–Jun 66	Jul 66–Dec 70	Jan 62–Dec 70
λ_1	9.57	9.31	9.32	9.67
λ_2	0.20	0.31	0.40	0.16
λ_3	0.09	0.20	0.17	0.07
ϕ_1	95.7%	93.1%	93.2%	96.7%

Source: Fase (1973). Reprinted with the permission of Elsevier.

The results in Table 4A.1 are presented for the whole period using the monthly data, for two monthly sub-samples, and for the whole period using data sampled quarterly instead of monthly. The results show clearly that the first principal component is sufficient to describe the common variation in these Dutch interest rate series. The first component is able to explain over 90% of the variation in all four cases, as given in the last row of Table 4A.1. Clearly, the estimated eigenvalues are fairly stable across the sample periods and are relatively invariant to the frequency of sampling of the data. The factor loadings (coefficient estimates) for the first two ordered components are given in Table 4A.2.

Table 4A.2 Factor loadings of the first and second principal components for Dutch interest rates, 1962–70

j	Debt instrument	α_{j1}	α_{j2}
1	Call money	0.95	−0.22
2	3-month Treasury paper	0.98	0.12
3	1-year Treasury paper	0.99	0.15
4	2-year Treasury paper	0.99	0.13
5	3-year Treasury paper	0.99	0.11
6	5-year Treasury paper	0.99	0.09
7	Loans to local authorities: 3-month	0.99	−0.08
8	Loans to local authorities: 1-year	0.99	−0.04
9	Eurodollar deposits	0.96	−0.26
10	Netherlands Bank official discount rate	0.96	−0.03
	Eigenvalue, λ_i	9.57	0.20
	Proportion of variability explained by eigenvalue i, ϕ_i(%)	95.7	2.0

Source: Fase (1973). Reprinted with the permission of Elsevier.

As Table 4A.2 shows, the loadings on each factor making up the first principal component are all positive. Since each series has been standardised to have zero mean and unit variance, the coefficients α_{j1} and α_{j2} can be interpreted as the correlations between the interest rate j and the first and second principal components, respectively. The factor loadings for each interest rate series on the first component are all very close to one. Fase (1973) therefore argues that the first component can be interpreted simply as an equally weighted combination of all of the market interest rates. The second component, which explains much less of the variability of the rates, shows a factor loading pattern of positive coefficients for the Treasury paper series and negative or almost zero values for the other series. Fase (1973) argues that this is owing to the characteristics of the Dutch Treasury instruments that they rarely change hands and have low transactions costs, and therefore have less sensitivity to general interest rate movements. Also, they are not subject to default risks in the same way as, for example Eurodollar deposits. Therefore, the second principal component is broadly interpreted as relating to default risk and transactions costs.

Principal components can be useful in some circumstances, although the technique has limited applicability for the following reasons

- A change in the units of measurement of x will change the principal components. It is thus usual to transform all of the variables to have zero mean and unit variance prior to applying PCA.
- The principal components usually have no theoretical motivation or interpretation whatsoever.
- The r principal components retained from the original k are the ones that explain most of the variation in x, but these components might not be the most useful as explanations for y.

SELF-STUDY QUESTIONS

1. By using examples from the relevant statistical tables, explain the relationship between the t- and the F-distributions.

For questions 2–5, assume that the econometric model is of the form

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 x_{4t} + \beta_5 x_{5t} + u_t \quad \textbf{(4.54)}$$

2. Which of the following hypotheses about the coefficients can be tested using a *t*-test? Which of them can be tested using an *F*-test? In each case, state the number of restrictions.

(a) $H_0 : \beta_3 = 2$

(b) $H_0 : \beta_3 + \beta_4 = 1$

(c) $H_0 : \beta_3 + \beta_4 = 1$ and $\beta_5 = 1$

(d) $H_0 : \beta_2 = 0$ and $\beta_3 = 0$ and $\beta_4 = 0$ and $\beta_5 = 0$

(e) $H_0 : \beta_2\beta_3 = 1$

3. Which of the above null hypotheses constitutes 'THE' regression *F*-statistic in the context of equation (4.54)? Why is this null hypothesis always of interest whatever the regression relationship under study? What exactly would constitute the alternative hypothesis in this case?

4. Which would you expect to be bigger – the unrestricted residual sum of squares or the restricted residual sum of squares, and why?

5. You decide to investigate the relationship given in the null hypothesis of Question 2, part (c). What would constitute the restricted regression? The regressions are carried out on a sample of 96 quarterly observations, and the residual sums of squares for the restricted and unrestricted regressions are 102.87 and 91.41, respectively. Perform the test. What is your conclusion?

6. You estimate a regression of the form given by equation (4.55) below in order to evaluate the effect of various firm-specific factors on the returns of a sample of firms. You run a cross-sectional regression with 200 firms

$$r_i = \beta_0 + \beta_1 S_i + \beta_2 MB_i + \beta_3 PE_i + \beta_4 BETA_i + u_i \quad \textbf{(4.55)}$$

where: r_i is the percentage annual return for the stock

S_i is the size of firm i measured in terms of sales revenue

MB_i is the market to book ratio of the firm

PE_i is the price/earnings (P/E) ratio of the firm

$BETA_i$ is the stock's CAPM beta coefficient

You obtain the following results (with standard errors in parentheses)

$$\hat{r}_i = 0.080 + 0.801S_i + 0.321MB_i + 0.164PE_i - 0.084BETA_i \quad (4.56)$$
$$(0.064) \quad (0.147) \quad (0.136) \quad (0.420) \quad (0.120)$$

Calculate the t-ratios. What do you conclude about the effect of each variable on the returns of the security? On the basis of your results, what variables would you consider deleting from the regression? If a stock's beta increased from 1 to 1.2, what would be the expected effect on the stock's return? Is the sign on beta as you would have expected? Explain your answers in each case.

7. A researcher estimates the following econometric models including a lagged dependent variable

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 y_{t-1} + u_t \quad \textbf{(4.57)}$$

$$\Delta y_t = \gamma_1 + \gamma_2 x_{2t} + \gamma_3 x_{3t} + \gamma_4 y_{t-1} + v_t \quad \textbf{(4.58)}$$

where u_t and v_t are iid disturbances.

Will these models have the same value of (a) The residual sum of squares (RSS), (b) R^2, (c) Adjusted R^2? Explain your answers in each case.

8. A researcher estimates the following two econometric models

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + u_t \quad \textbf{(4.59)}$$

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 x_{4t} + v_t \quad \textbf{(4.60)}$$

where u_t and v_t are iid disturbances and x_{3t} is an irrelevant variable which does not enter into the data generating process for y_t. Will the value of (a) R^2, (b) Adjusted R^2, be higher for the second model than the first? Explain your answers.

9. What are the units of R^2?

10. What are quantile regressions and why are they useful?

11. A researcher wishes to examine the link between the returns on two assets A and B in situations where the price of B is falling rapidly. To do this, he orders the data according to changes in the price of B and drops the top 80% of ordered observations. He then runs a regression of the returns of A on the returns of B for the remaining lowest 20% of observations. Would this be a good way to proceed? Explain your answer.

[1] For further reading on quantile regression, Koenker and Hallock (2001) represents a very accessible, albeit brief, introduction to quantile regressions and their applications. A more thorough treatment is given in the book by Koenker (2005).

5

Classical Linear Regression Model Assumptions and Diagnostic Tests

LEARNING OUTCOMES

In this chapter, you will learn how to

- Describe the steps involved in testing regression residuals for heteroscedasticity and autocorrelation
- Explain the impact of heteroscedasticity or autocorrelation on the optimality of OLS parameter and standard error estimation
- Distinguish between the Durbin–Watson and Breusch–Godfrey tests for autocorrelation
- Highlight the advantages and disadvantages of dynamic models
- Test for whether the functional form of the model employed is appropriate
- Determine whether the residual distribution from a regression differs significantly from normality
- Investigate whether the model parameters are stable
- Appraise different philosophies of how to build an econometric model

5.1 Introduction

Recall from Chapter 3 that five assumptions were made relating to the classical linear regression model (CLRM). These were required to show that the estimation technique, ordinary least squares (OLS), had a number of desirable properties, and also so that hypothesis tests regarding the coefficient estimates could validly be conducted. Specifically, it was assumed that

(1) $E(u_t) = 0$

(2) $var(u_t) = \sigma^2 < \infty$

(3) $cov(u_i, u_j) = 0$

(4) $cov(u_t, x_t) = 0$

(5) $u_t \sim N(0, \sigma^2)$

These assumptions will now be studied further, in particular looking at the following

- How can violations of the assumptions be detected?
- What are the most likely causes of the violations in practice?
- What are the consequences for the model if an assumption is violated but this fact is ignored and the researcher proceeds regardless?

The answer to the last of these questions is that, in general, the model could encounter any combination of three problems

- The coefficient estimates $(\hat{\beta}s)$ are wrong
- The associated standard errors are wrong
- The distributions that were assumed for the test statistics are inappropriate.

A pragmatic approach to 'solving' problems associated with the use of models where one or more of the assumptions is not supported by the data will then be adopted. Such solutions usually operate such that

- The assumptions are no longer violated, or
- The problems are side-stepped, so that alternative techniques are used which are still valid.

5.2 Statistical Distributions for Diagnostic Tests

The text below discusses various regression diagnostic (misspecification) tests that are based on the calculation of a test statistic. These tests can be constructed in several ways, and the precise approach to constructing the test statistic will determine the distribution that the test statistic is assumed to follow. Two particular approaches are in common usage and their results are given by the statistical packages: the Lagrange Multiplier (LM) test and the Wald test. Further details concerning these procedures are given in Chapter 9. For now, all that readers require to know is that LM test statistics in the context of the diagnostic tests presented here follow a χ^2 distribution with degrees of freedom equal to the number of restrictions placed on the model, and denoted *m*. The Wald version of the test follows an *F*-distribution with $(m, T - k)$ degrees of freedom. Asymptotically, these two tests are equivalent, although their results will differ somewhat in small samples. They are equivalent as the sample size increases towards infinity since there is a direct relationship between the χ^2- and *F*-distributions. Asymptotically, an *F*-variate will tend towards a χ^2 variate divided by its degrees of freedom

$$F(m, T-k) \rightarrow \frac{\chi^2(m)}{m} \quad \text{as} \quad T \rightarrow \infty$$

Computer packages typically present results using both approaches, although only one of the two will be illustrated for each test below. They will usually give the same conclusion, although if they do not, the *F*-version is usually considered preferable for finite samples, since it is sensitive to sample size (one of its degrees of freedom parameters depends on sample size) in a way that the χ^2-version is not.

5.3 Assumption (1): $E(u_t) = 0$

The first assumption required is that the average value of the errors is zero. In fact, if a constant term is included in the regression equation, this assumption will never be violated. But what if financial theory suggests that, for a particular application, there should be no intercept so that the regression line is forced through the origin? If the regression did not include an intercept, and the average value of the errors was non-zero, several undesirable consequences could arise. First, R^2, defined as ESS/TSS can be negative, implying that the sample average, $\bar{y}$, 'explains' more of the variation in *y* than the explanatory variables. Second, and more fundamentally, a regression with no intercept parameter could lead to potentially severe biases in the slope coefficient estimates. To see this, consider Figure 5.1.

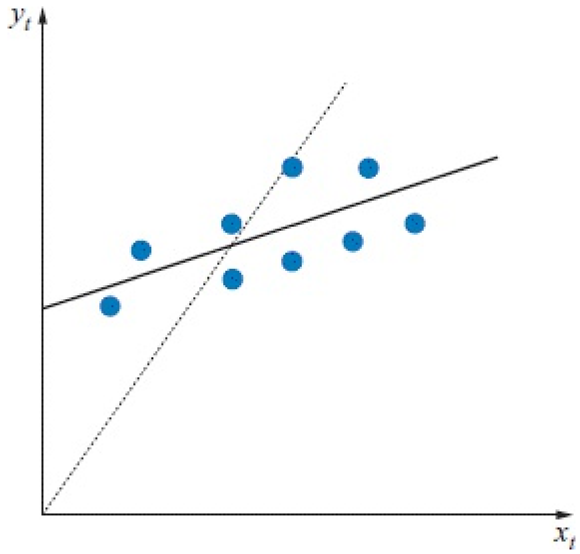


Figure 5.1 Effect of no intercept on a regression line

The solid line shows the regression estimated including a constant term, while the dotted line shows the effect of suppressing (i.e., setting to zero) the constant term. The effect is that the estimated line in this case is forced through the origin, so that the estimate of the slope coefficient $(\hat{\beta})$ is biased. Additionally, R^2 and $\bar{R}^2$ are usually meaningless in such a context. This arises since the mean value of the dependent variable, $\bar{y}$, will not be equal to the mean of the fitted values from the model, i.e., the mean of $\hat{y}$ if there is no constant in the regression.

5.4 Assumption (2): var(u_t) = $\sigma^2 < \infty$

It has been assumed thus far that the variance of the errors is constant, σ^2 – this is known as the *assumption of homoscedasticity*. If the errors do not have a constant variance, they are said to be *heteroscedastic*. To consider one illustration of heteroscedasticity, suppose that a regression had been estimated and the residuals, $\hat{u}_t$, have been calculated and then plotted against one of the explanatory variables, x_{2t}, as shown in Figure 5.2.

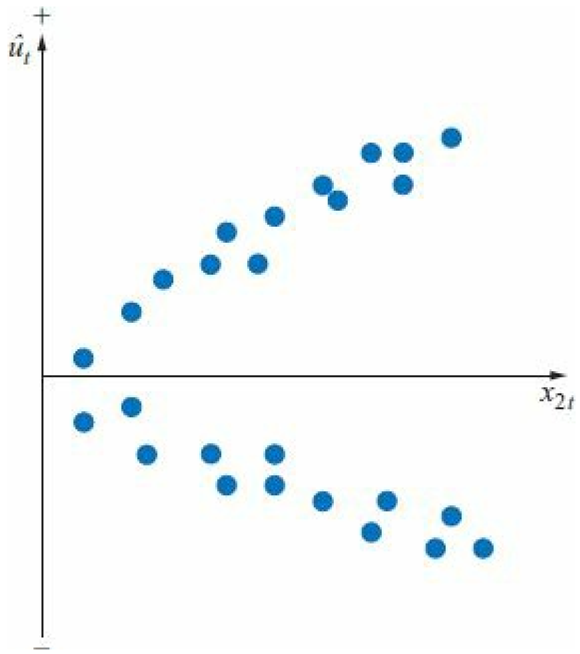


Figure 5.2 Graphical illustration of heteroscedasticity

It is clearly evident that the errors in figure 5.2 are heteroscedastic – that is, although their mean value is roughly constant, their variance is increasing systematically with x_{2t}.

5.4.1 Detection of Heteroscedasticity

How can one tell whether the errors are heteroscedastic or not? It is possible to use a graphical method as above, but unfortunately one rarely knows the cause or the form of the heteroscedasticity, so that a plot is likely to reveal nothing. For example, if the variance of the errors was an increasing function of x_{3t}, and the researcher had plotted the residuals against x_{2t}, he would be unlikely to see any pattern and would thus wrongly conclude that the errors had constant variance. It is also possible that the variance of the errors changes over time rather than systematically with one of the explanatory variables; this phenomenon is known as 'ARCH' and is described in Chapter 9.

Fortunately, there are a number of formal statistical tests for heteroscedasticity, and one of the simplest such methods is the Goldfeld–Quandt (1965) test. Their approach is based on splitting the total sample of length T into two sub-samples of length T_1 and T_2. The regression model is estimated on each sub-sample and the two residual variances are calculated as $s_1^2 = \hat{u}_1'\hat{u}_1/(T_1 - k)$ and $s_2^2 = \hat{u}_2'\hat{u}_2/(T_2 - k)$, respectively. The null hypothesis is that the variances of the disturbances are equal, which can be written

H_0: $\sigma_1^2 = \sigma_2^2$, against a two-sided alternative. The test statistic, denoted *GQ*, is simply the ratio of the two residual variances where the larger of the two variances must be placed in the numerator (i.e., s_1^2 is the higher sample variance for the sample with length T_1, even if it comes from the second sub-sample)

$$GQ = \frac{s_1^2}{s_2^2} \tag{5.1}$$

The test statistic is distributed as an $F(T_1 - k, T_2 - k)$ under the null hypothesis, and the null of a constant variance is rejected if the test statistic exceeds the critical value.

The *GQ* test is simple to construct but its conclusions may be contingent upon a particular, and probably arbitrary, choice of where to split the sample. Clearly, the test is likely to be more powerful when this choice is made on theoretical grounds – for example, before and after a major structural event. Suppose that it is thought that the variance of the disturbances is related to some observable variable z_t (which may or may not be one of the regressors). A better way to perform the test would be to order the sample according to values of z_t (rather than through time) and then to split the reordered sample into T_1 and T_2.

An alternative method that is sometimes used to sharpen the inferences from the test and to increase its power is to omit some of the observations from the centre of the sample so as to introduce a degree of separation between the two sub-samples.

A further popular test is White's (1980) general test for heteroscedasticity. The test is particularly useful because it makes few assumptions about the likely form of the heteroscedasticity. The test is carried out as in Box 5.1.

BOX 5.1 Conducting White's test

(1) Assume that the regression model estimated is of the standard linear form, e.g.

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + u_t \tag{5.2}$$

To test var(u_t) = σ^2, estimate the model above, obtaining the residuals, $\hat{u}_t$.

(2) Then run the auxiliary regression

$$\hat{u}_t^2 = \alpha_1 + \alpha_2 x_{2t} + \alpha_3 x_{3t} + \alpha_4 x_{2t}^2 + \alpha_5 x_{3t}^2 + \alpha_6 x_{2t} x_{3t} + v_t \quad \textbf{(5.3)}$$

where v_t is a normally distributed disturbance term independent of u_t. This regression is of the squared residuals on a constant, the original explanatory variables, the squares of the explanatory variables and their cross-products. To see why the squared residuals are the quantity of interest, recall that for a random variable u_t, the variance can be written

$$\text{var}(u_t) = \text{E}[(u_t - \text{E}(u_t))^2] \quad \textbf{(5.4)}$$

Under the assumption that E(u_t) = 0, the second part of the RHS of this expression disappears

$$\text{var}(u_t) = \text{E}\left[u_t^2\right] \quad \textbf{(5.5)}$$

Once again, it is not possible to know the squares of the population disturbances, u_t^2, so their sample counterparts, the squared residuals, are used instead.

The reason that the auxiliary regression takes this form is that it is desirable to investigate whether the variance of the residuals (embodied in $\hat{u}_t^2$) varies systematically with any known variables relevant to the model. Relevant variables will include the original explanatory variables, their squared values and their cross-products. Note also that this regression should include a constant term, even if the original regression did not. This is as a result of the fact that $\hat{u}_t^2$ will always have a non-zero mean, even if $\hat{u}_t$ has a zero mean.

(3) Given the auxiliary regression, as stated above, the test can be conducted using two different approaches. First, it is possible to use the *F*-test framework described in Chapter 4. This would involve estimating model (5.3) as the unrestricted regression and then running a restricted regression of $\hat{u}_t^2$ on a constant only. The *RSS* from each specification would then be used as inputs to the standard *F*-test formula.

With many diagnostic tests, an alternative approach can be adopted that does not require the estimation of a second (restricted) regression. This approach is known as a Lagrange Multiplier (LM) test, which centres around the value of R^2 for the auxiliary regression. If one or more coefficients in model (5.3) is statistically significant, the value of R^2 for that equation will be relatively high, while if none of the variables is significant, R^2 will be relatively low. The LM test would thus operate by obtaining R^2 from the auxiliary regression and multiplying it by the number of observations, T. It can be shown that

$$TR^2 \sim \chi^2(m)$$

where m is the number of regressors in the auxiliary regression (excluding the constant term), equivalent to the number of restrictions that would have to be placed under the F-test approach.

(4) The test is one of the joint null hypothesis that $\alpha_2 = 0$, and $\alpha_3 = 0$, and $\alpha_4 = 0$, and $\alpha_5 = 0$, and $\alpha_6 = 0$. For the LM test, if the χ^2-test statistic from step (3) is greater than the corresponding value from the statistical table then reject the null hypothesis that the errors are homoscedastic.

5.4.2 Consequences of Using OLS in the Presence of Heteroscedasticity

What happens if the errors are heteroscedastic, but this fact is ignored and the researcher proceeds with estimation and inference? In this case, OLS estimators will still give unbiased (and also consistent) coefficient estimates, but they are no longer best linear unbiased estimators (BLUE) – that is, they no longer have the minimum variance among the class of unbiased estimators. The reason is that the error variance, σ^2, plays no part in the proof that the OLS estimator is consistent and unbiased, but σ^2 does appear in the formulae for the coefficient variances. If the errors are heteroscedastic, the formulae presented for the coefficient standard errors no longer hold. For a very accessible algebraic treatment of the consequences of heteroscedasticity, see Hill, Griffiths and Judge (1997, pp. 217–18).

EXAMPLE 5.1

Suppose that the model (5.2) above has been estimated using 120 observations, and the R^2 from the auxiliary regression (5.3) is 0.234. The test statistic will be given by $TR^2 = 120 \times 0.234 = 28.8$, which will follow a $\chi^2(5)$ under the null hypothesis. The 5% critical value from the χ^2 table is 11.07. The test statistic is therefore more than the critical value and hence the null hypothesis is rejected. It would be concluded that there is significant evidence of heteroscedasticity, so that it would not be plausible to assume that the variance of the errors is constant in this case.

So, the upshot is that if OLS is still used in the presence of heteroscedasticity, the standard errors could be wrong and hence any inferences made could be misleading. In general, the OLS standard errors will be too large for the intercept when the errors are heteroscedastic. The effect of heteroscedasticity on the slope standard errors will depend on its form. For example, if the variance of the errors is positively related to the square of an explanatory variable (which is often the case in practice), the OLS standard error for the slope will be too low. On the other hand, the OLS slope standard errors will be too big when the variance of the errors is inversely related to an explanatory variable.

5.4.3 Dealing with Heteroscedasticity

If the form (i.e., the cause) of the heteroscedasticity is known, then an alternative estimation method which takes this into account can be used. One possibility is called generalised least squares (GLS). For example, suppose that the error variance was related to z_t by the expression

$$\text{var}(u_t) = \sigma^2 z_t^2 \tag{5.6}$$

All that would be required to remove the heteroscedasticity would be to divide the regression equation through by z_t

$$\frac{y_t}{z_t} = \beta_1 \frac{1}{z_t} + \beta_2 \frac{x_{2t}}{z_t} + \beta_3 \frac{x_{3t}}{z_t} + v_t \tag{5.7}$$

where $v_t = \frac{u_t}{z_t}$ is an error term.

Now, if $\text{var}(u_t) = \sigma^2 z_t^2$, $\text{var}(v_t) = \text{var}\left(\frac{u_t}{z_t}\right) = \frac{\text{var}(u_t)}{z_t^2} = \frac{\sigma^2 z_t^2}{z_t^2} = \sigma^2$ for known z.

Therefore, the disturbances from equation (5.7) will be homoscedastic. Note that this latter regression does not include a constant since β_1 is multiplied by $(1/z_t)$. GLS can be viewed as OLS applied to transformed data that satisfy the OLS assumptions. GLS is also known as weighted least squares (WLS), since under GLS a weighted sum of the squared residuals is minimised, whereas under OLS it is an unweighted sum.

However, researchers are typically unsure of the exact cause of the heteroscedasticity, and hence this technique is usually infeasible in practice. Two other possible 'solutions' for heteroscedasticity are shown in Box 5.2.

BOX 5.2 'Solutions' for Heteroscedasticity

(1) *Transforming the variables into logs or reducing by some other measure of 'size'*. It is sometimes said that a log transform is appropriate when its standard deviation is proportional to its mean. This has the effect of rescaling the data to 'pull in' extreme observations. The regression would then be conducted upon the natural logarithms or the transformed data. Taking logarithms also has the effect of making a multiplicative model, such as the exponential regression model discussed previously (with a multiplicative error term), into an additive one. However, logarithms of a variable cannot be taken in situations where the variable can take on zero or negative values, for the log will not be defined in such cases.

(2) *Using heteroscedasticity-consistent standard error estimates*. Most standard econometrics software packages have an option (usually called something like 'robust') that allows the user to employ standard error estimates that have been modified to account for the heteroscedasticity following White (1980). The effect of using the correction is that, if the variance of the errors is positively related to the square of an explanatory variable, the standard errors for the slope coefficients are increased relative to the usual OLS standard errors, which would make hypothesis testing more 'conservative', so that more evidence would be required against the null hypothesis before it would be rejected.

Examples of tests for heteroscedasticity in the context of the single index market model are given in Fabozzi and Francis (1980). Their results are strongly suggestive of the presence of heteroscedasticity, and they examine various factors that may constitute the form of the heteroscedasticity.

5.5 Assumption (3): cov(u_i, u_j) = 0 for $i \neq j$

Assumption 3 that is made of the CLRM's disturbance terms is that the covariance between the error terms over time (or cross-sectionally, for that type of data) is zero. In other words, it is assumed that the errors are uncorrelated with one another. If the errors are not uncorrelated with one another, it would be stated that they are 'autocorrelated' or that they are 'serially correlated'. A test of this assumption is therefore required.

Again, the population disturbances cannot be observed, so tests for autocorrelation are conducted on the residuals, $\hat{u}$. Before one can proceed to see how formal tests for autocorrelation are formulated, the concept of the lagged value of a variable needs to be defined.

5.5.1 The Concept of a Lagged Value

The lagged value of a variable (which may be y_t, x_t, or u_t) is simply the value that the variable took during a previous period. So for example, the value of y_t lagged one period, written y_{t-1}, can be constructed by shifting all of the observations forward one period in a spreadsheet, as illustrated in Table 5.1.

Table 5.1 Constructing a series of lagged values and first differences

t	y_t	y_{t-1}	Δy_t
2006M09	0.8	−	−
2006$M$10	1.3	0.8	$(1.3 - 0.8) = 0.5$
2006$M$11	−0.9	1.3	$(-0.9 - 1.3) = -2.2$
2006$M$12	0.2	−0.9	$(0.2 - -0.9) = 1.1$
2007M01	−1.7	0.2	$(-1.7 -0.2) = -1.9$
2007M02	2.3	−1.7	$(2.3 - -1.7) = 4.0$
2007M03	0.1	2.3	$(0.1 - 2.3) = -2.2$
2007M04	0.0	0.1	$(0.0 - 0.1) = -0.1$
.	.	.	.
.	.	.	.
.	.	.	.

So, the value in the 2006$M$10 row and the y_{t-1} column shows the value that y_t took in the previous period, 2006M09, which was 0.8. The last column in Table 5.1 shows another quantity relating to y, namely the 'first difference'. The first difference of y, also known as the change in y, and denoted Δy_t, is calculated as the difference between the values of y in this period and in the previous period. This is calculated as

$$\Delta y_t = y_t - y_{t-1} \quad \textbf{(5.8)}$$

Note that when one-period lags or first differences of a variable are constructed, the first observation is lost. Thus a regression of Δy_t using the above data would begin with the October 2006 data point. It is also possible to produce two-period lags, three-period lags and so on. These would be accomplished in the obvious way.

5.5.2 Graphical Tests for Autocorrelation

In order to test for autocorrelation, it is necessary to investigate whether any relationships exist between the current value of $\hat{u}$, $\hat{u}_t$, and any of its previous values, $\hat{u}_{t-1}$, $\hat{u}_{t-2}$, ...The first step is to consider possible relationships between the current residual and the immediately previous one, $\hat{u}_{t-1}$, via a graphical exploration. Thus $\hat{u}_t$ is plotted against $\hat{u}_{t-1}$, and $\hat{u}_t$ is plotted over time. Some stereotypical patterns that may be found in the residuals are discussed below.

Figures 5.3 and 5.4 show positive autocorrelation in the residuals, which is indicated by a cyclical residual plot over time. This case is known as *positive autocorrelation* since on average if the residual at time $t - 1$ is positive, the residual at time t is likely to be also positive; similarly, if the residual at $t - 1$ is negative, the residual at t is also likely to be negative. Figure 5.3 shows that most of the dots representing observations are in the first and third quadrants, while Figure 5.4 shows that a positively autocorrelated series of residuals will not cross the time-axis very frequently.

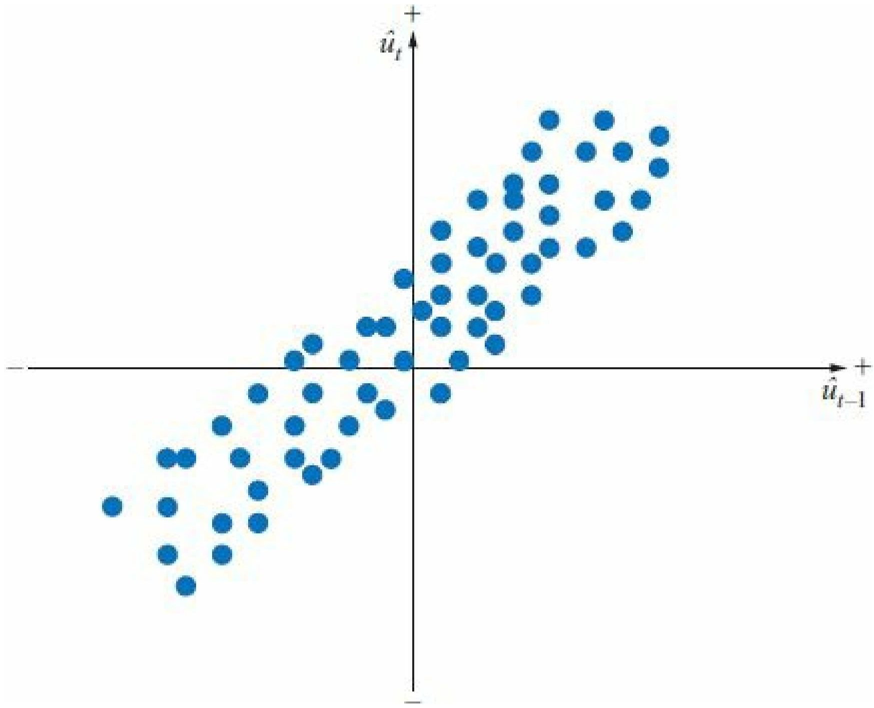


Figure 5.3 Plot of $\hat{u}_t$ against $\hat{u}_{t-1}$, showing positive autocorrelation

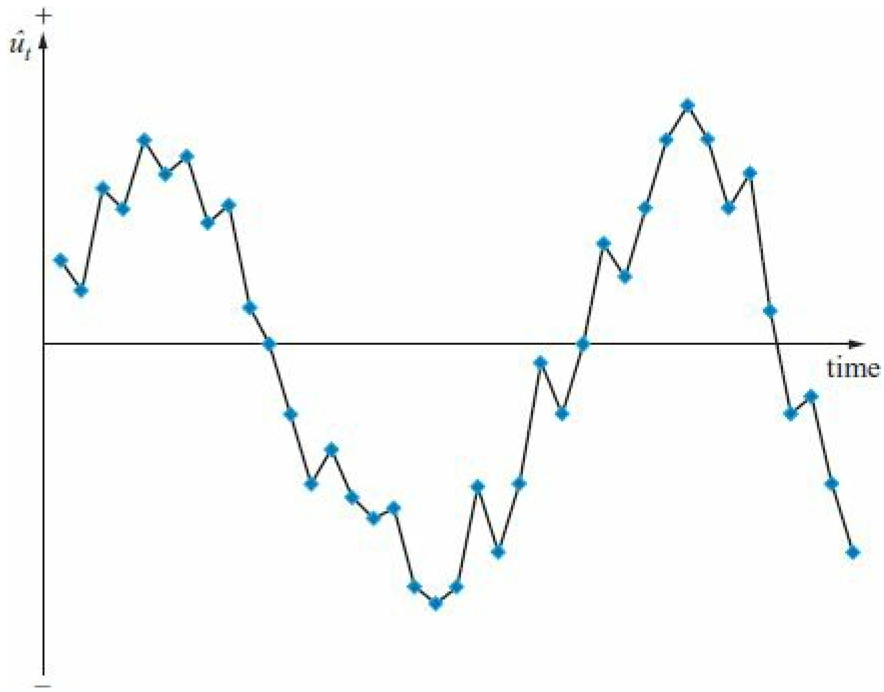


Figure 5.4 Plot of $\hat{u}_t$ over time, showing positive autocorrelation

Figures 5.5 and 5.6 show negative autocorrelation, indicated by an alternating pattern in the residuals. This case is known as negative autocorrelation since on average if the residual at time $t - 1$ is positive, the residual at time t is likely to be negative; similarly, if the residual at $t - 1$ is negative, the residual at t is likely to be positive. Figure 5.5 shows that most of the dots are in the second and fourth quadrants, while Figure 5.6 shows that a negatively autocorrelated series of residuals will cross the time-axis more frequently than if they were distributed randomly.

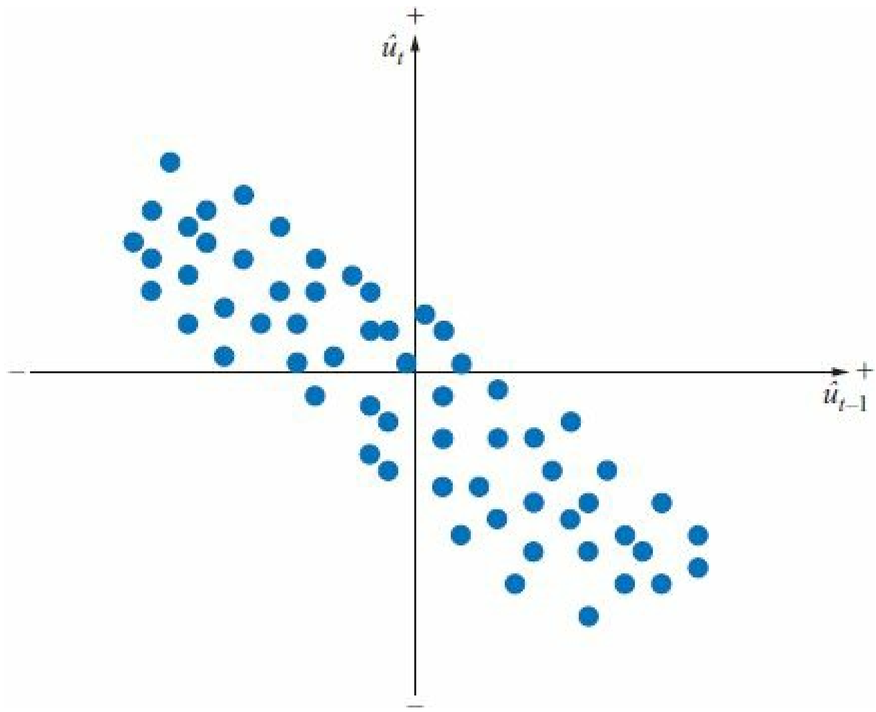


Figure 5.5 Plot of $\hat{u}_t$ against $\hat{u}_{t-1}$, showing negative autocorrelation

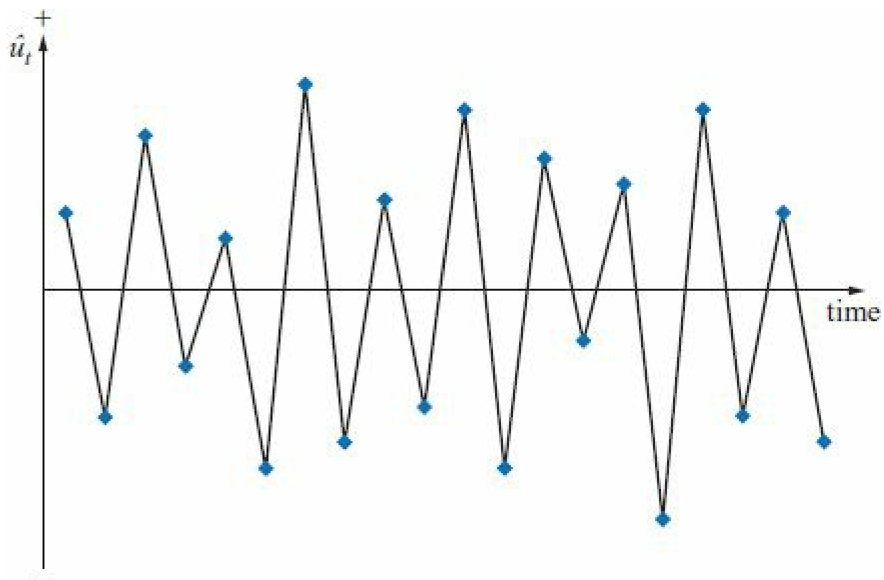


Figure 5.6 Plot of $\hat{u}_t$ over time, showing negative autocorrelation

Finally, Figures 5.7 and 5.8 show no pattern in the residuals at all: this is what is desirable to see. In the plot of $\hat{u}_t$ against $\hat{u}_{t-1}$ Figure 5.7), the points are randomly spread across all four quadrants, and the time-series plot of the residuals (Figure 5.8) does not cross the *x*-axis either too frequently or too little.

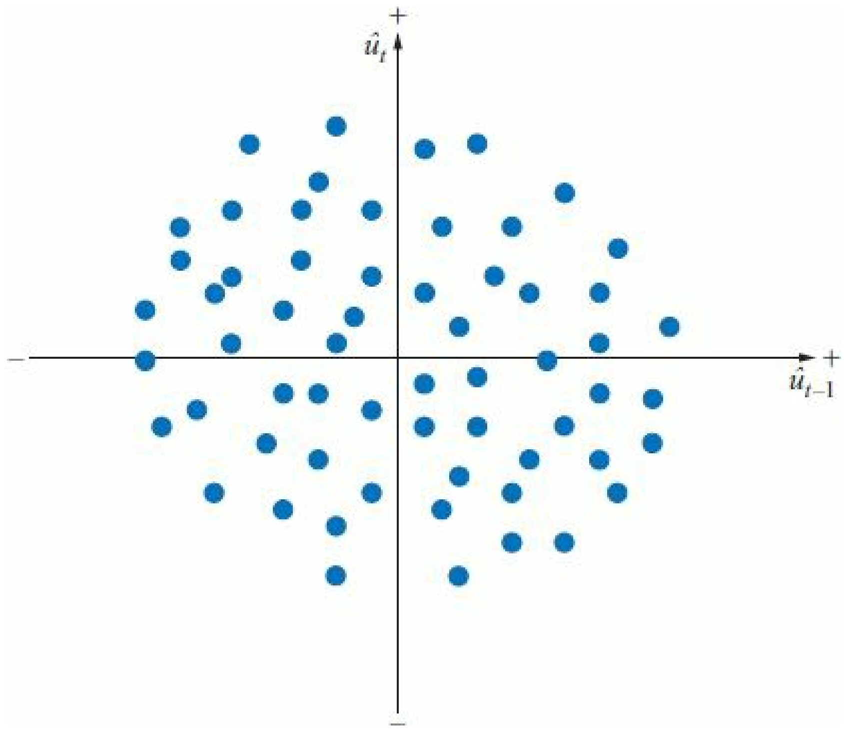


Figure 5.7 Plot of $\hat{u}_t$ against $\hat{u}_{t-1}$, showing no autocorrelation

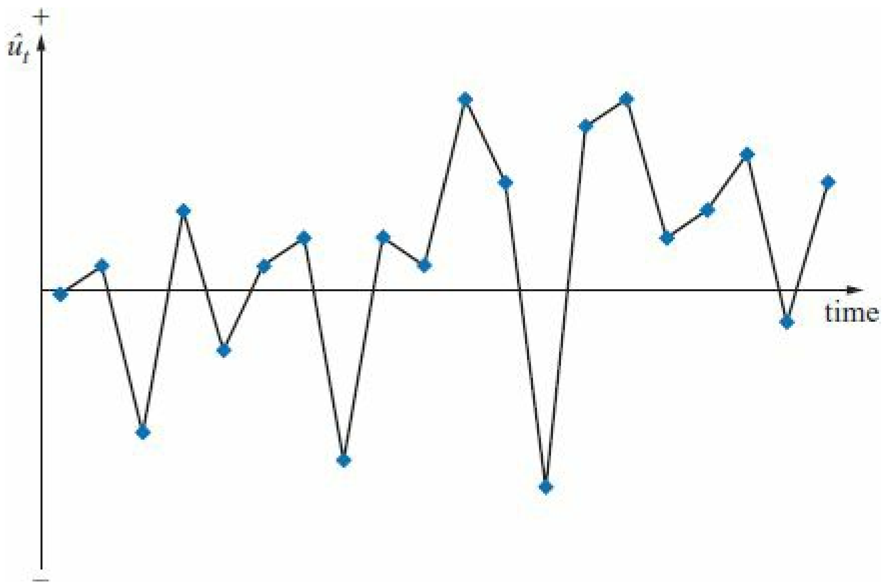


Figure 5.8 Plot of $\hat{u}_t$ over time, showing no autocorrelation

5.5.3 Detecting Autocorrelation: The Durbin–Watson Test

Of course, a first step in testing whether the residual series from an estimated model are autocorrelated would be to plot the residuals as above, looking for any patterns. Graphical methods may be difficult to interpret in practice, however, and hence a formal statistical test should also be applied. The simplest test is due to Durbin and Watson (1951).

Durbin–Watson (*DW*) is a test for first order autocorrelation – i.e., it tests only for a relationship between an error and its immediately previous value. One way to motivate the test and to interpret the test statistic would be in the context of a regression of the time *t* error on its previous value

$$u_t = \rho u_{t-1} + v_t \tag{5.9}$$

where $v_t \sim N(0, \sigma_v^2)$. The *DW* test statistic has as its null and alternative hypotheses

$$H_0 : \rho = 0 \quad \text{and} \quad H_1 : \rho \neq 0$$

Thus, under the null hypothesis, the errors at time $t - 1$ and t are independent of one another, and if this null were rejected, it would be concluded that there was evidence of a relationship between successive residuals. In fact, it is not necessary to run the regression given by equation (5.9) since the test statistic can be calculated using quantities that are already available after the first regression has been run

$$DW = \frac{\sum_{t=2}^{T} (\hat{u}_t - \hat{u}_{t-1})^2}{\sum_{t=2}^{T} \hat{u}_t^2} \tag{5.10}$$

The denominator of the test statistic is simply (the number of observations −1) × the variance of the residuals. This arises since if the average of the residuals is zero

$$\text{var}(\hat{u}_t) = E(\hat{u}_t^2) = \frac{1}{T-1}\sum_{t=2}^{T} \hat{u}_t^2$$

so that

$$\sum_{t=2}^{T} \hat{u}_t^2 = \text{var}(\hat{u}_t) \times (T-1)$$

The numerator 'compares' the values of the error at times $t - 1$ and t. If there is positive autocorrelation in the errors, this difference in the numerator will be relatively small, while if there is negative autocorrelation, with the sign of the error changing very frequently, the numerator will be relatively large. No autocorrelation would result in a value for the numerator between small and large.

It is also possible to express the *DW* statistic as an approximate function of the estimated value of ρ

$$DW \approx 2(1 - \hat{\rho}) \tag{5.11}$$

where $\hat{\rho}$ is the estimated correlation coefficient that would have been obtained from an estimation of equation (5.9). To see why this is the case, consider that the numerator of equation (5.10) can be written as the parts of a quadratic

$$\sum_{t=2}^{T} (\hat{u}_t - \hat{u}_{t-1})^2 = \sum_{t=2}^{T} \hat{u}_t^2 + \sum_{t=2}^{T} \hat{u}_{t-1}^2 - 2\sum_{t=2}^{T} \hat{u}_t \hat{u}_{t-1} \tag{5.12}$$

Consider now the composition of the first two summations on the RHS of equation (5.12). The first of these is

$$\sum_{t=2}^{T} \hat{u}_t^2 = \hat{u}_2^2 + \hat{u}_3^2 + \hat{u}_4^2 + \cdots + \hat{u}_T^2$$

while the second is

$$\sum_{t=2}^{T} \hat{u}_{t-1}^2 = \hat{u}_1^2 + \hat{u}_2^2 + \hat{u}_3^2 + \cdots + \hat{u}_{T-1}^2$$

Thus, the only difference between them is that they differ in the first and last terms in the summation, so

$$\sum_{t=2}^{T} \hat{u}_t^2$$

contains $\hat{u}_T^2$ but not $\hat{u}_1^2$, while

$$\sum_{t=2}^{T} \hat{u}_{t-1}^2$$

contains $\hat{u}_1^2$ but not $\hat{u}_T^2$. As the sample size, *T*, increases towards infinity, the difference between these two will become negligible. Hence, the expression in equation (5.12), the numerator of equation (5.10), is approximately

$$2\sum_{t=2}^{T} \hat{u}_t^2 - 2\sum_{t=2}^{T} \hat{u}_t \hat{u}_{t-1}$$

Replacing the numerator of equation (5.10) with this expression leads to

$$DW \approx \frac{2\sum_{t=2}^{T}\hat{u}_t^2 - 2\sum_{t=2}^{T}\hat{u}_t\hat{u}_{t-1}}{\sum_{t=2}^{T}\hat{u}_t^2} = 2\left(1 - \frac{\sum_{t=2}^{T}\hat{u}_t\hat{u}_{t-1}}{\sum_{t=2}^{T}\hat{u}_t^2}\right) \quad \textbf{(5.13)}$$

The covariance between u_t and u_{t-1} can be written as $E[(u_t - E(u_t))(u_{t-1} - E(u_{t-1}))]$. Under the assumption that $E(u_t) = 0$ (and therefore that $E(u_{t-1}) = 0$), the covariance will be $E[u_t\ u_{t-1}]$. For the sample residuals, this covariance will be evaluated as

$$\frac{1}{T-1}\sum_{t=2}^{T}\hat{u}_t\hat{u}_{t-1}$$

Thus, the sum in the numerator of the expression on the right of equation (5.13) can be seen as $T - 1$ times the covariance between $\hat{\rho} = 0$, and $\hat{\rho} = 1$, while the sum in the denominator of the expression on the right of equation (5.13) can be seen from the previous exposition as $T - 1$ times the variance of $\hat{\rho} = -1$, Thus, it is possible to write

$$DW \approx 2\left(1 - \frac{(T-1)\operatorname{cov}(\hat{u}_t, \hat{u}_{t-1})}{(T-1)\operatorname{var}(\hat{u}_t)}\right) = 2\left(1 - \frac{\operatorname{cov}(\hat{u}_t, \hat{u}_{t-1})}{\operatorname{var}(\hat{u}_t)}\right) = 2\left(1 - \operatorname{corr}(\hat{u}_t, \hat{u}_{t-1})\right) \quad \textbf{(5.14)}$$

so that the DW test statistic is approximately equal to $2(1-\hat{\rho})$. Since $\hat{\rho}$ is a correlation, it implies that $-1 \leq \hat{\rho} \leq 1$. That is, $\hat{\rho}$ is bounded to lie between −1 and +1. Substituting in these limits for $\hat{\rho}$ to calculate DW from equation (5.11) would give the corresponding limits for DW as $0 \leq DW \leq 4$. Consider now the implication of DW taking one of three important values (0, 2, and 4):

- $\hat{\rho} = 0$, $DW = 2$ This is the case where there is no autocorrelation in the residuals. So roughly speaking, the null hypothesis would not be rejected if DW is near 2 → i.e., there is little evidence of autocorrelation.
- $\hat{\rho} = 1$, $DW = 0$ This corresponds to the case where there is perfect positive autocorrelation in the residuals.
- $\hat{\rho} = -1$, $DW = 4$ This corresponds to the case where there is perfect

negative autocorrelation in the residuals.

The *DW* test does not follow a standard statistical distribution such as a *t*, *F*, or χ^2. *DW* has 2 critical values: an upper critical value (d_U) and a lower critical value (d_L), and there is also an intermediate region where the null hypothesis of no autocorrelation can neither be rejected nor not rejected! The rejection, non-rejection and inconclusive regions are shown on the number line in Figure 5.9.

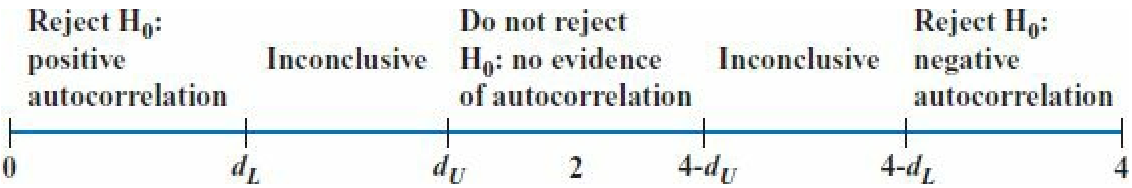


Figure 5.9 Rejection and non-rejection regions for *DW* test

So, to reiterate, the null hypothesis is rejected and the existence of positive autocorrelation presumed if *DW* is less than the lower critical value; the null hypothesis is rejected and the existence of negative autocorrelation presumed if *DW* is greater than 4 minus the lower critical value; the null hypothesis is not rejected and no significant residual autocorrelation is presumed if *DW* is between the upper and 4 minus the upper limits.

EXAMPLE 5.2

A researcher wishes to test for first order serial correlation in the residuals from a linear regression. The *DW* test statistic value is 0.86. There are eighty quarterly observations in the regression, which is of the form

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 x_{4t} + u_t \quad \textbf{(5.15)}$$

The relevant critical values for the test (see Table A2.6 in the appendix of statistical distributions at the end of this book), are $d_L = 1.42$, $d_U = 1.57$, so $4 - d_U = 2.43$ and $4 - d_L = 2.58$. The test statistic is clearly lower than the lower critical value and hence the null hypothesis of no autocorrelation is rejected and it would be concluded that the residuals from the model appear to be positively autocorrelated.

5.5.4 Conditions Which Must be Fulfilled for *DW* to be a Valid Test

In order for the *DW* test to be valid for application, three conditions must be fulfilled (Box 5.3).

BOX 5.3 Conditions for DW to be a valid test

- **(1)** There must be a constant term in the regression
- **(2)** The regressors must be non-stochastic – as assumption (4) of the CLRM (see Chapter 7)
- **(3)** There must be no lags of dependent variable (see Section 5.5.8) in the regression.

If the test were used in the presence of lags of the dependent variable or otherwise stochastic regressors, the test statistic would be biased towards 2, suggesting that in some instances the null hypothesis of no autocorrelation would not be rejected when it should be.

5.5.5 Another Test for Autocorrelation: The Breusch–Godfrey Test

Recall that *DW* is a test only of whether consecutive errors are related to one another. So, not only can the *DW* test not be applied if a certain set of circumstances are not fulfilled, there will also be many forms of residual autocorrelation that *DW* cannot detect. For example, if $\text{corr}(\hat{u}_t, \hat{u}_{t-1}) = 0$, but $\text{corr}(\hat{u}_t, \hat{u}_{t-2}) \neq 0$, *DW* as defined above will not find any autocorrelation. One possible solution would be to replace $\hat{u}_{t-1}$ in equation (5.10) with $\hat{u}_{t-2}$. However, pairwise examinations of the correlations $(\hat{u}_t,\ \hat{u}_{t-1}),\ (\hat{u}_t,\ \hat{u}_{t-2}),\ (\hat{u}_t,\ \hat{u}_{t-3}), \ldots$ will be tedious in practice and is not coded in econometrics software packages, which have been programmed to construct *DW* using only a one-period lag. In addition, the approximation in equation (5.11) will deteriorate as the difference between the two time indices increases. Consequently, the critical values should also be modified somewhat in these cases.

Therefore, it is desirable to examine a joint test for autocorrelation that will allow examination of the relationship between $\hat{u}_t$ and several of its lagged values at the same time. The Breusch–Godfrey test is a more general test for autocorrelation up to the *r*th order. The model for the errors under this test is

$$u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + \rho_3 u_{t-3} + \cdots + \rho_r u_{t-r} + v_t, \qquad v_t \sim N\left(0, \sigma_v^2\right) \tag{5.16}$$

The null and alternative hypotheses are:

$$H_0\colon \rho_1 = 0 \text{ and } \rho_2 = 0 \text{ and } \ldots \text{and } \rho_r = 0$$
$$H_1\colon \rho_1 \neq 0 \text{ or } \rho_2 \neq 0 \text{ or } \ldots \text{or } \rho_r \neq 0$$

So, under the null hypothesis, the current error is not related to any of its r previous values. The test is carried out as in Box 5.4.

BOX 5.4 Conducting a Breusch–Godfrey test

(1) Estimate the linear regression using OLS and obtain the residuals, $\hat{u}_t$

(2) Regress $\hat{u}_t$ on all of the regressors from stage 1 (the xs) plus $\hat{u}_{t-1}, \hat{u}_{t-2}, \ldots, \hat{u}_{t-r}$; the regression will thus be

$$\hat{u}_t = \gamma_1 + \gamma_2 x_{2t} + \gamma_3 x_{3t} + \gamma_4 x_{4t} + \rho_1 \hat{u}_{t-1} + \rho_2 \hat{u}_{t-2} + \rho_3 \hat{u}_{t-3} + \cdots + \rho_r \hat{u}_{t-r} + v_t,\ v_t \sim N\left(0, \sigma_v^2\right) \tag{5.17}$$

Obtain R^2 from this auxiliary regression

(3) Letting T denote the number of observations, the test statistic is given by

$$(T - r)R^2 \sim \chi_r^2$$

Note that $(T - r)$ pre-multiplies R^2 in the test for autocorrelation rather than T (as was the case for the heteroscedasticity test). This arises because the first r observations will effectively have been lost from the sample in order to obtain the r lags used in the test regression, leaving $(T - r)$ observations from which to estimate the auxiliary regression. If the test statistic exceeds the critical value from the chi-squared statistical tables, reject the null hypothesis of no autocorrelation. As with any joint test, only one part of the null hypothesis has to be rejected to lead to rejection of the hypothesis as a whole. So the error at time t has to be significantly related only to one of its previous r values in the sample for the null of no autocorrelation to be rejected. The test is more general than the DW test, and can be applied in a wider variety of circumstances since it does not impose the *DW* restrictions on the format of the first stage regression.

One potential difficulty with Breusch–Godfrey, however, is in determining an appropriate value of *r*, the number of lags of the residuals, to use in computing the test. There is no obvious answer to this, so it is typical to experiment with a range of values, and also to use the frequency of the data to decide. So, for example, if the data are monthly or quarterly, set *r* equal to 12 or 4, respectively. The argument would then be that errors at any given time would be expected to be related only to those errors in the previous year. Obviously, if the model is statistically adequate, no evidence of autocorrelation should be found in the residuals whatever value of *r* is chosen.

5.5.6 Consequences of Ignoring Autocorrelation if it is Present

In fact, the consequences of ignoring autocorrelation when it is present are similar to those of ignoring heteroscedasticity. The coefficient estimates derived using OLS are still unbiased, but they are inefficient, i.e., they are not BLUE, even at large sample sizes, so that the standard error estimates could be wrong. There thus exists the possibility that the wrong inferences could be made about whether a variable is or is not an important determinant of variations in *y*. In the case of positive serial correlation in the residuals, the OLS standard error estimates will be biased downwards relative to the true standard errors. That is, OLS will understate their true variability. This would lead to an increase in the probability of type I error – that is, a tendency to reject the null hypothesis sometimes when it is correct. Furthermore, R^2 is likely to be inflated relative to its 'correct' value if autocorrelation is present but ignored, since residual autocorrelation will lead to an underestimate of the true error variance (for positive autocorrelation).

5.5.7 Dealing with Autocorrelation

If the form of the autocorrelation is known, it would be possible to use a GLS procedure. One approach, which was once fairly popular, is known as the Cochrane–Orcutt procedure (see Box 5.5). Such methods work by assuming a particular form for the structure of the autocorrelation (usually a first order autoregressive process – see Chapter 6 for a general description of these models). The model would thus be specified as follows

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + u_t, \qquad u_t = \rho u_{t-1} + v_t$$

(5.18)

BOX 5.5 The Cochrane–Orcutt procedure

(1) Assume that the general model is of the form (5.18) above. Estimate the equation in (5.18) using OLS, ignoring the residual autocorrelation

(2) Obtain the residuals, and run the regression

$$\hat{u}_t = \rho \hat{u}_{t-1} + v_t \quad \textbf{(5.19)}$$

(3) Obtain $\hat{\rho}$ and construct y_t^* etc. using this estimate of $\hat{\rho}$

(4) Run the GLS regression (5.24).

Note that a constant is not required in the specification for the errors since $E(u_t) = 0$. If this model holds at time t, it is assumed to also hold for time $t - 1$, so that the model in equation (5.18) is lagged one period

$$y_{t-1} = \beta_1 + \beta_2 x_{2t-1} + \beta_3 x_{3t-1} + u_{t-1} \quad \textbf{(5.20)}$$

Multiplying equation (5.20) by ρ

$$\rho y_{t-1} = \rho\beta_1 + \rho\beta_2 x_{2t-1} + \rho\beta_3 x_{3t-1} + \rho u_{t-1} \quad \textbf{(5.21)}$$

Subtracting equation (5.21) from equation (5.18) would give

$$y_t - \rho y_{t-1} = \beta_1 - \rho\beta_1 + \beta_2 x_{2t} - \rho\beta_2 x_{2t-1} + \beta_3 x_{3t} - \rho\beta_3 x_{3t-1} + u_t - \rho u_{t-1} \quad \textbf{(5.22)}$$

Factorising, and noting that $v_t = u_t - \rho u_{t-1}$

$$(y_t - \rho y_{t-1}) = (1 - \rho)\beta_1 + \beta_2(x_{2t} - \rho x_{2t-1}) + \beta_3(x_{3t} - \rho x_{3t-1}) + v_t \quad \textbf{(5.23)}$$

Setting $y_t^* = y_t - \rho y_{t-1}, \beta_1^* = (1 - \rho)\beta_1, x_{2t}^* = (x_{2t} - \rho x_{2t-1})$, and $x_{3t}^* = (x_{3t} - \rho x_{3t-1})$, the model in equation (5.23) can be written

$$y_t^* = \beta_1^* + \beta_2 x_{2t}^* + \beta_3 x_{3t}^* + v_t \quad \textbf{(5.24)}$$

Since the final specification equation (5.24) contains an error term that is free from autocorrelation, OLS can be directly applied to it. This procedure is effectively an application of GLS. Of course, the construction of y_t^* etc. requires ρ to be known. In practice, this will never be the case so that ρ has to be estimated before equation (5.24) can be used.

A simple method would be to use the ρ obtained from rearranging the equation for the *DW* statistic given in equation (5.11). However, this is only an approximation, as the related algebra showed. This approximation may be poor in the context of small samples.

The Cochrane–Orcutt procedure is an alternative, which operates as in Box 5.5. This could be the end of the process. However, Cochrane and Orcutt (1949) argue that better estimates can be obtained by going through steps (2)–(4) again. That is, given the new coefficient estimates, $\beta_1^*, \beta_2, \beta_3$, etc. construct again the residual and regress it on its previous value to obtain a new estimate for $\hat{\rho}$. This would then be used to construct new values of the variables $y_t^*, x_{2t}^*, x_{3t}^*$ and a new equation (5.24) is estimated. This procedure would be repeated until the change in $\hat{\rho}$ between one iteration and the next is less than some fixed amount (e.g., 0.01). In practice, a small number of iterations (no more than five) will usually suffice.

However, the Cochrane–Orcutt procedure and similar approaches require a specific assumption to be made concerning the form of the model for the autocorrelation. Consider again equation (5.23). This can be rewritten taking ρy_{t-1} over to the RHS

$$y_t = (1-\rho)\beta_1 + \beta_2(x_{2t} - \rho x_{2t-1}) + \beta_3(x_{3t} - \rho x_{3t-1}) + \rho y_{t-1} + v_t \tag{5.25}$$

Expanding the brackets around the explanatory variable terms would give

$$y_t = (1-\rho)\beta_1 + \beta_2 x_{2t} - \rho\beta_2 x_{2t-1} + \beta_3 x_{3t} - \rho\beta_3 x_{3t-1} + \rho y_{t-1} + v_t \tag{5.26}$$

Now, suppose that an equation containing the same variables as (5.26) were estimated using OLS

$$y_t = \gamma_1 + \gamma_2 x_{2t} + \gamma_3 x_{2t-1} + \gamma_4 x_{3t} + \gamma_5 x_{3t-1} + \gamma_6 y_{t-1} + v_t \tag{5.27}$$

It can be seen that equation (5.26) is a restricted version of equation (5.27), with the restrictions imposed that the coefficient on x_{2t} in equation (5.26) multiplied by the negative of the coefficient on y_{t-1} gives the coefficient on x_{2t-1}, and that the coefficient on x_{3t} multiplied by the negative of the coefficient on y_{t-1} gives the coefficient on x_{3t-1}. Thus, the restrictions implied for equation (5.27) to get equation (5.26) are

$$\gamma_2\gamma_6 = -\gamma_3 \quad \text{and} \quad \gamma_4\gamma_6 = -\gamma_5$$

These are known as the *common factor restrictions*, and they should be tested before the Cochrane–Orcutt or similar procedure is implemented. If the restrictions hold, Cochrane–Orcutt can be validly applied. If not, however, Cochrane–Orcutt and similar techniques would be inappropriate, and the appropriate step would be to estimate an equation such as (5.27) directly using OLS. Note that in general there will be a common factor restriction for every explanatory variable (excluding a constant) x_{2t}, x_{3t}, …, x_{kt} in the regression. Hendry and Mizon (1978) argued that the restrictions are likely to be invalid in practice and therefore a dynamic model that allows for the structure of *y* should be used rather than a residual correction on a static model – see also Hendry (1980).

The White variance–covariance matrix of the coefficients (that is, calculation of the standard errors using the White correction for heteroscedasticity) is appropriate when the residuals of the estimated equation are heteroscedastic but serially uncorrelated. Newey and West (1987) develop a variance–covariance estimator that is consistent in the presence of both heteroscedasticity and autocorrelation. So an alternative approach to dealing with residual autocorrelation would be to use appropriately modified standard error estimates.

While White's correction to standard errors for heteroscedasticity as discussed above does not require any user input, the Newey–West procedure requires the specification of a truncation lag length to determine the number of lagged residuals used to evaluate the autocorrelation. The statistical software EViews, for example, uses INTEGER$[4(T/100)^{2/9}]$.

A more 'modern' view concerning autocorrelation is that it presents an opportunity rather than a problem. This view, associated with Sargan, Hendry and Mizon, suggests that serial correlation in the errors arises as a consequence of 'misspecified dynamics'. For another explanation of the reason why this stance is taken, recall that it is possible to express the dependent variable as the sum of the parts that can be explained using the model, and a part which cannot (the residuals)

$$y_t = \hat{y}_t + \hat{u}_t \qquad \textbf{(5.28)}$$

where $\hat{y}_t$ are the fitted values from the model $(= \hat{\beta}_1 + \hat{\beta}_2 x_{2t} + \hat{\beta}_3 x_{3t} + \cdots + \hat{\beta}_k x_{kt})$. Autocorrelation in the residuals is often caused by a dynamic structure in y that has not been modelled and so has not been captured in the fitted values. In other words, there exists a richer structure in the dependent variable y and more information in the sample about that structure than has been captured by the models previously estimated. What is required is a dynamic model that allows for this extra structure in y.

5.5.8 Dynamic Models

All of the models considered so far have been static in nature, e.g.,

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 x_{4t} + \beta_5 x_{5t} + u_t \tag{5.29}$$

In other words, these models have allowed for only a *contemporaneous relationship* between the variables, so that a change in one or more of the explanatory variables at time t causes an instant change in the dependent variable at time t. But this analysis can easily be extended to the case where the current value of y_t depends on previous values of y or on previous values of one or more of the variables, e.g.,

$$\begin{aligned} y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 x_{4t} + \beta_5 x_{5t} + \gamma_1 y_{t-1} + \gamma_2 x_{2t-1} \\ + \cdots + \gamma_k x_{kt-1} + u_t \end{aligned} \tag{5.30}$$

It is of course possible to extend the model even more by adding further lags, e.g., x_{2t-2}, y_{t-3}. Models containing lags of the explanatory variables (but no lags of the explained variable) are known as *distributed lag models*. Specifications with lags of both explanatory and explained variables are known as *autoregressive distributed lag* (ADL) models.

How many lags and of which variables should be included in a dynamic regression model? This is a tricky question to answer, but hopefully recourse to financial theory will help to provide an answer; for another response, see Section 5.14.

Another potential ‘remedy’ for autocorrelated residuals would be to switch to a model in first differences rather than in levels. As explained previously, the first difference of y_t, i.e., $y_t - y_{t-1}$ is denoted Δy_t; similarly, one can construct a series of first differences for each of the explanatory variables, e.g., $\Delta x_{2t} = x_{2t} - x_{2t-1}$, etc. Such a model has a number of other useful features (see Chapter 8 for more details) and could be expressed as

$$\Delta y_t = \beta_1 + \beta_2 \Delta x_{2t} + \beta_3 \Delta x_{3t} + u_t \tag{5.31}$$

Sometimes the change in y is purported to depend on previous values of the level of y or $x_i (i = 2, \ldots, k)$ as well as changes in the explanatory variables

$$\Delta y_t = \beta_1 + \beta_2 \Delta x_{2t} + \beta_3 \Delta x_{3t} + \beta_4 x_{2t-1} + \beta_5 y_{t-1} + u_t \tag{5.32}$$

5.5.9 Why Might Lags be Required in a Regression?

Lagged values of the explanatory variables or of the dependent variable (or both) may capture important dynamic structure in the dependent variable that might be caused by a number of factors. Two possibilities that are relevant in finance are as follows

- **Inertia of the dependent variable** Often a change in the value of one of the explanatory variables will not affect the dependent variable immediately during one time period, but rather with a lag over several time periods. For example, the effect of a change in market microstructure or government policy may take a few months or longer to work through since agents may be initially unsure of what the implications for asset pricing are, and so on. More generally, many variables in economics and finance will change only slowly. This phenomenon arises partly as a result of pure psychological factors – for example, in financial markets, agents may not fully comprehend the effects of a particular news announcement immediately, or they may not even believe the news. The speed and extent of reaction will also depend on whether the change in the variable is expected to be permanent or transitory. Delays in response may also arise as a result of technological or institutional factors. For example, the speed of technology will limit how quickly investors' buy or sell orders can be executed. Similarly, many investors have savings plans or other financial products where they are 'locked in' and therefore unable to act for a fixed period. It is also worth noting that dynamic structure is likely to be stronger and more prevalent the higher is the frequency of observation of the data.
- **Overreactions** It is sometimes argued that financial markets overreact to good and to bad news. So, for example, if a firm makes a profit warning, implying that its profits are likely to be down when formally reported later in the year, the markets might be anticipated to perceive this as implying that the value of the firm is less than was

previously thought, and hence that the price of its shares will fall. If there is an overreaction, the price will initially fall below that which is appropriate for the firm given this bad news, before subsequently bouncing back up to a new level (albeit lower than the initial level before the announcement).

Moving from a purely static model to one which allows for lagged effects is likely to reduce, and possibly remove, serial correlation which was present in the static model's residuals. However, other problems with the regression could cause the null hypothesis of no autocorrelation to be rejected, and these would not be remedied by adding lagged variables to the model

- **Omission of relevant variables, which are themselves autocorrelated** In other words, if there is a variable that is an important determinant of movements in *y*, but which has not been included in the model, and which itself is autocorrelated, this will induce the residuals from the estimated model to be serially correlated. To give a financial context in which this may arise, it is often assumed that investors assess one-step-ahead expected returns on a stock using a linear relationship

$$r_t = \alpha_0 + \alpha_1 \Omega_{t-1} + u_t \tag{5.33}$$

  where Ω_{t-1} is a set of lagged information variables (i.e., Ω_{t-1} is a vector of observations on a set of variables at time $t - 1$). However, equation (5.33) cannot be estimated since the actual information set used by investors to form their expectations of returns is not known. Ω_{t-1} is therefore proxied with an assumed sub-set of that information, Z_{t-1}. For example, in many popular arbitrage pricing specifications, the information set used in the estimated model includes unexpected changes in industrial production, the term structure of interest rates, inflation and default risk premia. Such a model is bound to omit some informational variables used by actual investors in forming expectations of returns, and if these are autocorrelated, it will induce the residuals of the estimated model to be also autocorrelated.
- **Autocorrelation owing to unparameterised seasonality** Suppose that the dependent variable contains a seasonal or cyclical pattern, where certain features periodically occur. This may arise, for

example, in the context of sales of gloves, where sales will be higher in the autumn and winter than in the spring or summer. Such phenomena are likely to lead to a positively autocorrelated residual structure that is cyclical in shape, such as that of Figure 5.4, unless the seasonal patterns are captured by the model. See Chapter 10 for a discussion of seasonality and how to deal with it.

- **If 'misspecification' error has been committed by using an inappropriate functional form** For example, if the relationship between y and the explanatory variables was a non-linear one, but the researcher had specified a linear regression model, this may again induce the residuals from the estimated model to be serially correlated.

5.5.10 The Long-Run Static Equilibrium Solution

Once a general model of the form given in equation (5.32) has been found, it may contain many differenced and lagged terms that make it difficult to interpret from a theoretical perspective. For example, if the value of x_2 were to increase in period t, what would be the effect on y in periods, t, $t + 1$, $t + 2$, and so on? One interesting property of a dynamic model that can be calculated is its long-run or static equilibrium solution.

The relevant definition of 'equilibrium' in this context is that a system has reached equilibrium if the variables have attained some steady state values and are no longer changing, i.e., if y and x are in equilibrium, it is possible to write

$$y_t = y_{t+1} = \ldots = y \text{ and } x_{2t} = x_{2t+1} = \ldots = x_2, \text{and so on.}$$

Consequently, $\Delta y_t = y_t - y_{t-1} = y - y = 0$, $\Delta x_{2t} = x_{2t} - x_{2t-1} = x_2 - x_2 = 0$, etc. since the values of the variables are no longer changing. So the way to obtain a long-run static solution from a given empirical model such as equation (5.32) is:

(1) Remove all time subscripts from the variables

(2) Set error terms equal to their expected values of zero, i.e., $E(u_t) = 0$

(3) Remove differenced terms (e.g., Δy_t) altogether

(4) Gather terms in x together and gather terms in y together

(5) Rearrange the resulting equation if necessary so that the dependent variable y is on the LHS and is expressed as a function of the independent variables

EXAMPLE 5.3

Calculate the long-run equilibrium solution for the following model

$$\Delta y_t = \beta_1 + \beta_2 \Delta x_{2t} + \beta_3 \Delta x_{3t} + \beta_4 x_{2t-1} + \beta_5 y_{t-1} + u_t \tag{5.34}$$

Applying first steps (1)–(3) above, the static solution would be given by

$$0 = \beta_1 + \beta_4 x_2 + \beta_5 y \tag{5.35}$$

Rearranging (5.35) to bring y to the LHS

$$\beta_5 y = -\beta_1 - \beta_4 x_2 \tag{5.36}$$

and finally, dividing through by β_5

$$y = -\frac{\beta_1}{\beta_5} - \frac{\beta_4}{\beta_5} x_2 \tag{5.37}$$

Equation (5.37) is the long-run static solution to equation (5.34). Note that this equation does not feature x_3, since the only term which contained x_3 was in first differenced form, so that x_3 does not influence the long-run equilibrium value of y.

5.5.11 Problems with Adding Lagged Regressors to 'Cure' Autocorrelation

In many instances, a move from a static model to a dynamic one will result in a removal of residual autocorrelation. The use of lagged variables in a regression model does, however, bring with it additional problems

- **Inclusion of lagged values of the dependent variable violates the assumption that the explanatory variables are non-stochastic** (assumption (4) of the CLRM), since by definition the value of y is determined partly by a random error term, and so its lagged values cannot be non-stochastic. In small samples, inclusion of lags of the dependent variable can lead to biased coefficient estimates, although they are still consistent, implying that the bias will disappear asymptotically (that is, as the sample size increases towards infinity).

- **What does an equation with a large number of lags actually mean?** A model with many lags may have solved a statistical problem (autocorrelated residuals) at the expense of creating an interpretational one (the empirical model containing many lags or differenced terms is difficult to interpret and may not test the original financial theory that motivated the use of regression analysis in the first place).

Note that if there is still autocorrelation in the residuals of a model including lags, then the OLS estimators will not even be consistent. To see why this occurs, consider the following regression model

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 y_{t-1} + u_t \quad \textbf{(5.38)}$$

where the errors, u_t, follow a first-order autoregressive process

$$u_t = \rho u_{t-1} + v_t \quad \textbf{(5.39)}$$

Substituting into equation (5.38) for u_t from equation (5.39)

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 y_{t-1} + \rho u_{t-1} + v_t \quad \textbf{(5.40)}$$

Now, clearly y_t depends upon y_{t-1}. Taking equation (5.38) and lagging it one period (i.e. subtracting one from each time index)

$$y_{t-1} = \beta_1 + \beta_2 x_{2t-1} + \beta_3 x_{3t-1} + \beta_4 y_{t-2} + u_{t-1} \quad \textbf{(5.41)}$$

It is clear from equation (5.41) that y_{t-1} is related to u_{t-1} since they both appear in that equation. Thus, the assumption that $E(X'u) = 0$ is not satisfied for equation (5.41) and therefore for equation (5.38). Thus the OLS estimator will not be consistent, so that even with an infinite quantity of data, the coefficient estimates would be biased.

5.5.12 Autocorrelation in Cross-Sectional Data

The possibility that autocorrelation may occur in the context of a time-series regression is quite intuitive. However, it is also plausible that autocorrelation could be present in certain types of cross-sectional data. For example, if the cross-sectional data comprise the profitability of banks in different regions of the US, autocorrelation may arise in a spatial sense, if there is a regional dimension to bank profitability that is not captured by the model. Thus the residuals from banks of the same region or in neighbouring regions may be correlated. Testing for autocorrelation in this case would be rather more complex than in the time-series context, and would involve the construction of a square, symmetric 'spatial contiguity matrix' or a 'distance matrix'. Both of these matrices would be $N \times N$, where N is the sample size. The former would be a matrix of zeros and ones, with one for element i, j when observation i occurred for a bank in the same region to, or sufficiently close to, region j and zero otherwise (i, j = 1, …, N). The distance matrix would comprise elements that measured the distance (or the inverse of the distance) between bank i and bank j. A potential solution to a finding of autocorrelated residuals in such a model would be again to use a model containing a lag structure, in this case known as a 'spatial lag'. Further details are contained in Anselin (1988).

5.6 Assumption (4): The x_t are Non-Stochastic

Fortunately, it turns out that the OLS estimator is consistent and unbiased in the presence of stochastic regressors, provided that the regressors are not correlated with the error term of the estimated equation. To see this, recall that

$$\hat{\beta} = (X'X)^{-1}X'y \quad \text{and} \quad y = X\beta + u \tag{5.42}$$

Thus

$$\hat{\beta} = (X'X)^{-1}X'(X\beta + u) \tag{5.43}$$

$$\hat{\beta} = (X'X)^{-1}X'X\beta + (X'X)^{-1}X'u \tag{5.44}$$

$$\hat{\beta} = \beta + (X'X)^{-1}X'u \tag{5.45}$$

Taking expectations, and provided that X and u are independent,[1]

$$E(\hat{\beta}) = E(\beta) + E((X'X)^{-1}X'u) \tag{5.46}$$

$$E(\hat{\beta}) = \beta + E[(X'X)^{-1}X']E(u) \tag{5.47}$$

Since E(u) = 0, this expression will be zero and therefore the estimator is still unbiased, even if the regressors are stochastic.

However, if one or more of the explanatory variables is contemporaneously correlated with the disturbance term, the OLS estimator will not even be consistent. This results from the estimator assigning explanatory power to the variables where in reality it is arising from the correlation between the error term and y_t. Suppose for illustration that x_{2t} and u_t are positively correlated. When the disturbance term happens to take a high value, y_t will also be high (because $y_t = \beta_1 + \beta_2 x_{2t} + \cdots + u_t$). But if x_{2t} is positively correlated with u_t, then x_{2t} is also likely to be high. Thus the OLS estimator will incorrectly attribute the high value of y_t to a high value of x_{2t}, where in reality y_t is high simply because u_t is high, which will result in biased and inconsistent parameter estimates and a fitted line that appears to capture the features of the data much better than it does in reality.

5.7 Assumption (5): The Disturbances are Normally Distributed

Recall that the normality assumption ($u_t \sim N(0, \sigma^2)$) is required in order to conduct single or joint hypothesis tests about the model parameters.

5.7.1 Testing for Departures from Normality

One of the most commonly applied tests for normality is the Bera–Jarque (hereafter BJ) test. BJ uses the property of a normally distributed random variable that the entire distribution is characterised by the first two moments – the mean and the variance. Recall from Chapter 2 that standardised third and fourth moments of a distribution are known as its *skewness* and *kurtosis*. A normal distribution is not skewed and is defined to have a coefficient of kurtosis of 3. It is possible to define a coefficient of excess kurtosis, equal to the coefficient of kurtosis minus 3; a normal distribution will thus have a coefficient of excess kurtosis of zero. Bera and Jarque (1981) formalise these ideas by testing whether the coefficient of skewness and the coefficient of excess kurtosis are jointly zero. Denoting the errors by u and their variance by σ^2, it can be proved that the coefficients of skewness and kurtosis can be expressed, respectively, as

$$b_1 = \frac{E[u^3]}{(\sigma^2)^{3/2}} \quad \text{and} \quad b_2 = \frac{E[u^4]}{(\sigma^2)^2} \tag{5.48}$$

The kurtosis of the normal distribution is 3 so its excess kurtosis ($b_2 - 3$) is zero.

The BJ test statistic is given by

$$W = T\left[\frac{b_1^2}{6} + \frac{(b_2 - 3)^2}{24}\right] \quad \textbf{(5.49)}$$

where T is the sample size. The test statistic asymptotically follows a $\chi^2(2)$ under the null hypothesis that the distribution of the series is symmetric and mesokurtic.

b_1 and b_2 can be estimated using the residuals from the OLS regression, $\hat{u}$. The null hypothesis is of normality, and this would be rejected if the residuals from the model were either significantly skewed or leptokurtic/platykurtic (or both).

5.7.2 What Should be Done if Evidence of Non-Normality is Found?

It is not obvious what should be done! It is, of course, possible to employ an estimation method that does not assume normality, but such a method may be difficult to implement, and one can be less sure of its properties. It is thus desirable to stick with OLS if possible, since its behaviour in a variety of circumstances has been well researched. For sample sizes that are sufficiently large, violation of the normality assumption is virtually inconsequential. Appealing to a central limit theorem, the test statistics will asymptotically follow the appropriate distributions even in the absence of error normality.[2]

It is possible that a log-transform of the dependent variable might help to make the distribution of the residuals closer to a normal. This might be useful if the data are strongly positively skewed. For example, if we have cross-sectional data and we want to model company size, this is likely to be positively skewed, with the bulk of firms being of a certain size and a small number being very much larger than the rest.

Alternatively, in economic or financial modelling, it is quite often the case that one or two very extreme residuals cause a rejection of the normality assumption. Such observations would appear in the tails of the distribution, and would therefore lead u^4, which enters into the definition of kurtosis, to be very large. Such observations that do not fit in with the pattern of the remainder of the data are known as *outliers*. If this is the case, one way to improve the chances of error normality is to use dummy variables or some other method to effectively remove those observations.

In the time-series context, suppose that a monthly model of asset returns from 1980–90 had been estimated, and the residuals plotted, and that a particularly large outlier has been observed for October 1987, shown in Figure 5.10.

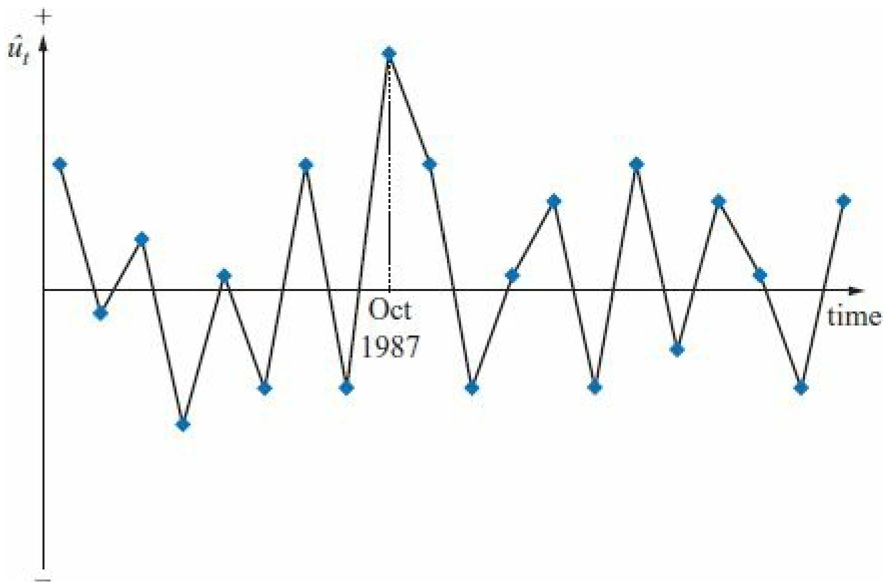


Figure 5.10 Regression residuals from stock return data, showing large outlier for October 1987

A new variable called $D87M10_t$ could be defined as $D87M10_t = 1$ during October 1987 and zero otherwise. The observations for the dummy variable would appear as in Box 5.6. The dummy variable would then be used just like any other variable in the regression model, e.g.,

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 D87M10_t + u_t \tag{5.50}$$

This type of dummy variable that takes the value one for only a single observation has an effect exactly equivalent to knocking out that observation from the sample altogether, by forcing the residual for that observation to zero. The estimated coefficient on the dummy variable will be equal to the residual that the dummied observation would have taken if the dummy variable had not been included.

BOX 5.6 Observations for the dummy variable

Time	**Value of dummy variable $D87M10_t$**
1986*M*12	0
1987*M*01	0
⋮	⋮
1987*M*09	0
1987*M*10	1
1987*M*11	0
⋮	⋮

However, many econometricians would argue that dummy variables to remove outlying residuals can be used to artificially improve the characteristics of the model – in essence fudging the results. Removing outlying observations will reduce standard errors, reduce the *RSS*, and therefore increase R^2, thus improving the apparent fit of the model to the data. The removal of observations is also hard to reconcile with the notion in statistics that each data point represents a useful piece of information.

The other side of this argument is that observations that are 'a long way away' from the rest, and seem not to fit in with the general pattern of the rest of the data are known as *outliers*. Outliers can have a serious effect on coefficient estimates since, by definition, OLS will receive a big penalty, in the form of an increased *RSS* for points that are a long way from the fitted line. Consequently, OLS will try extra hard to minimise the distances of points that would have otherwise been a long way from the line. A graphical depiction of the possible effect of an outlier on OLS estimation, is given in Figure 5.11.

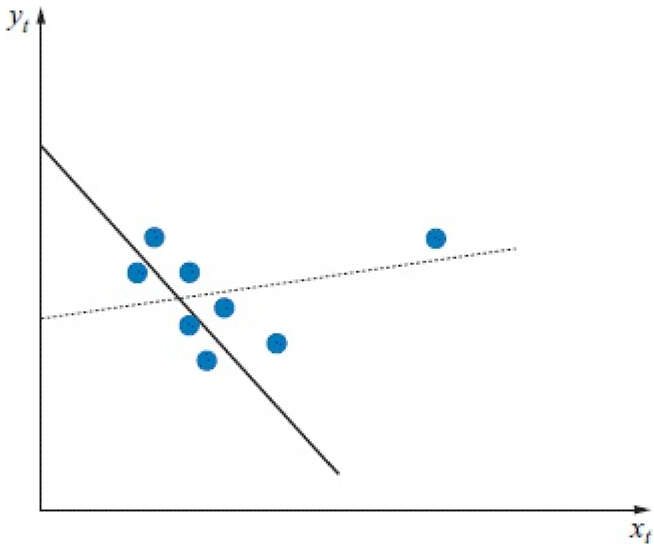


Figure 5.11 Possible effect of an outlier on OLS estimation

In Figure 5.11, one point is a long way away from the rest. If this point is included in the estimation sample, the fitted line will be the dotted one, which has a slight positive slope. If this observation were removed, the full line would be the one fitted. Clearly, the slope is now large and negative. OLS would not select this line if the outlier is included since the observation is a long way from the others and hence when the residual (the distance from the point to the fitted line) is squared, it would lead to a big increase in the *RSS*. Note that outliers could be detected by plotting y against x only in the context of a bivariate regression. In the case where there are more explanatory variables, outliers are easiest identified by plotting the residuals over time, as in Figure 5.10, etc.

So, it can be seen that a trade-off potentially exists between the need to remove outlying observations that could have an undue impact on the OLS estimates and cause residual non-normality on the one hand, and the notion that each data point represents a useful piece of information on the other. The latter is coupled with the fact that removing observations at will could artificially improve the fit of the model. A sensible way to proceed is by introducing dummy variables to the model only if there is both a statistical need to do so and a theoretical justification for their inclusion. This justification would normally come from the researcher's knowledge of the historical events that relate to the dependent variable and the model over the relevant sample period. Dummy variables may be justifiably used to remove observations corresponding to 'one-off' or extreme events that are considered highly unlikely to be repeated, and the information content of which is deemed of no relevance for the data as a whole. Examples may include stock market crashes, financial panics, government crises, and so on.

Non-normality in financial data could also arise from certain types of heteroscedasticity, known as ARCH – see Chapter 9. In this case, the non-normality is intrinsic to all of the data and therefore outlier removal would not make the residuals of such a model normal.

Another important use of dummy variables is in the modelling of seasonality in financial data, and accounting for so-called 'calendar anomalies', such as day-of-the-week effects and weekend effects. These are discussed in Chapter 10.

5.8 Multicollinearity

An implicit assumption that is made when using the OLS estimation method is that the explanatory variables are not correlated with one another. If there is no relationship between the explanatory variables, they would be said to be *orthogonal* to one another. If the explanatory variables were orthogonal to one another, adding or removing a variable from a regression equation would not cause the values of the coefficients on the other variables to change.

In any practical context, the correlation between explanatory variables will be nonzero, although this will generally be relatively benign in the sense that a small degree of association between explanatory variables will almost always occur but will not cause too much loss of precision. However, a problem occurs when the explanatory variables are very highly correlated with each other, and this problem is known as *multicollinearity*. It is possible to distinguish between two classes of multicollinearity: perfect multicollinearity and near multicollinearity.

Perfect multicollinearity occurs when there is an exact relationship between two or more variables. In this case, it is not possible to estimate all of the coefficients in the model. Perfect multicollinearity will usually be observed only when the same explanatory variable is inadvertently used twice in a regression. For illustration, suppose that two variables were employed in a regression function such that the value of one variable was always twice that of the other (e.g., suppose $x_3 = 2x_2$). If both x_3 and x_2 were used as explanatory variables in the same regression, then the model parameters cannot be estimated. Since the two variables are perfectly related to one another, together they contain only enough information to estimate one parameter, not two. Technically, the difficulty would occur in trying to invert the $(X'X)$ matrix since it would not be of full rank (two of the columns would be linearly dependent on one another), so that the inverse of $(X'X)$ would not exist and hence the OLS estimates $\hat{\beta} = (X'X)^{-1}X'y$ could not be calculated.

Near multicollinearity is much more likely to occur in practice, and would arise when there was a non-negligible, but not perfect, relationship between two or more of the explanatory variables. Note that a high correlation between the dependent variable and one of the independent variables is not multicollinearity.

Visually, we could think of the difference between near and perfect multicollinearity as follows. Suppose that the variables x_{2t} and x_{3t} were highly correlated. If we produced a scatter plot of x_{2t} against x_{3t}, then perfect multicollinearity would correspond to all of the points lying exactly on a straight line, while near multicollinearity would correspond to the points lying close to the line, and the closer they were to the line (taken altogether), the stronger would be the relationship between the two variables.

5.8.1 Measuring Near Multicollinearity

Testing for multicollinearity is surprisingly difficult, and hence all that is presented here is a simple method to investigate the presence or otherwise of the most easily detected forms of near multicollinearity. This method simply involves looking at the matrix of correlations between the individual variables. Suppose that a regression equation has three explanatory variables (plus a constant term), and that the pairwise correlations between these explanatory variables are

corr	x_2	x_3	x_4
x_2	–	0.2	0.8
x_3	0.2	–	0.3
x_4	0.8	0.3	–

Clearly, if multicollinearity was suspected, the most likely culprit would be a high correlation between x_2 and x_4. Of course, if the relationship involves three or more variables that are collinear – e.g., $x_2 + x_3 \approx x_4$ – then multicollinearity would be very difficult to detect.

A more formal method for measuring the extent of multicollinearity is by calculating the *variance inflation factors* (VIF), which provide an estimate of to what extent the variance of a parameter estimate increases because the explanatory variables are correlated. For example, if a VIF for a particular variable is 4, this suggests that the variance of the parameter estimate is 4 times larger than would be the case if it were independent from the other explanatory variables in the model (so that the standard error is twice as large (i.e., the square root of 4). The VIF can be calculated for variable *i* as

$$VIF = \frac{1}{(1 - R_i^2)} \tag{5.51}$$

where R_i^2 is the R^2 value from an auxiliary regression of the explanatory variable *i* on an intercept plus all of the other explanatory variables from the model.

The larger the VIF, the more serious is the collinearity between the explanatory variable under test and the others in the model. It is clear from equation (5.51) that since (under some assumptions), the R^2 will be positive, the minimum value of VIF is one, and this would take place if the variable under study is independent of all of the other explanatory variables. As a rule of thumb, usually if the VIF is below 5, multicollinearity is usually assumed to be negligible, whereas if it is greater than or equal to 5, the problem is sufficiently serious that some remedial action is warranted. Some researchers use a threshold of 10, rather than 5, to indicate whether multicollinearity is sufficiently large to be cause for concern.

5.8.2 Problems if Near Multicollinearity is Present but Ignored

First, R^2 will be high but the individual coefficients will have high standard errors, so that the regression ‘looks good’ as a whole, but the individual variables are not significant.[3] This arises in the context of very closely related explanatory variables as a consequence of the difficulty in observing the individual contribution of each variable to the overall fit of the regression. Second, the regression becomes very sensitive to small changes in the specification, so that adding or removing an explanatory variable leads to large changes in the coefficient values or significances of the other variables. Finally, near multicollinearity will thus make confidence intervals for the parameters very wide, and significance tests might therefore give inappropriate conclusions, and so make it difficult to draw sharp inferences.

5.8.3 Solutions to the Problem of Multicollinearity

A number of alternative estimation techniques have been proposed that are valid in the presence of multicollinearity – for example, ridge regression, or principal components. Principal components analysis (PCA) was discussed briefly in Appendix 4.1 to Chapter 4. Many researchers do not use these techniques, however, as they can be complex, their properties are less well understood than those of the OLS estimator and, above all, many econometricians would argue that multicollinearity is more a problem with the data than with the model or estimation method.

Other, more ad hoc methods for dealing with the possible existence of near multicollinearity include

- **Ignore it**, if the model is otherwise adequate, i.e., statistically and in terms of each coefficient being of a plausible magnitude and having an appropriate sign. Sometimes, the existence of multicollinearity does not reduce the *t*-ratios on variables that would have been significant without the multicollinearity sufficiently to make them insignificant. It is worth stating that the presence of near multicollinearity does not affect the BLUE properties of the OLS estimator – i.e., it will still be consistent, unbiased and efficient since the presence of near multicollinearity does not violate any of the CLRM assumptions (1)–(4). However, in the presence of near multicollinearity, it will be hard to obtain small standard errors. This will not matter if the aim of the model-building exercise is to produce forecasts from the estimated model, since the forecasts will be unaffected by the presence of near multicollinearity so long as this relationship between the explanatory variables continues to hold over the forecasted sample.
- **Drop one of the collinear variables**, so that the problem disappears. However, this may be unacceptable to the researcher if there were strong *a priori* theoretical reasons for including both variables in the model. Also, if the removed variable was relevant in the data generating process for *y*, an omitted variable bias would result (see Section 5.10).
- **Transform the highly correlated variables into a ratio** and include only the ratio and not the individual variables in the regression. Again, this may be unacceptable if financial theory suggests that changes in the dependent variable should occur following changes in the individual explanatory variables, and not a ratio of them.
- Finally, as stated above, it is also often said that near multicollinearity

is *more a problem with the data than with the model*, so that there is insufficient information in the sample to obtain estimates for all of the coefficients. This is why near multicollinearity leads coefficient estimates to have wide standard errors, which is exactly what would happen if the sample size were small. An increase in the sample size will usually lead to an increase in the accuracy of coefficient estimation and consequently a reduction in the coefficient standard errors, thus enabling the model to better dissect the effects of the various explanatory variables on the explained variable. A further possibility, therefore, is for the researcher to **go out and collect more data** – for example, by taking a longer run of data, or switching to a higher frequency of sampling. Of course, it may be infeasible to increase the sample size if all available data are being utilised already. A further method of increasing the available quantity of data as a potential remedy for near multicollinearity would be to **use a pooled sample**. This would involve the use of data with both cross-sectional and time-series dimensions (see Chapter 11).

5.9 Adopting the Wrong Functional Form

A further implicit assumption of the classical linear regression model is that the appropriate 'functional form' is linear. This means that the appropriate model is assumed to be linear in the parameters, and that in the bivariate case, the relationship between *y* and *x* can be represented by a straight line. However, this assumption may not always be upheld. Whether the model should be linear can be formally tested using Ramsey's (1969) RESET test, which is a general test for misspecification of functional form. Essentially, the method works by using higher order terms of the fitted values (e.g., $\hat{y}_t^2, \hat{y}_t^3,$ etc.) in an auxiliary regression. The auxiliary regression is thus one where y_t, the dependent variable from the original regression, is regressed on powers of the fitted values together with the original explanatory variables

$$y_t = \alpha_1 + \alpha_2\hat{y}_t^2 + \alpha_3\hat{y}_t^3 + \cdots + \alpha_p\hat{y}_t^p + \sum \beta_i x_{it} + v_t \quad \textbf{(5.52)}$$

Higher order powers of the fitted values of *y* can capture a variety of nonlinear relationships, since they embody higher order powers and cross-products of the original explanatory variables, e.g.,

$$\hat{y}_t^2 = (\hat{\beta}_1 + \hat{\beta}_2 x_{2t} + \hat{\beta}_3 x_{3t} + \cdots + \hat{\beta}_k x_{kt})^2 \quad \textbf{(5.53)}$$

We are interested in testing the joint null hypothesis that $\alpha_2 = 0$ and $\alpha_3 = 0$ and … and $\alpha_p = 0$. Note that in some applications, equation (5.52) is broken into two stages: first a standard linear regression is undertaken and the residuals (call these $\hat{u}_t$ following the conventional notation) are collected. Then the $\hat{u}_t$ become the dependent variable in a second stage regression that includes the powers of the fitted values $(\hat{y}_t^2, \hat{y}_t^3)$, etc., with only the latter and a constant included in the auxiliary regression

$$\hat{u}_t = \alpha_1 + \alpha_2 \hat{y}_t^2 + \alpha_3 \hat{y}_t^3 + \cdots + \alpha_p \hat{y}_t^p + v_t \qquad \textbf{(5.54)}$$

The residuals in the auxiliary regression, $\hat{v}_t$ would be the same in both cases. The value of R^2 is obtained from the regression (5.52), and the test statistic, given by TR^2, is distributed asymptotically as a $\chi^2(p - 1)$. Note that the degrees of freedom for this test will be $(p - 1)$ and not p. This arises because p is the highest order term in the fitted values used in the auxiliary regression and thus the test will involve $p - 1$ terms, one for the square of the fitted value, one for the cube, …, one for the pth power. If the value of the test statistic is greater than the χ^2 critical value, reject the null hypothesis that the functional form was correct.

5.9.1 What if the Functional Form is Found to be Inappropriate?

One possibility would be to switch to a non-linear model, but the RESET test presents the user with no guide as to what a better specification might be! Also, non-linear models in the parameters typically preclude the use of OLS, and require the use of a non-linear estimation technique. Some non-linear models can still be estimated using OLS, provided that they are linear in the parameters. For example, if the true model is of the form

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{2t}^2 + u_t \qquad \textbf{(5.55)}$$

– that is, a second order polynomial (i.e., a quadratic) in x – and the researcher assumes that the relationship between y_t and x_t is linear (so x_{2t}^2 is missing from the specification), this is simply a special case of omitted variables, with the usual problems (see Section 5.10) and obvious remedy.

Sometimes a quadratic form of equation such as that in equation (5.55) is useful to allow for a relationship where y increases with x_2 at an increasing rate, or where y initially increases as x increases but then the increase tails off and eventually reverses as x increases further. These two situations are displayed in Figure 5.12. In the left-hand graph, y increases at an increasing rate (at least over the relevant range of values of x) and both β_2 and β_3 would be positive. By contrast, the right-hand figure shows the situation where β_2 is positive but β_3 is negative. As we saw in Chapter 3, in this situation we would obtain a ∩=shape for the curve since the squared term would come to dominate the overall behaviour of the function as x increases. To offer an example of where this might be relevant, it has been suggested in the research literature that the relationship between age and attitude to risk is non-linear, so that risk tolerance rises with age for a certain range (e.g., from 18–40 years) and then declines thereafter – we could capture this with a quadratic where x is years of age and y is some measure or risk tolerance.

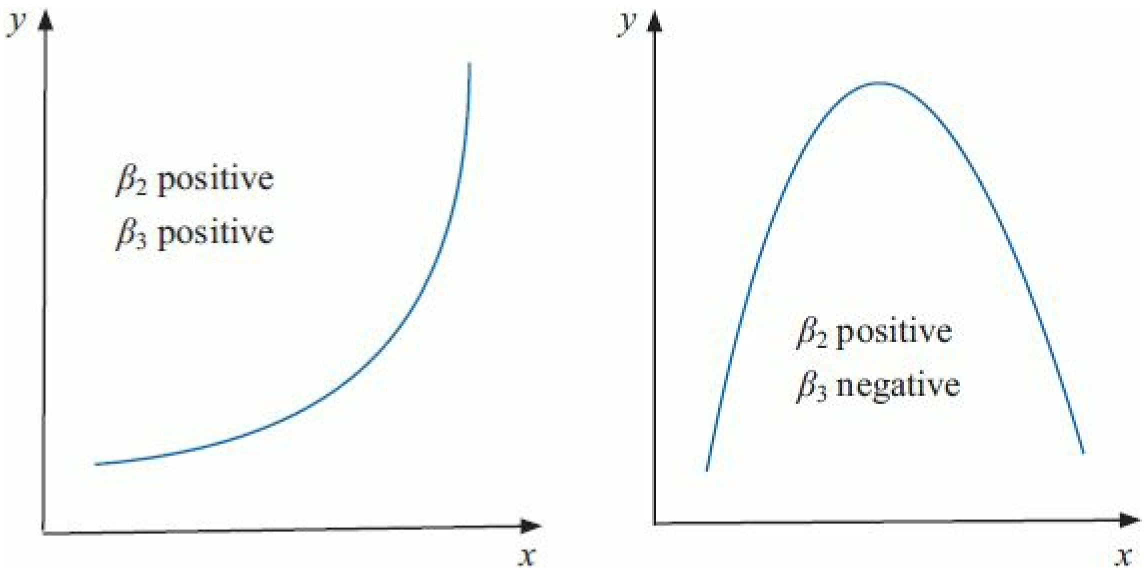


Figure 5.12 Relationship between y and x_2 in a quadratic regression for different values of β_2 and β_3

We could add even higher order terms to equation (5.55), such as a cubic (x_{2t}^3) or quartic (x_{2t}^4) term. It might be that a cubic would be useful to capture something like a point of inflection, where the relationship between x and y hits a stationary point but it is rare that we would be able to justify any higher order term than a quadratic from the perspective of its relevance.

An alternative possibility is that the model may be multiplicatively non-linear or have a more complex relationship that cannot be captured by simply adding higher order powers of explanatory variables to the regression model. Therefore, another approach that is sensible in this case would be to transform the data into logarithms. This will linearise many previously multiplicative models into additive ones. For example, consider again the exponential growth model

$$y_t = \beta_1 x_t^{\beta_2} u_t \tag{5.56}$$

Taking logs, this becomes

$$\ln(y_t) = \ln(\beta_1) + \beta_2 \ln(x_t) + \ln(u_t) \tag{5.57}$$

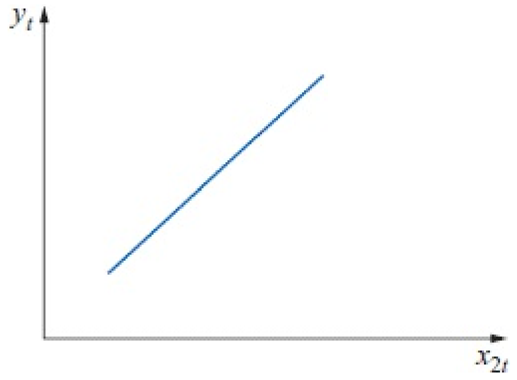
or

$$Y_t = \alpha + \beta_2 X_t + v_t \tag{5.58}$$

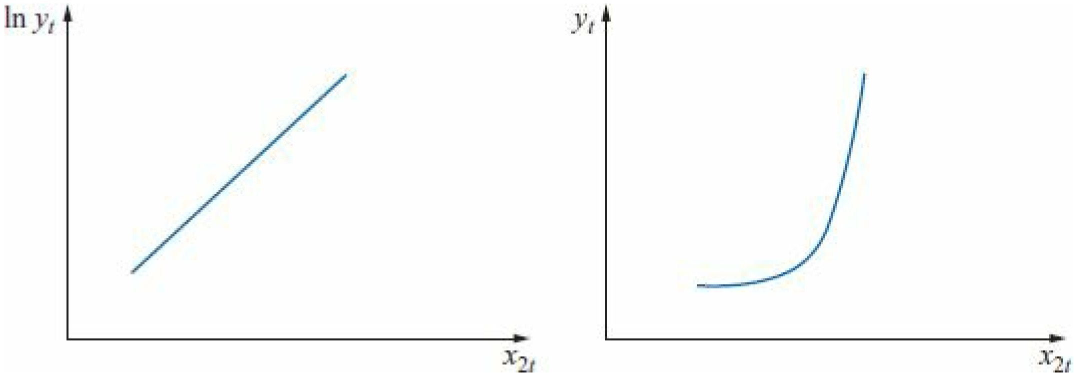
where $Y_t = \ln(y_t)$, $\alpha = \ln(\beta_1)$, $X_t = \ln(x_t)$, $v_t = \ln(u_t)$. Thus a simple logarithmic transformation makes this model a standard linear bivariate regression equation that can be estimated using OLS.

Loosely following the treatment given in Stock and Watson (2011), the following list shows four different functional forms for models that are either linear or can be made linear following a logarithmic transformation to one or more of the dependent or independent variables, examining only a bivariate specification for simplicity. Care is needed when interpreting the coefficient values in each case.

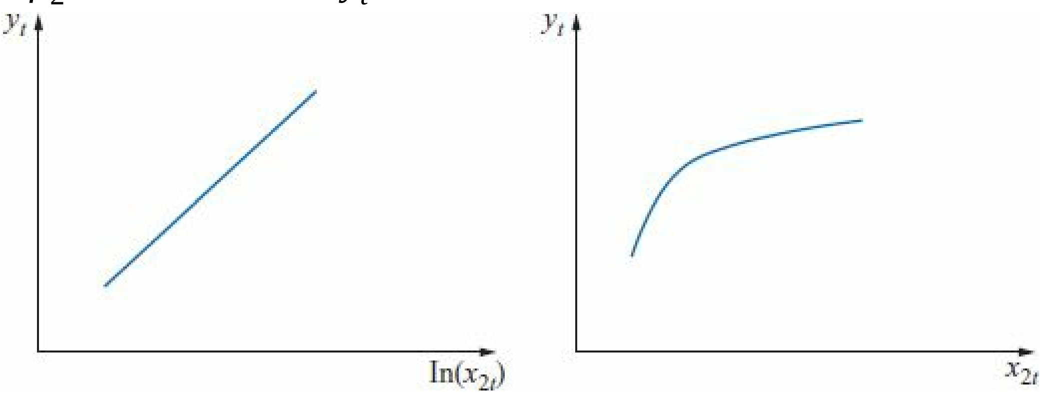
(1) Linear model: $y_t = \beta_1 + \beta_2 x_{2t} + u_t$; a 1-unit increase in x_{2t} causes a β_2-unit increase in y_t.



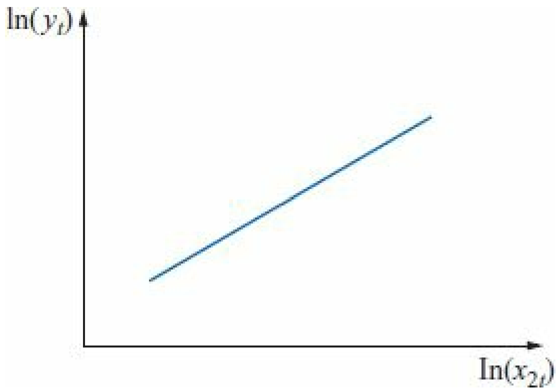
(2) Log-linear: $ln(y_t) = \beta_1 + \beta_2 x_{2t} + u_t$; a 1-unit increase in x_{2t} causes a $100 \times \beta_2$% increase in y_t.



(3) Linear-log: $y_t = \beta_1 + \beta_2 ln(x_{2t}) + u_t$; a 1% increase in x_{2t} causes a $0.01 \times \beta_2$- unit increase in y_t.



(4) Double log: $ln(y_t) = \beta_1 + \beta_2 ln(x_{2t}) + u_t$; a 1% increase in x_{2t} causes a β_2% increase in y_t. Note that to plot y against x_2 would be more complex since the shape would depend on the size of β_2.



Note also that we cannot use R^2 or adjusted R^2 to determine which of these four types of model is most appropriate since the dependent variables are different across some of the models.

5.10 Omission of an Important Variable

What would be the effects of excluding from the estimated regression a variable that is a determinant of the dependent variable? For example, suppose that the true, but unknown, data generating process is represented by

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 x_{4t} + \beta_5 x_{5t} + u_t \tag{5.59}$$

but the researcher estimated a model of the form

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 x_{4t} + u_t \tag{5.60}$$

so that the variable x_{5t} is omitted from the model. The consequence would be that the estimated coefficients on all the other variables will be biased and inconsistent unless the excluded variable is uncorrelated with all the included variables. Even if this condition is satisfied, the estimate of the coefficient on the constant term will be biased, which would imply that any forecasts made from the model would be biased. The standard errors will also be biased (upwards), and hence hypothesis tests could yield inappropriate inferences. Further intuition is offered in Dougherty (1992, pp. 168–73).

5.11 Inclusion of an Irrelevant Variable

Suppose now that the researcher makes the opposite error to Section 5.10, i.e., that the true data generating process (DGP) was represented by

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 x_{4t} + u_t \tag{5.61}$$

but the researcher estimates a model of the form

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 x_{4t} + \beta_5 x_{5t} + u_t \tag{5.62}$$

thus incorporating the superfluous or irrelevant variable x_{5t}. As x_{5t} is irrelevant, the expected value of β_5 is zero, although in any practical application, its estimated value is very unlikely to be exactly zero. The consequence of including an irrelevant variable would be that the coefficient estimators would still be consistent and unbiased, but the estimators would be inefficient. This would imply that the standard errors for the coefficients are likely to be inflated relative to the values which they would have taken if the irrelevant variable had not been included. Variables which would otherwise have been marginally significant may no longer be so in the presence of irrelevant variables. In general, it can also be stated that the extent of the loss of efficiency will depend positively on the absolute value of the correlation between the included irrelevant variable and the other explanatory variables.

Summarising the last two sections it is evident that when trying to determine whether to err on the side of including too many or too few variables in a regression model, there is an implicit trade-off between inconsistency and efficiency; many researchers would argue that while in an ideal world, the model will incorporate precisely the correct variables – no more and no less – the former problem is more serious than the latter and therefore in the real world, one should err on the side of incorporating marginally significant variables.

5.12 Parameter Stability Tests

So far, regressions of a form such as

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + u_t \qquad \textbf{(5.63)}$$

have been estimated. These regressions embody the implicit assumption that the parameters (β_1, β_2 and β_3) are constant for the entire sample, both for the data period used to estimate the model, and for any subsequent period used in the construction of forecasts.

This implicit assumption can be tested using parameter stability tests. The idea is essentially to split the data into sub-periods and then to estimate up to three models, for each of the sub-parts and for all the data and then to 'compare' the *RSS* of each of the models. There are two types of test that will be considered, namely the Chow (analysis of variance) test and predictive failure tests.

5.12.1 The Chow Test

The steps involved are shown in Box 5.7. Note that it is also possible to use a dummy variables approach to calculating both Chow and predictive failure tests. In the case of the Chow test, the unrestricted regression would contain dummy variables for the intercept and for all of the slope coefficients (see also Chapter 10). For example, suppose that the regression is of the form

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + u_t \qquad \textbf{(5.64)}$$

If the split of the total of T observations is made so that the sub-samples contain T_1 and T_2 observations (where $T_1 + T_2 = T$), the unrestricted regression would be given by

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 D_t + \beta_5 D_t x_{2t} + \beta_6 D_t x_{3t} + v_t \quad \textbf{(5.65)}$$

where $D_t = 1$ for $t \in T_1$ and zero otherwise. In other words, D_t takes the value one for observations in the first sub-sample and zero for observations in the second subsample. The Chow test viewed in this way would then be a standard F-test of the joint restriction H_0: $\beta_4 = 0$ and $\beta_5 = 0$ and $\beta_6 = 0$, with equations (5.64) and (5.65) being the unrestricted and restricted regressions, respectively.

BOX 5.7 Conducting a Chow test

(1) *Split the data into two sub-periods* Estimate the regression over the whole period and then for the two sub-periods separately (three regressions). Obtain the *RSS* for each regression.

(2) *The restricted regression is now the regression for the whole period* while the 'unrestricted regression' comes in two parts: one for each of the sub-samples. It is thus possible to form an *F*-test, which is based on the difference between the *RSS*s. The statistic is

$$\text{test statistic} = \frac{RSS - (RSS_1 + RSS_2)}{RSS_1 + RSS_2} \times \frac{T - 2k}{k} \quad \textbf{(5.66)}$$

where RSS = residual sum of squares for whole sample

RSS_1 = residual sum of squares for sub-sample 1

RSS_2 = residual sum of squares for sub-sample 2

T = number of observations

$2k$ = number of regressors in the 'unrestricted' regression (since it comes in two parts)

k = number of regressors in (each) 'unrestricted' regression

The unrestricted regression is the one where the restriction has not been imposed on the model. Since the restriction is that the coefficients are equal across the sub-samples, the restricted regression will be the single regression for the whole sample.

Thus, the test is one of how much the residual sum of squares for the whole sample (RSS) is bigger than the sum of the residual sums of squares for the two sub-samples ($RSS_1 + RSS_2$). If the coefficients do not change much between the samples, the residual sum of squares will not rise much upon imposing the restriction. Thus the test statistic in equation (5.66) can be considered a straightforward application of the standard F-test formula discussed in Chapter 4. The restricted residual sum of squares in equation (5.66) is RSS, while the unrestricted residual sum of squares is ($RSS_1 + RSS_2$). The number of restrictions is equal to the number of coefficients that are estimated for each of the regressions, i.e., k. The number of regressors in the unrestricted regression (including the constants) is $2k$, since the unrestricted regression comes in two parts, each with k regressors.

(3) *Perform the test* If the value of the test statistic is greater than the critical value from the F-distribution, which is an $F(k, T - 2k)$, then reject the null hypothesis that the parameters are stable over time.

EXAMPLE 5.4

Suppose that it is now January 1993. Consider the following regression for the standard CAPM β for the returns on a stock

$$r_{gt} = \alpha + \beta r_{Mt} + u_t \tag{5.67}$$

where r_{gt} and r_{Mt} are excess returns on Glaxo shares and on a market portfolio, respectively. Suppose that you are interested in estimating beta using monthly data from 1981 to 1992, to aid a stock selection decision. Another researcher expresses concern that the October 1987 stock market crash fundamentally altered the risk–return relationship. Test this conjecture using a Chow test. The model for each sub-period is

$1981M1{-}1987M10$

$$\hat{r}_{gt} = 0.24 + 1.2 r_{Mt} \quad T = 82 \quad RSS_1 = 0.03555 \tag{5.68}$$

1987*M*11–1992*M*12

$$\hat{r}_{gt} = 0.68 + 1.53r_{Mt} \quad T = 62 \quad RSS_2 = 0.00336 \tag{5.69}$$

1981*M*1–1992*M*12

$$\hat{r}_{gt} = 0.39 + 1.37r_{Mt} \quad T = 144 \quad RSS = 0.0434 \tag{5.70}$$

The null hypothesis is

$$H_0: \alpha_1 = \alpha_2 \text{ and } \beta_1 = \beta_2$$

where the subscripts 1 and 2 denote the parameters for the first and second sub-samples, respectively. The test statistic will be given by

$$\begin{aligned} \textit{test statistic} &= \frac{0.0434 - (0.0355 + 0.00336)}{0.0355 + 0.00336} \times \frac{144 - 4}{2} \\ &= 7.698 \end{aligned} \tag{5.71}$$

The test statistic should be compared with a 5%, $F(2, 140) = 3.06$. H_0 is rejected at the 5% level and hence it is concluded that the restriction that the coefficients are the same in the two periods cannot be employed. The appropriate modelling response would probably be to employ only the second part of the data in estimating the CAPM beta relevant for investment decisions made in early 1993.

5.12.2 The Predictive Failure Test

A problem with the Chow test is that it is necessary to have enough data to do the regression on both sub-samples, i.e., $T_1 \gg k$, $T_2 \gg k$. This may not hold in the situation where the total number of observations available is small. Even more likely is the situation where the researcher would like to examine the effect of splitting the sample at some point very close to the start or very close to the end of the sample. An alternative formulation of a test for the stability of the model is the predictive failure test, which requires estimation for the full sample and one of the sub-samples only. The predictive failure test works by estimating the regression over a 'long' sub-period (i.e., most of the data) and then using those coefficient estimates for predicting values of *y* for the other period. These predictions for *y* are then implicitly compared with the actual values. Although it can be expressed in several different ways, the null hypothesis for this test is that the prediction errors for all of the forecasted observations are zero.

To calculate the test:

- **Run the regression for the whole period** (the restricted regression) and obtain the *RSS*.
- **Run the regression for the 'large' sub-period** and obtain the *RSS* (called RSS_1). Note that in this book, the number of observations for the long estimation sub-period will be denoted by T_1 (even though it may come second). The test statistic is given by

$$\text{test statistic} = \frac{RSS - RSS_1}{RSS_1} \times \frac{T_1 - k}{T_2} \tag{5.72}$$

where T_2 = number of observations that the model is attempting to 'predict'. The test statistic will follow an $F(T_2, T_1 - k)$.

For an intuitive interpretation of the predictive failure test statistic formulation, consider an alternative way to test for predictive failure using a regression containing dummy variables. A separate dummy variable would be used for each observation that was in the prediction sample. The unrestricted regression would then be the one that includes the dummy variables, which will be estimated using all T observations, and will have $(k + T_2)$ regressors (the k original explanatory variables, and a dummy variable for each prediction observation, i.e., a total of T_2 dummy variables). Thus the numerator of the last part of equation (5.72) would be the total number of observations (T) minus the number of regressors in the unrestricted regression $(k + T_2)$. Noting also that $T - (k + T_2) = (T_1 - k)$, since $T_1 + T_2 = T$, this gives the numerator of the last term in equation (5.72). The restricted regression would then be the original regression containing the explanatoryvariables but none of the dummy variables. Thus the number of restrictions would be the number of observations in the prediction period, which would be equivalent to the number of dummy variables included in the unrestricted regression, T_2.

To offer an illustration, suppose that the regression is again of the form of (5.64), and that the last three observations in the sample are used for a predictive failure test. The unrestricted regression would include three dummy variables, one for each of the observations in T_2

$$r_{gt} = \alpha + \beta r_{Mt} + \gamma_1 D1_t + \gamma_2 D2_t + \gamma_3 D3_t + u_t \quad (5.73)$$

where $D1_t = 1$ for observation $T-2$ and zero otherwise, $D2_t = 1$ for observation $T-1$ and zero otherwise, $D3_t = 1$ for observation T and zero otherwise. In this case, $k = 2$, and $T_2 = 3$. The null hypothesis for the predictive failure test in this regression is that the coefficients on all of the dummy variables are zero (i.e., $H_0 : \gamma_1 = 0$ and $\gamma_2 = 0$ and $\gamma_3 = 0$). Both approaches to conducting the predictive failure test described above are equivalent, although the dummy variable regression is likely to take more time to set up.

However, for both the Chow and the predictive failure tests, the dummy variables approach has the one major advantage that it provides the user with more information. This additional information comes from the fact that one can examine the significances of the coefficients on the individual dummy variables to see which part of the joint null hypothesis is causing a rejection. For example, in the context of the Chow regression, is it the intercept or the slope coefficients that are significantly different across the two sub-samples? In the context of the predictive failure test, use of the dummy variables approach would show for which period(s) the prediction errors are significantly different from zero.

5.12.3 Backward versus Forward Predictive Failure Tests

There are two types of predictive failure tests – forward tests and backwards tests. Forward predictive failure tests are where the last few observations are kept back for forecast testing. For example, suppose that observations for 1980Q1–2013Q4 are available. A forward predictive failure test could involve estimating the model over 1980Q1–2012Q4 and forecasting 2013Q1–2013Q4. Backward predictive failure tests attempt to ‘back-cast’ the first few observations, e.g., if data for 1980Q1–2013Q4 are available, and the model is estimated over 1981Q1–2013Q4 and back-cast 1980Q1–1980Q4. Both types of test offer further evidence on the stability of the regression relationship over the whole sample period.

EXAMPLE 5.5

Suppose that the researcher decided to determine the stability of the estimated model for stock returns over the whole sample in Example 5.4 by using a predictive failure test of the last two years of observations. The following models would be estimated

1981$M$1–1992$M$12 (whole sample)

$$\hat{r}_{gt} = 0.39 + 1.37 r_{Mt} \qquad T = 144 \qquad RSS = 0.0434 \tag{5.74}$$

1981$M$1–1990$M$12 (‘long sub-sample’)

$$\hat{r}_{gt} = 0.32 + 1.31 r_{Mt} \qquad T = 120 \qquad RSS_1 = 0.0420 \tag{5.75}$$

Can this regression adequately ‘forecast’ the values for the last two years? The test statistic would be given by

$$\begin{aligned} \text{test statistic} &= \frac{0.0434 - 0.0420}{0.0420} \times \frac{120 - 2}{24} \\ &= 0.164 \end{aligned} \tag{5.76}$$

Compare the test statistic with an $F(24, 118) = 1.66$ at the 5% level. So the null hypothesis that the model can adequately predict the last few observations would not be rejected. It would thus be concluded that the model did not suffer from predictive failure during the 1991$M$1–1992$M$12 period.

5.12.4 How Can the Appropriate Sub-Parts to Use be Decided?

As a rule of thumb, some or all of the following methods for selecting where the overall sample split occurs could be used:

- Plot the dependent variable over time and split the data accordingly to *any obvious structural changes in the series*, as illustrated in Figure 5.13. It is clear that y underwent a large fall in its value around observation 175, and it is possible that this may have caused a change in its behaviour. A Chow test could be conducted with the sample split at this observation.
- Split the data according to *any known important historical events* (e.g., a stock market crash, change in market microstructure, new government elected). The argument is that a major change in the underlying environment in which y is measured is more likely to cause a structural change in the model’s parameters than a relatively trivial change.
- Use all but the last few observations and do a *forward predictive failure test* on those.
- Use all but the first few observations and do a *backward predictive*

failure test on those.

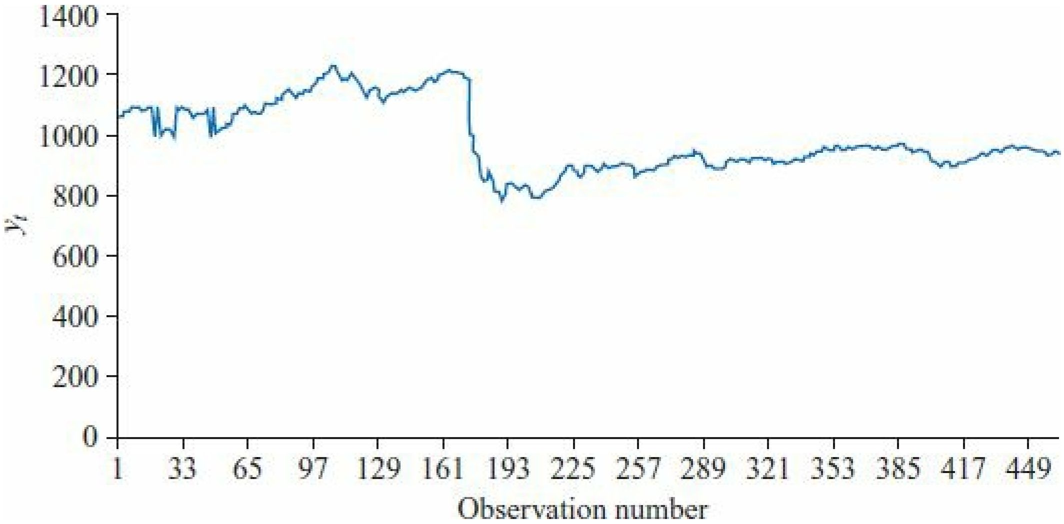


Figure 5.13 Plot of a variable showing suggestion for break date

If a model is good, it will survive a Chow or predictive failure test with any break date. If the Chow or predictive failure tests are failed, two approaches could be adopted. Either the model is respecified, for example, by including additional variables, or separate estimations are conducted for each of the sub-samples. On the other hand, if the Chow and predictive failure tests show no rejections, it is empirically valid to pool all of the data together in a single regression. This will increase the sample size and therefore the number of degrees of freedom relative to the case where the sub-samples are used in isolation.

5.12.5 The QLR Test

The Chow and predictive failure tests will work satisfactorily if the date of a structural break in a financial time series can be specified. But more often, a researcher will not know the break date in advance, or may know only that it lies within a given range (sub-set) of the sample period. In such circumstances, a modified version of the Chow test, known as the *Quandt likelihood ratio (QLR) test*, named after Quandt (1960), can be used instead. The test works by automatically computing the usual Chow *F*-test statistic repeatedly with different break dates, then the break date giving the largest *F*-statistic value is chosen. While the test statistic is of the *F*-variety, it will follow a non-standard distribution rather than an *F*-distribution since we are selecting the largest from a number of *F*-statistics rather than examining a single one.

The test is well behaved only when the range of possible break dates is sufficiently far from the end points of the whole sample, so it is usual to 'trim' the sample by (typically) 5% at each end. To illustrate, suppose that the full sample comprises 200 observations; then we would test for a structural break between observations 31 and 170 inclusive. The critical values will depend on how much of the sample is trimmed away, the number of restrictions under the null hypothesis (the number of regressors in the original regression as this is effectively a Chow test), and the significance level.

5.12.6 Stability Tests Based on Recursive Estimation

An alternative to the QLR test for use in the situation where a researcher believes that a series may contain a structural break but is unsure of the date is to perform a recursive estimation. This is sometimes known as *recursive least squares* (RLS). The procedure is appropriate only for time-series data or cross-sectional data that have been ordered in some sensible way (for example, a sample of annual stock returns, ordered by market capitalisation). Recursive estimation simply involves starting with a sub-sample of the data, estimating the regression, then sequentially adding one observation at a time and rerunning the regression until the end of the sample is reached. It is common to begin the initial estimation with the very minimum number of observations possible, which will be $k + 1$. So at the first step, the model is estimated using observations 1 to $k + 1$; at the second step, observations 1 to $k + 2$ are used and so on; at the final step, observations 1 to T are used. The final result will be the production of $T - k$ separate estimates of every parameter in the regression model.

It is to be expected that the parameter estimates produced near the start of the recursive procedure will appear rather unstable since these estimates are being produced using so few observations, but the key question is whether they then gradually settle down or whether the volatility continues through the whole sample. Seeing the latter would be an indication of parameter instability.

It should be evident that RLS in itself is not a statistical test for parameter stability as such, but rather it provides qualitative information which can be plotted and thus gives a very visual impression of how stable the parameters appear to be. But two important stability tests, known as the *CUSUM* and *CUSUMSQ* tests, are derived from the residuals of the recursive estimation (known as the recursive residuals).[4] The CUSUM statistic is based on a normalised (i.e., scaled) version of the cumulative sums of the residuals. Under the null hypothesis of perfect parameter stability, the CUSUM statistic is zero however many residuals are included in the sum (because the expected value of a disturbance is always zero). A set of ±2 standard error bands is usually plotted around zero and any statistic lying outside the bands is taken as evidence of parameter instability.

The CUSUMSQ test is based on a normalised version of the cumulative sums of squared residuals. The scaling is such that under the null hypothesis of parameter stability, the CUSUMSQ statistic will start at zero and end the sample with a value of 1. Again, a set of ±2 standard error bands is usually plotted around zero and any statistic lying outside these is taken as evidence of parameter instability.

5.13 Measurement Errors

As stated above, one of the of the assumptions of the classical linear regression model is that the explanatory variables are non-stochastic. One way in which this assumption can be violated is when there is a two-way causal relationship between the explanatory and explained variable, and this situation (*simultaneous equations bias*) is discussed in detail in Chapter 7. A further situation where the assumption will not apply is when there is *measurement error* in one or more of the explanatory variables. Sometimes this is also known as the *errors-in-variables* problem. Measurement errors can occur in a variety of circumstances – for example, macroeconomic variables are almost always estimated quantities (GDP, inflation and so on), as is most information contained in company accounts. Similarly, it is sometimes the case that we cannot observe or obtain data on a variable we require and so we need to use a *proxy variable* – for instance, many models include expected quantities (e.g., expected inflation) but since we cannot typically measure expectations, we need to use a proxy. More generally, measurement error could be present in the dependent or independent variables, and each of these cases is considered in the following sub-sections.

5.13.1 Measurement Error in the Explanatory Variable(s)

For simplicity, suppose that we wish to estimate a model containing just one explanatory variable, x_t

$$y_t = \beta_1 + \beta_2 x_t + u_t \qquad \textbf{(5.77)}$$

where u_t is a disturbance term. Suppose further that x_t is measured with error so that instead of observing its true value, we observe a noisy version, $\tilde{x}_t$ that comprises the actual x_t plus some additional noise, v_t, that is independent of x_t and u_t

$$\tilde{x}_t = x_t + v_t \tag{5.78}$$

Taking equation (5.77) and substituting in for x_t from equation (5.78), we get

$$y_t = \beta_1 + \beta_2(\tilde{x}_t - v_t) + u_t \tag{5.79}$$

We can rewrite this equation by separately expressing the composite error term, $(u_t - \beta_2 v_t)$

$$y_t = \beta_1 + \beta_2 \tilde{x}_t + (u_t - \beta_2 v_t) \tag{5.80}$$

It should be clear from equations (5.78) and (5.80) that the explanatory variable measured with error, $(\tilde{x})$ and the composite error term $(u_t - \beta_2 v_t)$ are correlated since both depend on v_t. Thus the requirement that the explanatory variables are non-stochastic does not hold. This causes the parameters to be estimated inconsistently. It can be shown that the size of the bias in the estimates will be a function of the variance of the noise in x_t as a proportion of the overall disturbance variance. It can be further shown that if β_2 is positive, the bias will be negative but if β_2 is negative, the bias will be positive – in other words, the parameter estimate will always be biased towards zero as a result of the measurement noise.

The impact of this estimation bias when the explanatory variables are measured with error can be quite important and is a serious issue in particular when testing asset pricing models. The standard approach to testing the CAPM pioneered by Fama and MacBeth (1973) comprises two stages (discussed more fully in Chapter 14). Stage one is to run separate time-series regressions for each firm to estimate the betas and the second stage involves running a cross-sectional regression of the stock returns on the betas. Since the betas are estimated at the first stage rather than being directly observable, they will surely contain measurement error. In the finance literature, the effect of this has sometimes been termed *attenuation bias*. Early tests of the CAPM showed that the relationship between beta and returns was positive but smaller than expected, and this is precisely what would happen as a result of measurement error in the betas. Various approaches to solving this issue have been proposed, the most common of which is to use portfolio betas in place of individual stock betas in the second stage. The hope is that this will smooth out the estimation error in the betas. An alternative approach attributed to Shanken (1992) is to modify the standard errors in the second-stage regression to adjust directly for the measurement errors in the betas. More discussion of this issue will be presented in Chapter 15.

5.13.2 Measurement Error in the Explained Variable

Measurement error in the explained variable is much less serious than in the explanatory variable(s); recall that one of the motivations for the inclusion of the disturbance term in a regression model is that it can capture measurement errors in y. Thus, when the explained variable is measured with error, the disturbance term will in effect be a composite of the usual disturbance term and another source of noise from the measurement error. In such circumstances, the parameter estimates will still be consistent and unbiased and the usual formulae for calculating standard errors will still be appropriate. The only consequence is that the additional noise means that the standard errors will be enlarged relative to the situation where there was no measurement error in y.

5.14 A Strategy for Constructing Econometric Models and a Discussion of Model-Building Philosophies

The objective of many econometric model-building exercises is to build a statistically adequate empirical model which satisfies the assumptions of the CLRM, is parsimonious, has the appropriate theoretical interpretation, and has the right ‘shape’ (i.e., all signs on coefficients are ‘correct’ and all sizes of coefficients are ‘correct’).

But how might a researcher go about achieving this objective? A common approach to model building is the ‘LSE’ or general-to-specific methodology associated with Sargan and Hendry. This approach essentially involves starting with a large model which is statistically adequate and restricting and rearranging the model to arrive at a parsimonious final formulation. Hendry’s approach (see Gilbert, 1986) argues that a good model is consistent with the data and with theory. A good model will also encompass rival models, which means that it can explain all that rival models can and more. The Hendry methodology suggests the extensive use of diagnostic tests to ensure the statistical adequacy of the model.

An alternative philosophy of econometric model-building, which pre-dates Hendry's research, is that of starting with the simplest model and adding to it sequentially so that it gradually becomes more complex and a better description of reality. This approach, associated principally with Koopmans (1937), is sometimes known as a 'specific-to-general' or 'bottoms-up' modelling approach. Gilbert (1986) termed this the 'Average Economic Regression' since most applied econometric work had been tackled in that way. This term was also having a joke at the expense of a top economics journal that published many papers using such a methodology.

Hendry and his co-workers have severely criticised this approach, mainly on the grounds that diagnostic testing is undertaken, if at all, almost as an after-thought and in a very limited fashion. However, if diagnostic tests are not performed, or are performed only at the end of the model-building process, all earlier inferences are potentially invalidated. Moreover, if the specific initial model is generally misspecified, the diagnostic tests themselves are not necessarily reliable in indicating the source of the problem. For example, if the initially specified model omits relevant variables which are themselves autocorrelated, introducing lags of the included variables would not be an appropriate remedy for a significant *DW* test statistic. Thus the eventually selected model under a specific-to-general approach could be sub-optimal in the sense that the model selected using a general-to-specific approach might represent the data better. Under the Hendry approach, diagnostic tests of the statistical adequacy of the model come first, with an examination of inferences for financial theory drawn from the model left until after a statistically adequate model has been found.

According to Hendry and Richard (1982), a final acceptable model should satisfy several criteria (adapted slightly here). The model should:

- be logically plausible
- be consistent with underlying financial theory, including satisfying any relevant parameter restrictions
- have regressors that are uncorrelated with the error term
- have parameter estimates that are stable over the entire sample
- have residuals that are white noise (i.e., completely random and exhibiting no patterns)
- be capable of explaining the results of all competing models and more

The last of these is known as the *encompassing principle*. A model that nests within it a smaller model always trivially encompasses it. But a small model is particularly favoured if it can explain all of the results of a larger model; this is known as *parsimonious encompassing*.

The advantages of the general-to-specific approach are that it is statistically sensible and also that the theory on which the models are based usually has nothing to say about the lag structure of a model. Therefore, the lag structure incorporated in the final model is largely determined by the data themselves. Furthermore, the statistical consequences from excluding relevant variables are usually considered more serious than those from including irrelevant variables.

The general-to-specific methodology is conducted as follows. The first step is to form a 'large' model with lots of variables on the RHS. This is known as a generalised unrestricted model (GUM), which should originate from financial theory, and which should contain all variables thought to influence the dependent variable. At this stage, the researcher is required to ensure that the model satisfies all of the assumptions of the CLRM. If the assumptions are violated, appropriate actions should be taken to address or allow for this, e.g., taking logs, adding lags, adding dummy variables.

It is important that the steps above are conducted prior to any hypothesis testing. It should also be noted that the diagnostic tests presented above should be cautiously interpreted as general rather than specific tests. In other words, rejection of a particular diagnostic test null hypothesis should be interpreted as showing that there is something wrong with the model. So, for example, if the RESET test or White's test show a rejection of the null, such results should not be immediately interpreted as implying that the appropriate response is to find a solution for inappropriate functional form or heteroscedastic residuals, respectively. It is quite often the case that one problem with the model could cause several assumptions to be violated simultaneously. For example, an omitted variable could cause failures of the RESET, heteroscedasticity and autocorrelation tests. Equally, a small number of large outliers could cause non-normality and residual autocorrelation (if they occur close together in the sample) and heteroscedasticity (if the outliers occur for a narrow range of the explanatory variables). Moreover, the diagnostic tests themselves do not operate optimally in the presence of other types of misspecification since they essentially assume that the model is correctly specified in all other respects. For example, it is not clear that tests for heteroscedasticity will behave well if the residuals are autocorrelated.

Once a model that satisfies the assumptions of the CLRM has been obtained, it could be very big, with large numbers of lags and independent variables. The next stage is therefore to reparameterise the model by knocking out very insignificant regressors. Also, some coefficients may be insignificantly different from each other, so that they can be combined. At each stage, it should be checked whether the assumptions of the CLRM are still upheld. If this is the case, the researcher should have arrived at a statistically adequate empirical model that can be used for testing underlying financial theories, forecasting future values of the dependent variable, or for formulating policies.

However, needless to say, the general-to-specific approach also has its critics. For small or moderate sample sizes, it may be impractical. In such instances, the large number of explanatory variables will imply a small number of degrees of freedom. This could mean that none of the variables is significant, especially if they are highly correlated. This being the case, it would not be clear which of the original long list of candidate regressors should subsequently be dropped. Moreover, in any case the decision on which variables to drop may have profound implications for the final specification of the model. A variable whose coefficient was not significant might have become significant at a later stage if other variables had been dropped instead.

In theory, sensitivity of the final specification to the various possible paths of variable deletion should be carefully checked. However, this could imply checking many (perhaps even hundreds) of possible specifications. It could also lead to several final models, none of which appears noticeably better than the others.

The general-to-specific approach, if followed faithfully to the end, will hopefully lead to a statistically valid model that passes all of the usual model diagnostic tests and contains only statistically significant regressors. However, the final model could also be a bizarre creature that is devoid of any theoretical interpretation. There would also be more than just a passing chance that such a model could be the product of a statistically vindicated data mining exercise. Such a model would closely fit the sample of data at hand, but could fail miserably when applied to other samples if it is not based soundly on theory.

There now follows another example of the use of the classical linear regression model in finance, based on an examination of the determinants of sovereign credit ratings by Cantor and Packer (1996).

5.15 Determinants of Sovereign Credit Ratings

5.15.1 Background

Sovereign credit ratings are an assessment of the riskiness of debt issued by governments. They embody an estimate of the probability that the borrower will default on her obligation. Two famous US ratings agencies, Moody's and Standard and Poor's (S&P), provide ratings for many governments. Although the two agencies use different symbols to denote the given riskiness of a particular borrower, the ratings of the two agencies are comparable. Gradings are split into two broad categories: investment grade and speculative grade. Investment grade issuers have good or adequate payment capacity, while speculative grade issuers either have a high degree of uncertainty about whether they will make their payments, or are already in default. The highest grade offered by the agencies, for the highest quality of payment capacity, is 'triple A', which Moody's denotes 'Aaa' and S&P denotes 'AAA'. The lowest grade issued to a sovereign in the Cantor and Packer sample was B3 (Moody's) or B–(S&P). Thus the number of grades of debt quality from the highest to the lowest given to governments in their sample is 16.

The central aim of Cantor and Packer's paper is an attempt to explain and model how the agencies arrived at their ratings. Although the ratings themselves are publicly available, the models or methods used to arrive at them are shrouded in secrecy. The agencies also provide virtually no explanation as to what the relative weights of the factors that make up the rating are. Thus, a model of the determinants of sovereign credit ratings could be useful in assessing whether the ratings agencies appear to have acted rationally. Such a model could also be employed to try to predict the rating that would be awarded to a sovereign that has not previously been rated and when a re-rating is likely to occur. The paper continues, among other things, to consider whether ratings add to publicly available information, and whether it is possible to determine what factors affect how the sovereign yields react to ratings announcements.

5.15.2 Data

Cantor and Packer (1996) obtain a sample of government debt ratings for forty-nine countries as of September 1995 that range between the above gradings. The ratings variable is quantified, so that the highest credit quality (Aaa/AAA) in the sample is given a score of 16, while the lowest rated sovereign in the sample is given a score of 1 (B3/B–). This score forms the dependent variable. The factors that are used to explain the variability in the ratings scores are macroeconomic variables. All of these variables embody factors that are likely to influence a government's ability and willingness to service its debt costs. Ideally, the model would also include proxies for socio-political factors, but these are difficult to measure objectively and so are not included. It is not clear in the paper from where the list of factors was drawn. The included variables (with their units of measurement) are:

- *Per capita income* (in 1994 US dollars, thousands). Cantor and Packer argue that *per capita* income determines the tax base, which in turn influences the government's ability to raise revenue.
- *GDP growth* (annual 1991–4 average, %). The growth rate of increase in GDP is argued to measure how much easier it will become to service debt costs in the future.
- *Inflation* (annual 1992–4 average, %). Cantor and Packer argue that high inflation suggests that inflationary money financing will be used to service debt when the government is unwilling or unable to raise the required revenue through the tax system.
- *Fiscal balance* (average annual government budget surplus as a proportion of GDP 1992–4, %). Again, a large fiscal deficit shows that the government has a relatively weak capacity to raise additional revenue and to service debt costs.
- *External balance* (average annual current account surplus as a proportion of GDP 1992–4, %). Cantor and Packer argue that a persistent current account deficit leads to increasing foreign indebtedness, which may be unsustainable in the long run.
- *External debt* (foreign currency debt as a proportion of exports in 1994, %). Reasoning as for external balance (which is the change in external debt over time).
- *Dummy for economic development* (=1 for a country classified by the International Monetary Fund (IMF) as developed, 0 otherwise). Cantor and Packer argue that credit ratings agencies perceive developing countries as relatively more risky beyond that suggested by the values of the other factors listed above.
- *Dummy for default history* (=1 if a country has defaulted, 0 otherwise). It is argued that countries that have previously defaulted experience a large fall in their credit rating.

The income and inflation variables are transformed to their logarithms.

The model is linear and estimated using OLS. Some readers of this book who have a background in econometrics will note that strictly, OLS is not an appropriate technique when the dependent variable can take on only one of a certain limited set of values (in this case, 1, 2, 3, …16). In such applications, a technique such as ordered probit (not covered in this text) would usually be more appropriate. Cantor and Packer argue that any approach other than OLS is infeasible given the relatively small sample size (forty-nine), and the large number (sixteen) of ratings categories.

The results from regressing the rating value on the variables listed above are presented in their exhibit 5, adapted and presented here as Table 5.2. Four regressions are conducted, each with identical independent variables but a different dependent variable. Regressions are conducted for the rating score given by each agency separately, with results presented in columns (4) and (5) of Table 5.2. Occasionally, the ratings agencies give different scores to a country – for example, in the case of Italy, Moody's gives a rating of 'A1', which would generate a score of 12 on a 16-scale. S&P, on the other hand, gives a rating of 'AA', which would score 14 on the 16-scale, two gradings higher. Thus a regression with the average score across the two agencies, and with the difference between the two scores as dependent variables, is also conducted, and presented in columns (3) and (6), respectively, of Table 5.2.

Table 5.2 Determinants and impacts of sovereign credit ratings

Explanatory variable (1)	Expected sign (2)	Dependent variable Average rating (3)	Moody's rating (4)	S&P rating (5)	Differ Moody' (6
Intercept	?	1.442 (0.663)	3.408 (1.379)	−0.524 (−0.223)	3.93 (2.5
Per capita income	+	1.242*** (5.302)	1.027*** (4.041)	1.458*** (6.048)	−0.43 (−2.6
GDP growth	+	0.151 (1.935)	0.130 (1.545)	0.171** (2.132)	−0.0 (0.7
Inflation	−	−0.611*** (−2.839)	−0.630*** (−2.701)	−0.591*** (−2.671)	−0.0 (−0.2
Fiscal	+	0.073	0.049	0.097*	−0.0

balance		(1.324)	(0.818)	(1.71)	(−1.2
External balance	+	0.003 (0.314)	0.006 (0.535)	0.001 (0.046)	0.0 (0.7
External debt	−	−0.013*** (−5.088)	−0.015*** (−5.365)	−0.011*** (−4.236)	−0.00 (−2.1
Development dummy	+	2.776*** (4.25)	2.957*** (4.175)	2.595*** (3.861)	0.3 (0.8
Default dummy	−	−2.042*** (−3.175)	−1.63** (−2.097)	−2.622*** (−3.962)	1.15 (2.6
Adjusted R^2		0.924	0.905	0.926	0.8

Notes: t-ratios in parentheses; *, ** and *** indicate significance at the 10%, 5% and 1% levels, respectively.

Source: Cantor and Packer (1996). Reprinted with permission from *Institutional Investor*.

5.15.3 Interpreting the Models

The models are difficult to interpret in terms of their statistical adequacy, since virtually no diagnostic tests have been undertaken. The values of the adjusted R^2, at over 90% for each of the three ratings regressions, are high for cross-sectional regressions, indicating that the model seems able to capture almost all of the variability of the ratings about their mean values across the sample. There does not appear to be any attempt at reparameterisation presented in the paper, so it is assumed that the authors reached this set of models after some searching.

In this particular application, the residuals have an interesting interpretation as the difference between the actual and fitted ratings. The actual ratings will be integers from 1 to 16, although the fitted values from the regression and therefore the residuals can take on any real value. Cantor and Packer argue that the model is working well as no residual is bigger than 3, so that no fitted rating is more than three categories out from the actual rating, and only four countries have residuals bigger than two categories. Furthermore, 70% of the countries have ratings predicted exactly (i.e., the residuals are less than 0.5 in absolute value).

Now, turning to interpret the models from a financial perspective, it is of interest to investigate whether the coefficients have their expected signs and sizes. The expected signs for the regression results of columns (3)–(5)

are displayed in column (2) of Table 5.2 (as determined by this author). As can be seen, all of the coefficients have their expected signs, although the fiscal balance and external balance variables are not significant or are only very marginally significant in all three cases. The coefficients can be interpreted as the average change in the rating score that would result from a unit change in the variable. So, for example, a rise in *per capita* income of $1,000 will on average increase the rating by 1.0 units according to Moody's and 1.5 units according to S&P. The development dummy suggests that, on average, a developed country will have a rating three notches higher than an otherwise identical developing country. And everything else equal, a country that has defaulted in the past will have a rating two notches lower than one that has always kept its obligation.

By and large, the ratings agencies appear to place similar weights on each of the variables, as evidenced by the similar coefficients and significances across columns (4) and (5) of Table 5.2. This is formally tested in column (6) of the table, where the dependent variable is the difference between Moody's and S&P ratings. Only three variables are statistically significantly differently weighted by the two agencies. S&P places higher weights on income and default history, while Moody's places more emphasis on external debt.

5.15.4 The Relationship Between Ratings and Yields

In this section of the paper, Cantor and Packer try to determine whether ratings have any additional information useful for modelling the cross-sectional variability of sovereign yield spreads over and above that contained in publicly available macroeconomic data. The dependent variable is now the log of the yield spread, i.e.,

ln(Yield on the sovereign bond – Yield on a US Treasury Bond)

One may argue that such a measure of the spread is imprecise, for the true credit spread should be defined by the entire credit quality curve rather than by just two points on it. However, leaving this issue aside, the results are presented in Table 5.3.

Table 5.3 Do ratings add to public information?

Variable	**Expected sign**	**Dependent variable: ln (yield spread)**		
		(1)	**(2)**	**(3)**

Intercept	?	2.105*** (16.148)	0.466 (0.345)	0.074 (0.071)
Average rating	–	–0.221*** (–19.175)		–0.218*** (–4.276)
Per capita income	–		–0.144 (–0.927)	0.226 (1.523)
GDP growth	–		–0.004 (–0.142)	0.029 (1.227)
Inflation	+		0.108 (1.393)	–0.004 (–0.068)
Fiscal balance	–		–0.037 (–1.557)	–0.02 (–1.045)
External balance	–		–0.038 (–1.29)	–0.023 (–1.008)
External debt	+		0.003*** (2.651)	0.000 (0.095)
Development dummy	–		–0.723*** (–2.059)	–0.38 (–1.341)
Default dummy	+		0.612*** (2.577)	0.085 (0.385)
Adjusted R^2		0.919	0.857	0.914

Notes: t-ratios in parentheses; *, **and *** indicate significance at the 10%, 5% and 1% levels, respectively.

Source: Cantor and Packer (1996). Reprinted with permission from *Institutional Investor*.

Three regressions are presented in Table 5.3, denoted specifications (1), (2) and (3). The first of these is a regression of the ln(spread) on only a constant and the average rating (column (1)), and this shows that ratings have a highly significant inverse impact on the spread. Specification (2) is a regression of the ln(spread) on the macroeconomic variables used in the previous analysis. The expected signs are given (as determined by this author) in column (2). As can be seen, all coefficients have their expected signs, although now only the coefficients belonging to the external debt and the two dummy variables are statistically significant. Specification (3) is a regression on both the average rating and the macroeconomic variables. When the rating is included with the macroeconomic factors, none of the latter is any longer significant – only the rating coefficient is statistically significantly different from zero. This message is also portrayed by the adjusted R^2 values, which are highest for the regression containing only the rating, and slightly lower for the regression containing the macroeconomic variables and the rating. One may also observe that, under specification (3), the coefficients on the *per capita* income, GDP growth and inflation variables now have the wrong sign. This is, in fact, never really an issue, for if a coefficient is not statistically significant, it is indistinguishable from zero in the context of hypothesis testing, and therefore it does not matter whether it is actually insignificant and positive or insignificant and negative. Only coefficients that are both of the wrong sign and statistically significant imply that there is a problem with the regression.

It would thus be concluded from this part of the paper that there is no more incremental information in the publicly available macroeconomic variables that is useful for predicting the yield spread than that embodied in the rating. The information contained in the ratings encompasses that contained in the macroeconomic variables.

5.15.5 What Determines How the Market Reacts to Ratings Announcements?

Cantor and Packer also consider whether it is possible to build a model to predict how the market will react to ratings announcements, in terms of the resulting change in the yield spread. The dependent variable for this set of regressions is now the change in the log of the relative spread, i.e., log[(yield – treasury yield)/treasury yield], over a two-day period at the time of the announcement. The sample employed for estimation comprises every announcement of a ratings change that occurred between 1987 and 1994; seventy-nine such announcements were made, spread over eighteen countries. Of these, thirty nine were actual ratings changes by one or more of the agencies, and forty were listed as likely in the near future to experience a regrading. Moody's calls this a 'watchlist', while S&P term it their 'outlook' list. The explanatory variables are mainly dummy variables for

- whether the announcement was positive – i.e., an upgrade
- whether there was an actual ratings change or just listing for probable

regrading

- whether the announcement was made by Moody's or S&P
- whether the bond was speculative grade or investment grade
- whether there had been another ratings announcement in the previous sixty days
- the ratings gap between the announcing and the other agency

The following cardinal variable was also employed:

- the change in the spread over the previous sixty days

The results are presented in Table 5.4, but in this text, only the final specification (numbered 5 in Cantor and Packer's exhibit 11) containing all of the variables described above is included.

Table 5.4 What determines reactions to ratings announcements?

Dependent variable: log relative spread	
Independent variable	Coefficient (*u*–ratio)
Intercept	–0.02 (–1.4)
Positive announcements	0.01 (0.34)
Ratings changes	–0.01 (–0.37)
Moody's announcements	0.02 (1.51)
Speculative grade	0.03** (2.33)
Change in relative spreads from day –60 to day –1	–0.06 (–1.1)
Rating gap	0.03* (1.7)
Other rating announcements from day –60 to day –1	0.05** (2.15)

Adjusted R^2	0.12

Note: * and ** denote significance at the 10% and 5% levels, respectively.

Source: Cantor and Packer (1996). Reprinted with permission from *Institutional Investor*.

As can be seen from Table 5.4, the models appear to do a relatively poor job of explaining how the market will react to ratings announcements. The adjusted R^2 value is only 12%, and this is the highest of the five specifications tested by the authors. Further, only two variables are significant and one marginally significant of the seven employed in the model. It can therefore be stated that yield changes are significantly higher following a ratings announcement for speculative than investment grade bonds, and that ratings changes have a bigger impact on yield spreads if there is an agreement between the ratings agencies at the time the announcement is made. Further, yields change significantly more if there has been a previous announcement in the past sixty days than if not. On the other hand, neither whether the announcement is an upgrade or a downgrade, nor whether it is an actual ratings change or a name on the watchlist, nor whether the announcement is made by Moody's or S&P, nor the amount by which the relative spread has already changed over the past sixty days, has any significant impact on how the market reacts to ratings announcements.

5.15.6 Conclusions

- To summarise, six factors appear to play a big role in determining sovereign credit ratings – incomes, GDP growth, inflation, external debt, industrialised or not and default history
- The ratings provide more information on yields than all of the macro-economic factors put together
- One cannot determine with any degree of confidence what factors determine how the markets will react to ratings announcements.

KEY CONCEPTS

The key terms to be able to define and explain from this chapter are

- homoscedasticity
- autocorrelation
- equilibrium solution

- skewness
- outlier
- multicollinearity
- irrelevant variable
- recursive least squares
- measurement error
- heteroscedasticity
- dynamic model
- robust standard errors
- kurtosis
- functional form
- omitted variable
- parameter stability
- general-to-specific approach

SELF-STUDY QUESTIONS

1. Are assumptions made concerning the unobservable error terms (u_t) or about their sample counterparts, the estimated residuals ($\hat{u}_t$)? Explain your answer.

2. What pattern(s) would one like to see in a residual plot and why?

3. A researcher estimates the following model for stock market returns, but thinks that there may be a problem with it. By calculating the *t*-ratios and considering their significance and by examining the value of R^2 or otherwise, suggest what the problem might be.

$$\hat{y}_t = \underset{(0.436)}{0.638} + \underset{(0.291)}{0.402}x_{2t} - \underset{(0.763)}{0.891}x_{3t} \quad R^2 = 0.96, \quad \bar{R}^2 = 0.89 \tag{5.81}$$

How might you go about solving the perceived problem?

4. (a) State in algebraic notation and explain the assumption about the CLRM's disturbances that is referred to by the term 'homoscedasticity'.

(b) What would the consequence be for a regression model if the errors were not homoscedastic?

(c) How might you proceed if you found that (b) were actually the case?

5. (a) What do you understand by the term 'autocorrelation'?

(b) An econometrician suspects that the residuals of her model might be autocorrelated. Explain the steps involved in testing this theory using the Durbin–Watson (*DW*) test.

(c) The econometrician follows your guidance (!!!) in part (b) and calculates a value for the Durbin–Watson statistic of 0.95. The regression has sixty quarterly observations and three explanatory variables (plus a constant term). Perform the test. What is your conclusion?

(d) In order to allow for autocorrelation, the econometrician decides to use a model in first differences with a constant

$$\Delta y_t = \beta_1 + \beta_2 \Delta x_{2t} + \beta_3 \Delta x_{3t} + \beta_4 \Delta x_{4t} + u_t \qquad \textbf{(5.82)}$$

By attempting to calculate the long-run solution to this model, explain what might be a problem with estimating models entirely in first differences.

(e) The econometrician finally settles on a model with both first differences and lagged levels terms of the variables

$$\Delta y_t = \beta_1 + \beta_2 \Delta x_{2t} + \beta_3 \Delta x_{3t} + \beta_4 \Delta x_{4t} + \beta_5 x_{2t-1} + \beta_6 x_{3t-1} + \beta_7 x_{4t-1} + v_t \qquad \textbf{(5.83)}$$

Can the Durbin–Watson test still validly be used in this case?

6. Calculate the long-run static equilibrium solution to the following dynamic econometric model

$$\Delta y_t = \beta_1 + \beta_2 \Delta x_{2t} + \beta_3 \Delta x_{3t} + \beta_4 y_{t-1} + \beta_5 x_{2t-1} + \beta_6 x_{3t-1} + \beta_7 x_{3t-4} + u_t \qquad \textbf{(5.84)}$$

7. What might Ramsey's RESET test be used for? What could be done if it were found that the RESET test has been failed?

8. (a) Why is it necessary to assume that the disturbances of a regression model are normally distributed?

(b) In a practical econometric modelling situation, how might the problem that the residuals are not normally distributed be addressed?

9. (a) Explain the term 'parameter structural stability'?

(b) A financial econometrician thinks that the stock market crash of October 1987 fundamentally changed the risk–return relationship given by the CAPM equation. He decides to test this hypothesis using a Chow test. The model is estimated using monthly data from January 1981–December 1995, and then two separate regressions are run for the sub-periods corresponding to data before and after the crash. The model is

$$r_t = \alpha + \beta r_{mt} + u_t \qquad \textbf{(5.85)}$$

so that the excess return on a security at time *t* is regressed upon the excess return on a proxy for the market portfolio at time *t*. The results for the three models estimated for a given stock are as follows:

1981*M*1–1995*M*12

$$r_t = 0.0215 + 1.491\, r_{mt} \qquad RSS = 0.189 \;\; T = 180 \qquad \textbf{(5.86)}$$

1981*M*1–1987*M*10

$$r_t = 0.0163 + 1.308\, r_{mt} \qquad RSS = 0.079 \;\; T = 82 \qquad \textbf{(5.87)}$$

1987*M*11–1995*M*12

$$r_t = 0.0360 + 1.613\, r_{mt} \qquad RSS = 0.082 \;\; T = 98 \qquad \textbf{(5.88)}$$

(c) What are the null and alternative hypotheses that are being tested here, in terms of α and β?

(d) Perform the test. What is your conclusion?

10. For the same model as above, and given the following results, do a forward and backward predictive failure test:

1981*M*1–1995*M*12

$$r_t = 0.0215 + 1.491\, r_{mt} \qquad RSS = 0.189 \;\; T = 180 \qquad \textbf{(5.89)}$$

1981*M*1–1994*M*12

$$r_t = 0.0212 + 1.478\, r_{mt} \qquad RSS = 0.148 \;\; T = 168$$

(5.90)

1982*M*1–1995*M*12

$$r_t = 0.0217 + 1.523\, r_{mt} \qquad RSS = 0.182\ T = 168 \tag{5.91}$$

What is your conclusion?

11. Why is it desirable to remove insignificant variables from a regression?

12. Explain why it is not possible to include an outlier dummy variable in a regression model when you are conducting a Chow test for parameter stability. Will the same problem arise if you were to conduct a predictive failure test? Why or why not?

13. (a) Explain the term 'measurement error'.

(b) How does measurement error arise?

(c) Is measurement error more serious if it is present in the dependent variable or the independent variable(s) of a regression? Explain your answer.

(d) What is the likely impact of measurement error on tests of the CAPM and what are the possible solutions?

1 A situation where X and u are not independent is discussed at length in Chapter 7.

2 The law of large numbers states that the average of a sample (which is a random variable) will converge to the population mean (which is fixed), and the central limit theorem states that the sample mean converges to a normal distribution.

3 Note that multicollinearity does not affect the value of R^2 in a regression.

4 Strictly, the CUSUM and CUSUMSQ statistics are based on the one-step-ahead prediction errors – i.e., the differences between y_t and its predicted value based on the parameters estimated at time $t - 1$. See Greene (2002, Chapter 7) for full technical details.

6

Univariate Time-Series Modelling and Forecasting

LEARNING OUTCOMES

In this chapter, you will learn how to

- Explain the defining characteristics of various types of stochastic processes
- Identify the appropriate time-series model for a given data series
- Produce forecasts for autoregressive moving average (ARMA) and exponential smoothing models
- Evaluate the accuracy of predictions using various metrics

6.1 Introduction

Univariate time-series models are a class of specifications where one attempts to model and to predict financial variables using only information contained in their own past values and possibly current and past values of an error term. This practice can be contrasted with *structural models*, which are multivariate in nature, and attempt to explain changes in a variable by reference to the movements in the current or past values of other (explanatory) variables. Time series models are usually a-theoretical, implying that their construction and use is not based upon any underlying theoretical model of the behaviour of a variable. Instead, time series models are an attempt to capture empirically relevant features of the observed data that may have arisen from a variety of different (but unspecified) structural models. An important class of time series models is the family of autoregressive integrated moving average (ARIMA) models, usually associated with Box and Jenkins (1976). Time series models may be useful when a structural model is inappropriate. For example, suppose that there is some variable y_t whose movements a researcher wishes to explain. It may be that the variables thought to drive movements of y_t are not observable or not measurable, or that these forcing variables are measured at a lower frequency of observation than y_t. For instance, y_t might be a series of daily stock returns, where possible explanatory variables could be macroeconomic indicators that are available monthly. Additionally, as will be examined later in this chapter, structural models are often not useful for out-of-sample forecasting. These observations motivate the consideration of pure time series models, which are the focus of this chapter.

The approach adopted for this topic is as follows. In order to define, estimate and use ARIMA models, one first needs to specify the notation and to define several important concepts. The chapter will then consider the properties and characteristics of a number of specific models from the ARIMA family. The book endeavours to answer the following question: 'For a specified time series model with given parameter values, what will be its defining characteristics?' Following this, the problem will be reversed, so that the reverse question is asked: 'Given a set of data, with characteristics that have been determined, what is a plausible model to describe that data?'

6.2 Some Notation and Concepts

The following sub-sections define and describe several important concepts in time-series analysis. Each will be elucidated and drawn upon later in the chapter. The first of these concepts is the notion of whether a series is *stationary* or not. Determining whether a series is stationary or not is very important, for the stationarity or otherwise of a series can strongly influence its behaviour and properties. Further detailed discussion of stationarity, testing for it, and implications of it not being present, are covered in Chapter 8.

6.2.1 A Strictly Stationary Process

A strictly stationary process is one where, for any $t_1, t_2, \ldots, t_T \in Z$, any $k \in Z$ and $T = 1,2, \ldots$

$$F y_{t_1}, y_{t_2}, \ldots, y_{t_T}(y_1, \ldots, y_T) = F y_{t_1+k}, y_{t_2+k}, \ldots, y_{t_T+k}(y_1, \ldots, y_T) \quad \textbf{(6.1)}$$

where F denotes the joint distribution function of the set of random variables (Tong, 1990, p.3). It can also be stated that the probability measure for the sequence $\{y_t\}$ is the same as that for $\{y_{t+k}\}$ $\forall$ k (where '$\forall$ k' means 'for all values of k'). In other words, a series is strictly stationary if the distribution of its values remains the same as time progresses, implying that the probability that y falls within a particular interval is the same now as at any time in the past or the future.

6.2.2 A Weakly Stationary Process

If a series satisfies (6.2)–(6.4) for $t = 1, 2, \ldots, \infty$, it is said to be weakly or covariance stationary

$$E(y_t) = \mu \quad \textbf{(6.2)}$$

$$E(y_t - \mu)(y_t - \mu) = \sigma^2 < \infty \quad \textbf{(6.3)}$$

$$E(y_{t_1} - \mu)(y_{t_2} - \mu) = \gamma_{t_2 - t_1} \quad \forall\, t_1, t_2 \quad \textbf{(6.4)}$$

These three equations state that a stationary process should have a constant mean, a constant variance and a constant autocovariance structure, respectively. Definitions of the mean and variance of a random variable are probably well known to readers, but the autocovariances may not be.

The autocovariances determine how y is related to its previous values, and for a stationary series they depend only on the difference between t_1 and t_2, so that the covariance between y_t and y_{t-1} is the same as the covariance between y_{t-10} and y_{t-11}, etc. The moment

$$E(y_t - E(y_t))(y_{t-s} - E(y_{t-s})) = \gamma_s, s = 0, 1, 2, \ldots \quad \textbf{(6.5)}$$

is known as the *autocovariance function*. When $s = 0$, the autocovariance at lag zero is obtained, which is the autocovariance of y_t with y_t, i.e., the variance of y. These covariances, γ_s, are also known as autocovariances since they are the covariances of y with its own previous values. The autocovariances are not a particularly useful measure of the relationship between y and its previous values, however, since the values of the autocovariances depend on the units of measurement of y_t, and hence the values that they take have no immediate interpretation.

It is thus more convenient to use the autocorrelations, which are the autocovariances normalised by dividing by the variance

$$\tau_s = \frac{\gamma_s}{\gamma_0}, \quad s = 0, 1, 2, \ldots \tag{6.6}$$

The series τ_s now has the standard property of correlation coefficients that the values are bounded to lie between ±1. In the case that $s = 0$, the autocorrelation at lag zero is obtained, i.e., the correlation of y_t with y_t, which is of course 1. If τ_s is plotted against $s = 0, 1, 2, \ldots$, a graph known as the *autocorrelation function* (acf) or *correlogram* is obtained.

6.2.3 A White Noise Process

Roughly speaking, a white noise process is one with no discernible structure. A definition of a white noise process is

$$E(y_t) = \mu \tag{6.7}$$

$$\text{var}(y_t) = \sigma^2 \tag{6.8}$$

$$\gamma_{t-r} = \begin{cases} \sigma^2 & \text{if} \quad t = r \\ 0 & otherwise \end{cases} \tag{6.9}$$

Thus a white noise process has constant mean and variance, and zero autocovariances, except at lag zero. Another way to state this last condition would be to say that each observation is uncorrelated with all other values in the sequence. Hence the autocorrelation function for a white noise process will be zero apart from a single peak of 1 at $s = 0$. If $\mu = 0$, and the three conditions hold, the process is known as zero mean white noise.

If it is further assumed that y_t is distributed normally, then the sample autocorrelation coefficients are also approximately normally distributed

$$\hat{\tau}_s \sim approx.\ N(0, 1/T)$$

where T is the sample size, and $\hat{\tau}_s$ denotes the autocorrelation coefficient at lag s estimated from a sample. This result can be used to conduct significance tests for the autocorrelation coefficients by constructing a non-rejection region (like a confidence interval) for an estimated autocorrelation coefficient to determine whether it is significantly different from zero. For example, a 95% non-rejection region would be given by

$$\pm 1.96 \times \frac{1}{\sqrt{T}}$$

for $s \neq 0$. If the sample autocorrelation coefficient, $\hat{\tau}_s$, falls outside this region for a given value of s, then the null hypothesis that the true value of the coefficient at that lag s is zero is rejected.

It is also possible to test the joint hypothesis that all m of the τ_k correlation coefficients are simultaneously equal to zero using the Q-statistic developed by Box and Pierce (1970)

$$Q = T \sum_{k=1}^{m} \hat{\tau}_k^2 \tag{6.10}$$

where T = sample size, m = maximum lag length.

The correlation coefficients are squared so that the positive and negative coefficients do not cancel each other out. Since the sum of squares of independent standard normal variates is itself a χ^2 variate with degrees of freedom equal to the number of squares in the sum, it can be stated that the Q-statistic is asymptotically distributed as a χ_m^2 under the null hypothesis that all m autocorrelation coefficients are zero. As for any joint hypothesis test, only one autocorrelation coefficient needs to be statistically significant for the test to result in a rejection.

However, the Box–Pierce test has poor small sample properties, implying that it leads to the wrong decision too frequently for small samples. A variant of the Box–Pierce test, having better small sample properties, has been developed. The modified statistic is known as the Ljung–Box (1978) statistic

$$Q^* = T(T+2) \sum_{k=1}^{m} \frac{\hat{\tau}_k^2}{T-k} \sim \chi_m^2 \tag{6.11}$$

It should be clear from the form of the statistic that asymptotically (that is, as the sample size increases towards infinity), the $(T + 2)$ and $(T - k)$ terms in the Ljung–Box formulation will cancel out, so that the statistic is equivalent to the Box–Pierce test. This statistic is very useful as a portmanteau (general) test of linear dependence in time series.

EXAMPLE 6.1

Suppose that a researcher had estimated the first five autocorrelation coefficients using a series of length 100 observations, and found them to be

Lag	1	2	3	4	5
Autocorrelation coefficient	0.207	−0.013	0.086	0.005	−0.022

Test each of the individual correlation coefficients for significance, and test all five jointly using the Box–Pierce and Ljung–Box tests.

SOLUTION

A 95% confidence interval can be constructed for each coefficient using

$$\pm 1.96 \times \frac{1}{\sqrt{T}}$$

where T = 100 in this case. The decision rule is thus to reject the null hypothesis that a given coefficient is zero in the cases where the coefficient lies outside the range (−0.196, +0.196). For this example, it would be concluded that only the first autocorrelation coefficient is significantly different from zero at the 5% level.

Now, turning to the joint tests, the null hypothesis is that all of the first five autocorrelation coefficients are jointly zero, i.e.

$$H_0: \tau_1 = 0, \tau_2 = 0, \tau_3 = 0, \tau_4 = 0, \tau_5 = 0$$

The test statistics for the Box–Pierce and Ljung–Box tests are given respectively, as

$$\begin{aligned} Q &= 100 \times (0.207^2 + -0.013^2 + 0.086^2 + 0.005^2 + -0.022^2) \\ &= 5.09 \end{aligned} \qquad \textbf{(6.12)}$$

$$\begin{aligned} Q^* = 100 \times 102 \times \bigg(& \frac{0.207^2}{100-1} + \frac{-0.013^2}{100-2} + \frac{0.086^2}{100-3} \\ & + \frac{0.005^2}{100-4} + \frac{-0.022^2}{100-5} \bigg) = 5.26 \end{aligned} \qquad \textbf{(6.13)}$$

The relevant critical values are from a χ^2 distribution with five degrees of freedom, which are 11.1 at the 5% level, and 15.1 at the 1% level. Clearly, in both cases, the joint null hypothesis that all of the first five autocorrelation coefficients are zero cannot be rejected. Note that, in this instance, the individual test caused a rejection while the joint test did not. This is an unexpected result that may have arisen as a result of the low power of the joint test when four of the five individual autocorrelation coefficients are insignificant. Thus the effect of the significant autocorrelation coefficient is diluted in the joint test by the insignificant coefficients. The sample size used in this example is also modest relative to those commonly available in finance.

6.3 Moving Average Processes

The simplest class of time-series model that one could entertain is that of the moving average process. Let u_t (t = 1, 2, 3, ...) be a white noise process with $E(u_t) = 0$ and $\text{var}(u_t) = \sigma^2$. Then

$$y_t = \mu + u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2} + \cdots + \theta_q u_{t-q} \quad \textbf{(6.14)}$$

is a qth order moving average model, denoted MA(q). This can be expressed using sigma notation as

$$y_t = \mu + \sum_{i=1}^{q} \theta_i u_{t-i} + u_t \quad \textbf{(6.15)}$$

A moving average model is simply a linear combination of white noise processes, so that y_t depends on the current and previous values of a white noise disturbance term. Equation equation (6.15) will later have to be manipulated, and such a process is most easily achieved by introducing the lag operator notation. This would be written $Ly_t = y_{t-1}$ to denote that y_t is lagged once. In order to show that the ith lag of y_t is being taken (that is, the value that y_t took i periods ago), the notation would be $L^i y_t = y_{t-i}$. Note that in some books and studies, the lag operator is referred to as the 'backshift operator', denoted by B. Using the lag operator notation, (6.15) would be written as

$$y_t = \mu + \sum_{i=1}^{q} \theta_i L^i u_t + u_t \quad \textbf{(6.16)}$$

or as

$$y_t = \mu + \theta(L)u_t \tag{6.17}$$

where: $\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$.

In much of what follows, the constant (μ) is dropped from the equations. Removing μ considerably eases the complexity of algebra involved, and is inconsequential for it can be achieved without loss of generality. To see this, consider a sample of observations on a series, z_t that has a mean $\bar{z}$. A zero-mean series, y_t can be constructed by simply subtracting $\bar{z}$ from each observation z_t.

The distinguishing properties of the moving average process of order q given above are

(1) $$E(y_t) = \mu \tag{6.18}$$

(2) $$\text{var}(y_t) = \gamma_0 = \left(1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2\right)\sigma^2 \tag{6.19}$$

(3) covariances γ_s

$$= \begin{cases} (\theta_s + \theta_{s+1}\theta_1 + \theta_{s+2}\theta_2 + \cdots + \theta_q\theta_{q-s})\,\sigma^2 & \text{for } s = 1, 2, \ldots, q \\ 0 & \text{for } s > q \end{cases} \tag{6.20}$$

So, a moving average process has constant mean, constant variance, and autocovariances which may be non-zero to lag q and will always be zero thereafter. Each of these results will be derived below.

EXAMPLE 6.2

Consider the following MA(2) process

$$y_t = u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2} \tag{6.21}$$

where u_t is a zero mean white noise process with variance σ^2.

(1) Calculate the mean and variance of y_t.

(2) Derive the autocorrelation function for this process (i.e., express the autocorrelations, τ_1, τ_2, … as functions of the parameters θ_1 and θ_2).

(3) If $\theta_1 = -0.5$ and $\theta_2 = 0.25$, sketch the acf of y_t.

SOLUTION

(1) $$\text{If } E(u_t) = 0, \text{ then } E(u_{t-i}) = 0 \ \forall \ i \tag{6.22}$$

So the expected value of the error term is zero for all time periods. Taking expectations of both sides of equation (6.21) gives

$$\begin{aligned} E(y_t) &= E(u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2}) \\ &= E(u_t) + \theta_1 E(u_{t-1}) + \theta_2 E(u_{t-2}) = 0 \end{aligned} \tag{6.23}$$

$$\text{var}(y_t) = E[y_t - E(y_t)][y_t - E(y_t)] \tag{6.24}$$

but $E(y_t) = 0$, so that the last component in each set of square brackets in equation (6.24) is zero and this reduces to

$$\text{var}(y_t) = E[(y_t)(y_t)] \tag{6.25}$$

Replacing y_t in equation (6.25) with the RHS of (6.21)

$$\text{var}(y_t) = E[(u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2})(u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2})] \tag{6.26}$$

$$\text{var}(y_t) = E\left[u_t^2 + \theta_1^2 u_{t-1}^2 + \theta_2^2 u_{t-2}^2 + \textit{cross-products}\right] \tag{6.27}$$

But E[*cross-products*] = 0 since $\text{cov}(u_t, u_{t-s}) = 0$ for $s \neq 0$. '*Cross-products*' is thus a catchall expression for all of the terms in u which have different time subscripts, such as $u_{t-1}\ u_{t-2}$ or $u_{t-5}\ u_{t-20}$, etc. Again, one does not need to worry about these cross-product terms, since these are effectively the autocovariances of u_t, which will all be zero by definition since u_t is a random error process, which will have zero autocovariances (except at lag zero). So

$$\text{var}(y_t) = \gamma_0 = E\left[u_t^2 + \theta_1^2 u_{t-1}^2 + \theta_2^2 u_{t-2}^2\right] \tag{6.28}$$

$$\text{var}(y_t) = \gamma_0 = \sigma^2 + \theta_1^2 \sigma^2 + \theta_2^2 \sigma^2 \tag{6.29}$$

$$\text{var}(y_t) = \gamma_0 = \left(1 + \theta_1^2 + \theta_2^2\right)\sigma^2 \tag{6.30}$$

γ_0 can also be interpreted as the autocovariance at lag zero.

(2) Calculating now the acf of y_t, first determine the autocovariances and then the autocorrelations by dividing the autocovariances by the variance.

The autocovariance at lag 1 is given by

$$\gamma_1 = \mathrm{E}[y_t - \mathrm{E}(y_t)][y_{t-1} - \mathrm{E}(y_{t-1})] \quad \textbf{(6.31)}$$

$$\gamma_1 = \mathrm{E}[y_t][y_{t-1}] \quad \textbf{(6.32)}$$

$$\gamma_1 = \mathrm{E}[(u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2})(u_{t-1} + \theta_1 u_{t-2} + \theta_2 u_{t-3})] \quad \textbf{(6.33)}$$

Again, ignoring the cross-products, equation (6.33) can be written as

$$\gamma_1 = \mathrm{E}\left[(\theta_1 u_{t-1}^2 + \theta_1\theta_2 u_{t-2}^2)\right] \quad \textbf{(6.34)}$$

$$\gamma_1 = \theta_1\sigma^2 + \theta_1\theta_2\sigma^2 \quad \textbf{(6.35)}$$

$$\gamma_1 = (\theta_1 + \theta_1\theta_2)\sigma^2 \quad \textbf{(6.36)}$$

The autocovariance at lag 2 is given by

$$\gamma_2 = \mathrm{E}[y_t - \mathrm{E}(y_t)][y_{t-2} - \mathrm{E}(y_{t-2})] \quad \textbf{(6.37)}$$

$$\gamma_2 = \mathrm{E}[y_t][y_{t-2}] \quad \textbf{(6.38)}$$

$$\gamma_2 = \mathrm{E}[(u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2})(u_{t-2} + \theta_1 u_{t-3} + \theta_2 u_{t-4})] \quad \textbf{(6.39)}$$

$$\gamma_2 = \mathrm{E}\left[(\theta_2 u_{t-2}^2)\right] \quad \textbf{(6.40)}$$

$$\gamma_2 = \theta_2\sigma^2 \quad \textbf{(6.41)}$$

The autocovariance at lag 3 is given by

$$\gamma_3 = \mathrm{E}[y_t - \mathrm{E}(y_t)][y_{t-3} - \mathrm{E}(y_{t-3})] \quad \textbf{(6.42)}$$

$$\gamma_3 = \mathrm{E}[y_t][y_{t-3}] \quad \textbf{(6.43)}$$

$$\gamma_3 = \mathrm{E}[(u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2})(u_{t-3} + \theta_1 u_{t-4} + \theta_2 u_{t-5})] \quad \textbf{(6.44)}$$

$$\gamma_3 = 0 \quad \textbf{(6.45)}$$

So $\gamma_s = 0$ for $s > 2$. All autocovariances for the MA(2) process will be zero for any lag length, s, greater than 2.

The autocorrelation at lag 0 is given by

$$\tau_0 = \frac{\gamma_0}{\gamma_0} = 1 \quad \textbf{(6.46)}$$

The autocorrelation at lag 1 is given by

$$\tau_1 = \frac{\gamma_1}{\gamma_0} = \frac{(\theta_1 + \theta_1\theta_2)\sigma^2}{(1 + \theta_1^2 + \theta_2^2)\sigma^2} = \frac{(\theta_1 + \theta_1\theta_2)}{(1 + \theta_1^2 + \theta_2^2)} \quad \textbf{(6.47)}$$

The autocorrelation at lag 2 is given by

$$\tau_2 = \frac{\gamma_2}{\gamma_0} = \frac{(\theta_2)\sigma^2}{(1+\theta_1^2+\theta_2^2)\sigma^2} = \frac{\theta_2}{(1+\theta_1^2+\theta_2^2)} \tag{6.48}$$

The autocorrelation at lag 3 is given by

$$\tau_3 = \frac{\gamma_3}{\gamma_0} = 0 \tag{6.49}$$

The autocorrelation at lag s is given by

$$\tau_s = \frac{\gamma_s}{\gamma_0} = 0 \ \forall \ s > 2 \tag{6.50}$$

(3) For $\theta_1 = -0.5$ and $\theta_2 = 0.25$, substituting these into the formulae above gives the first two autocorrelation coefficients as $\tau_1 = -0.476$, $\tau_2 = 0.190$. Autocorrelation coefficients for lags greater than 2 will all be zero for an MA(2) model. Thus the acf plot will appear as in Figure 6.1.

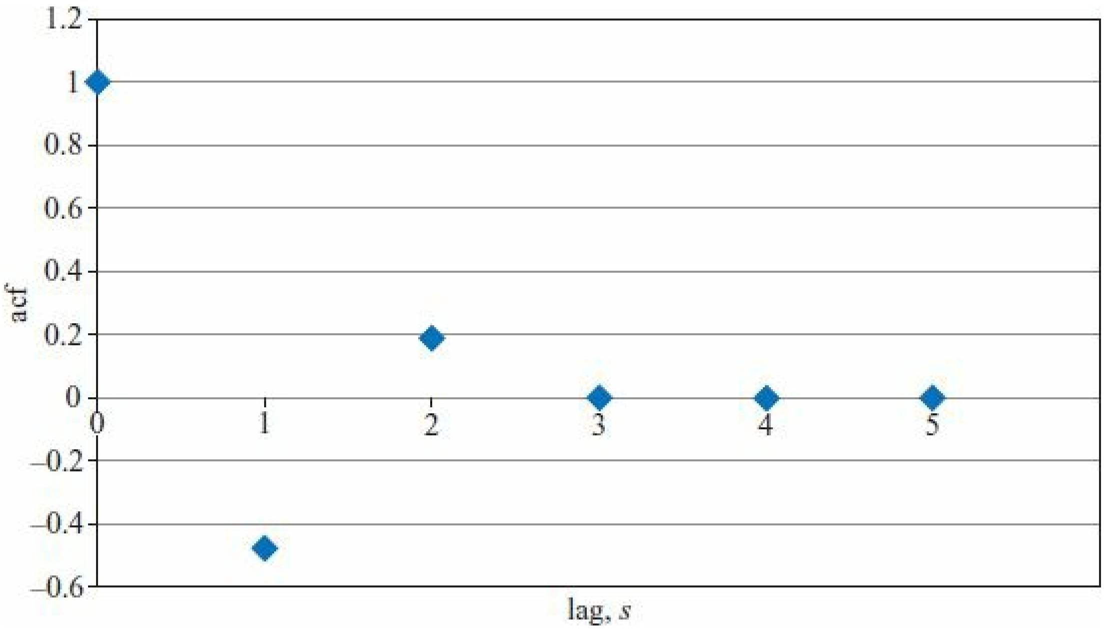


Figure 6.1 Autocorrelation function for sample MA(2) process

6.4 Autoregressive Processes

An autoregressive model is one where the current value of a variable, y, depends upon only the values that the variable took in previous periods plus an error term. An autoregressive model of order p, denoted as AR(p), can be expressed as

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + u_t \tag{6.51}$$

where u_t is a white noise disturbance term. A manipulation of expression (13.24) will be required to demonstrate the properties of an autoregressive model. This expression can be written more compactly using sigma notation

$$y_t = \mu + \sum_{i=1}^{p} \phi_i y_{t-i} + u_t \tag{6.52}$$

or using the lag operator, as

$$y_t = \mu + \sum_{i=1}^{p} \phi_i L^i y_t + u_t \tag{6.53}$$

or

$$\phi(L) y_t = \mu + u_t \tag{6.54}$$

where $\phi(L) = (1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)$

6.4.1 The Stationarity Condition

Stationarity is a desirable property of an estimated AR model, for several reasons. One important reason is that a model whose coefficients are non-stationary will exhibit the unfortunate property that previous values of the error term will have a non-declining effect on the current value of y_t as time progresses. This is arguably counter-intuitive and empirically implausible in many cases. More discussion on this issue will be presented in Chapter 8. Box 6.1 defines the stationarity condition algebraically.

BOX 6.1 The stationarity condition for an AR(p) model

Setting μ to zero in equation (6.54), for a zero mean AR(p) process, y_t, given by

$$\phi(L) y_t = u_t$$

(6.55)

it would be stated that the process is stationary if it is possible to write

$$y_t = \phi(L)^{-1} u_t \tag{6.56}$$

with $\phi\ (L)^{-1}$ converging to zero. This means that the autocorrelations will decline eventually as the lag length is increased. When the expansion $\phi\ (L)^{-1}$ is calculated, it will contain an infinite number of terms, and can be written as an MA(∞), e.g., $a_1 u_{t-1} + a_2 u_{t-2} + a_3 u_{t-3} + \cdots + u_t$. If the process given by equation (6.54) is stationary, the coefficients in the MA(∞) representation will decline eventually with lag length. On the other hand, if the process is non-stationary, the coefficients in the MA(∞) representation would not converge to zero as the lag length increases.

The condition for testing for the stationarity of a general AR(p) model is that the roots of the 'characteristic equation'

$$1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p = 0 \tag{6.57}$$

all lie outside the unit circle. The notion of a characteristic equation is so-called because its roots determine the characteristics of the process y_t – for example, the acf for an AR process will depend on the roots of this characteristic equation, which is a polynomial in z.

EXAMPLE 6.3

Is the following model stationary?

$$y_t = y_{t-1} + u_t \tag{6.58}$$

In order to test this, first write y_{t-1} in lag operator notation (i.e., as Ly_t), and take this term over to the LHS of equation (6.58), and factorise

$$y_t = Ly_t + u_t \tag{6.59}$$

$$y_t - Ly_t = u_t \tag{6.60}$$

$$y_t(1 - L) = u_t \tag{6.61}$$

Then the characteristic equation is

$$1 - z = 0, \tag{6.62}$$

having the root $z = 1$, which lies on, not outside, the unit circle. In fact, the particular AR(p) model given by equation (6.58) is a non-stationary process known as a random walk (see Chapter 8).

This procedure can also be adopted for autoregressive models with longer lag lengths and where the stationarity or otherwise of the process is less obvious. For example, is the following process for y_t stationary?

$$y_t = 3y_{t-1} - 2.75y_{t-2} + 0.75y_{t-3} + u_t \tag{6.63}$$

Again, the first stage is to express this equation using the lag operator notation, and then taking all the terms in y over to the LHS

$$y_t = 3Ly_t - 2.75L^2y_t + 0.75L^3y_t + u_t \tag{6.64}$$

$$(1 - 3L + 2.75L^2 - 0.75L^3)y_t = u_t \tag{6.65}$$

The characteristic equation is

$$1 - 3z + 2.75z^2 - 0.75z^3 = 0 \tag{6.66}$$

which fortunately factorises to

$$(1 - z)(1 - 1.5z)(1 - 0.5z) = 0 \tag{6.67}$$

so that the roots are $z = 1$, $z = 2/3$, and $z = 2$. Only one of these lies outside the unit circle and hence the process for y_t described by equation (6.63) is not stationary.

6.4.2 Wold's Decomposition Theorem

Wold's decomposition theorem states that any stationary series can be decomposed into the sum of two unrelated processes, a purely deterministic part and a purely stochastic part, which will be an MA(∞). A simpler way of stating this in the context of AR modelling is that any stationary autoregressive process of order p with no constant and no other terms can be expressed as an infinite order moving average model. This result is important for deriving the autocorrelation function for an autoregressive process.

For the AR(p) model, given in, for example, equation (6.51) (with μ set to zero for simplicity) and expressed using the lag polynomial notation, $\phi(L)y_t = u_t$, the Wold decomposition is

$$y_t = \psi(L)u_t \tag{6.68}$$

where $\psi(L) = \phi(L)^{-1} = (1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)^{-1}$

The characteristics of an autoregressive process are as follows. The (unconditional) mean of y is given by

$$E(y_t) = \frac{\mu}{1 - \phi_1 - \phi_2 - \cdots - \phi_p} \tag{6.69}$$

The autocovariances and autocorrelation functions can be obtained by solving a set of simultaneous equations known as the Yule–Walker equations. The Yule–Walker equations express the correlogram (the τ s) as a function of the autoregressive coefficients (the ϕs)

$$\begin{aligned}
\tau_1 &= \phi_1 + \tau_1\phi_2 + \cdots + \tau_{p-1}\phi_p \\
\tau_2 &= \tau_1\phi_1 + \phi_2 + \cdots + \tau_{p-2}\phi_p \\
&\vdots \qquad \vdots \qquad \vdots \\
\tau_p &= \tau_{p-1}\phi_1 + \tau_{p-2}\phi_2 + \cdots + \phi_p
\end{aligned} \tag{6.70}$$

For any AR model that is stationary, the autocorrelation function will decay geometrically to zero.[1] These characteristics of an autoregressive process will be derived from first principles below using an illustrative example.

EXAMPLE 6.4

Consider the following simple AR(1) model

$$y_t = \mu + \phi_1 y_{t-1} + u_t \tag{6.71}$$

(1) Calculate the (unconditional) mean of y_t.

For the remainder of the question, set the constant to zero ($\mu = 0$) for simplicity.

(2) Calculate the (unconditional) variance of y_t.

(3) Derive the autocorrelation function for this process.

SOLUTION

(i) The unconditional mean will be given by the expected value of expression (6.71)

$$E(y_t) = E(\mu + \phi_1 y_{t-1}) \quad \textbf{(6.72)}$$

$$E(y_t) = \mu + \phi_1 E(y_{t-1}) \quad \textbf{(6.73)}$$

But also

$$y_{t-1} = \mu + \phi_1 y_{t-2} + u_{t-1} \quad \textbf{(6.74)}$$

So, replacing y_{t-1} in (6.73) with the RHS of (6.72)

$$E(y_t) = \mu + \phi_1(\mu + \phi_1 E(y_{t-2})) \quad \textbf{(6.75)}$$

$$E(y_t) = \mu + \phi_1\mu + \phi_1^2 E(y_{t-2}) \quad \textbf{(6.76)}$$

Lagging equation (6.74) by a further one period

$$y_{t-2} = \mu + \phi_1 y_{t-3} + u_{t-2} \quad \textbf{(6.77)}$$

Repeating the steps given above one more time

$$E(y_t) = \mu + \phi_1\mu + \phi_1^2(\mu + \phi_1 E(y_{t-3})) \quad \textbf{(6.78)}$$

$$E(y_t) = \mu + \phi_1\mu + \phi_1^2\mu + \phi_1^3 E(y_{t-3}) \quad \textbf{(6.79)}$$

Hopefully, readers will by now be able to see a pattern emerging. Making n such substitutions would give

$$E(y_t) = \mu\left(1 + \phi_1 + \phi_1^2 + \cdots + \phi_1^{n-1}\right) + \phi_1^n E(y_{t-n}) \quad \textbf{(6.80)}$$

So long as the model is stationary, i.e., $|\phi_1| < 1$, then $\phi_1^\infty = 0$. Therefore, taking limits as $n \to \infty$, then $lim_{n\to\infty}\phi_1^t E(y_{t-n}) = 0$, and so

$$E(y_t) = \mu\left(1 + \phi_1 + \phi_1^2 + \cdots\right) \quad \textbf{(6.81)}$$

Recall the rule of algebra that the finite sum of an infinite number of geometrically declining terms in a series is given by ‘first term in series divided by (1 minus common difference)', where the common difference is the quantity that each term in the series is multiplied by to arrive at the next term. It can thus be stated from (6.81) that

$$E(y_t) = \frac{\mu}{1-\phi_1} \tag{6.82}$$

Thus the expected or mean value of an autoregressive process of order one is given by the intercept parameter divided by one minus the autoregressive coefficient.

(ii) Calculating now the variance of y_t, with μ set to zero

$$y_t = \phi_1 y_{t-1} + u_t \tag{6.83}$$

This can be written equivalently as

$$y_t(1-\phi_1 L) = u_t \tag{6.84}$$

From Wold's decomposition theorem, the AR(p) can be expressed as an MA(∞)

$$y_t = (1-\phi_1 L)^{-1} u_t \tag{6.85}$$

$$y_t = \left(1+\phi_1 L + \phi_1^2 L^2 + \cdots\right) u_t \tag{6.86}$$

or

$$y_t = u_t + \phi_1 u_{t-1} + \phi_1^2 u_{t-2} + \phi_1^3 u_{t-3} + \cdots \tag{6.87}$$

So long as $|\phi_1| < 1$, i.e., so long as the process for y_t is stationary, this sum will converge.

From the definition of the variance of any random variable y, it is possible to write

$$\text{var}(y_t) = \text{E}[y_t - \text{E}(y_t)][y_t - \text{E}(y_t)] \tag{6.88}$$

but $\text{E}(y_t) = 0$, since μ is set to zero to obtain equation (6.83) above. Thus

$$\text{var}(y_t) = \text{E}[(y_t)(y_t)] \tag{6.89}$$

$$\text{var}(y_t) = \text{E}\left[\left(u_t + \phi_1 u_{t-1} + \phi_1^2 u_{t-2} + \cdots\right)\left(u_t + \phi_1 u_{t-1} + \phi_1^2 u_{t-2} + \cdots\right)\right] \tag{6.90}$$

$$\text{var}(y_t) = \text{E}\left[u_t^2 + \phi_1^2 u_{t-1}^2 + \phi_1^4 u_{t-2}^2 + \cdots + \textit{cross-products}\right] \tag{6.91}$$

As discussed above, the ‘cross-products’ can be set to zero.

$$\text{var}(y_t) = \gamma_0 = \text{E}\left[u_t^2 + \phi_1^2 u_{t-1}^2 + \phi_1^4 u_{t-2}^2 + \cdots\right] \tag{6.92}$$

$$\text{var}(y_t) = \sigma^2 + \phi_1^2\sigma^2 + \phi_1^4\sigma^2 + \cdots \tag{6.93}$$

$$\text{var}(y_t) = \sigma^2\left(1 + \phi_1^2 + \phi_1^4 + \cdots\right) \tag{6.94}$$

Provided that $|\phi_1| < 1$, the infinite sum in equation (6.94) can be written as

$$\text{var}(y_t) = \frac{\sigma^2}{(1 - \phi_1^2)} \tag{6.95}$$

(iii) Turning now to the calculation of the autocorrelation function, the auto-covariances must first be calculated. This is achieved by following similar algebraic manipulations as for the variance above, starting with the definition of the autocovariances for a random variable. The autocovariances for lags 1, 2, 3, …, s, will be denoted by $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_s$, as previously.

$$\gamma_1 = \text{cov}\,(y_t, y_{t-1}) = \text{E}[y_t - E(y_t)][y_{t-1} - E(y_{t-1})] \tag{6.96}$$

Since μ has been set to zero, $\text{E}(y_t) = 0$ and $\text{E}(y_{t-1}) = 0$, so

$$\gamma_1 = \text{E}[y_t y_{t-1}] \tag{6.97}$$

under the result above that $\text{E}(y_t) = \text{E}(y_t - 1) = 0$. Thus

$$\gamma_1 = \text{E}\left[\left(u_t + \phi_1 u_{t-1} + \phi_1^2 u_{t-2} + \cdots\right)\left(u_{t-1} + \phi_1 u_{t-2} + \phi_1^2 u_{t-3} + \cdots\right)\right] \tag{6.98}$$

$$\gamma_1 = \text{E}\left[\phi_1 u_{t-1}^2 + \phi_1^3 u_{t-2}^2 + \cdots + \textit{cross-products}\right] \tag{6.99}$$

Again, the cross-products can be ignored so that

$$\gamma_1 = \phi_1\sigma^2 + \phi_1^3\sigma^2 + \phi_1^5\sigma^2 + \cdots \tag{6.100}$$

$$\gamma_1 = \phi_1\sigma^2\left(1 + \phi_1^2 + \phi_1^4 + \cdots\right) \tag{6.101}$$

$$\gamma_1 = \frac{\phi_1\sigma^2}{(1 - \phi_1^2)} \tag{6.102}$$

For the second autocovariance,

$$\gamma_2 = \text{cov}(y_t, y_{t-2}) = \text{E}[y_t - \text{E}(y_t)][y_{t-2} - \text{E}(y_{t-2})] \quad \textbf{(6.103)}$$

Using the same rules as applied above for the lag 1 covariance

$$\gamma_2 = \text{E}[y_t y_{t-2}] \quad \textbf{(6.104)}$$

$$\gamma_2 = \text{E}\left[\left(u_t + \phi_1 u_{t-1} + \phi_1^2 u_{t-2} + \cdots\right)\left(u_{t-2} + \phi_1 u_{t-3} + \phi_1^2 u_{t-4} + \cdots\right)\right] \quad \textbf{(6.105)}$$

$$\gamma_2 = \text{E}\left[\phi_1^2 u_{t-2}^2 + \phi_1^4 u_{t-3}^2 + \cdots + cross\text{-}products\right] \quad \textbf{(6.106)}$$

$$\gamma_2 = \phi_1^2 \sigma^2 + \phi_1^4 \sigma^2 + \cdots \quad \textbf{(6.107)}$$

$$\gamma_2 = \phi_1^2 \sigma^2 \left(1 + \phi_1^2 + \phi_1^4 + \cdots\right) \quad \textbf{(6.108)}$$

$$\gamma_2 = \frac{\phi_1^2 \sigma^2}{(1 - \phi_1^2)} \quad \textbf{(6.109)}$$

By now it should be possible to see a pattern emerging. If these steps were repeated for γ_3, the following expression would be obtained

$$\gamma_3 = \frac{\phi_1^3 \sigma^2}{(1 - \phi_1^2)} \quad \textbf{(6.110)}$$

and for any lag *s*, the autocovariance would be given by

$$\gamma_s = \frac{\phi_1^s \sigma^2}{(1 - \phi_1^2)} \quad \textbf{(6.111)}$$

The acf can now be obtained by dividing the covariances by the variance, so that

$$\tau_0 = \frac{\gamma_0}{\gamma_0} = 1 \quad \textbf{(6.112)}$$

$$\tau_1 = \frac{\gamma_1}{\gamma_0} = \frac{\left(\dfrac{\phi_1 \sigma^2}{(1 - \phi_1^2)}\right)}{\left(\dfrac{\sigma^2}{(1 - \phi_1^2)}\right)} = \phi_1 \quad \textbf{(6.113)}$$

$$\tau_2 = \frac{\gamma_2}{\gamma_0} = \frac{\left(\frac{\phi_1^2\sigma^2}{(1-\phi_1^2)}\right)}{\left(\frac{\sigma^2}{(1-\phi_1^2)}\right)} = \phi_1^2 \quad \textbf{(6.114)}$$

$$\tau_3 = \phi_1^3 \quad \textbf{(6.115)}$$

The autocorrelation at lag s is given by

$$\tau_s = \phi_1^s \quad \textbf{(6.116)}$$

which means that $\text{corr}(y_t, y_{t-s}) = \phi_1^s$. Note that use of the Yule–Walker equations would have given the same answer.

6.5 The Partial Autocorrelation Function

The partial autocorrelation function, or pacf (denoted τ_{kk}), measures the correlation between an observation k periods ago and the current observation, after controlling for observations at intermediate lags (i.e., all lags < k) – i.e. the correlation between y_t and y_{t-k}, after removing the effects of y_{t-k+1}, y_{t-k+2}, …, y_{t-1}. For example, the pacf for lag 3 would measure the correlation between y_t and y_{t-3} after controlling for the effects of y_{t-1} and y_{t-2}.

At lag 1, the autocorrelation and partial autocorrelation coefficients are equal, since there are no intermediate lag effects to eliminate. Thus, $\tau_{11} = \tau_1$, where τ_1 is the autocorrelation coefficient at lag 1.

At lag 2

$$\tau_{22} = \left(\tau_2 - \tau_1^2\right)/\left(1 - \tau_1^2\right) \quad \textbf{(6.117)}$$

where τ_1 and τ_2 are the autocorrelation coefficients at lags 1 and 2, respectively. For lags greater than two, the formulae are more complex and hence a presentation of these is beyond the scope of this book. There now proceeds, however, an intuitive explanation of the characteristic shape of the pacf for a moving average and for an autoregressive process.

In the case of an autoregressive process of order p, there will be direct connections between y_t and y_{t-s} for $s \le p$, but no direct connections for $s > p$. For example, consider the following AR(3) model

$$y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + u_t \quad \textbf{(6.118)}$$

There is a direct connection through the model between y_t and y_{t-1}, and between y_t and y_{t-2}, and between y_t and y_{t-3}, but not between y_t and y_{t-s}, for $s > 3$. Hence the pacf will usually have non-zero partial autocorrelation coefficients for lags up to the order of the model, but will have zero partial autocorrelation coefficients thereafter. In the case of the AR(3), only the first three partial autocorrelation coefficients will be non-zero.

What shape would the partial autocorrelation function take for a moving average process? One would need to think about the MA model as being transformed into an AR in order to consider whether y_t and y_{t-k}, $k = 1, 2, \ldots$, are directly connected. In fact, so long as the MA(q) process is invertible, it can be expressed as an AR(∞). Thus a definition of invertibility is now required.

6.5.1 The Invertibility Condition

An MA(q) model is typically required to have roots of the characteristic equation $\theta(z) = 0$ greater than one in absolute value. The invertibility condition is mathematically the same as the stationarity condition, but is different in the sense that the former refers to MA rather than AR processes. This condition prevents the model from exploding under an AR(∞) representation, so that $\theta^{-1}(L)$ converges to zero. Box 6.2 shows the invertibility condition for an MA(2) model.

BOX 6.2 The invertibility condition for an MA(2) model

In order to examine the shape of the pacf for moving average processes, consider the following MA(2) process for y_t

$$y_t = u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2} = \theta(L) u_t \quad \textbf{(6.119)}$$

Provided that this process is invertible, this MA(2) can be expressed as an AR(∞)

$$y_t = \sum_{i=1}^{\infty} c_i y_{t-i} + u_t \quad \textbf{(6.120)}$$

$$y_t = c_1 y_{t-1} + c_2 y_{t-2} + c_3 y_{t-3} + \cdots + u_t \quad \textbf{(6.121)}$$

It is now evident when expressed in this way that for a moving average model, there are direct connections between the current value of y and all of its previous values. Thus, the partial autocorrelation function for an MA(q) model will decline geometrically, rather than dropping off to zero after q lags, as is the case for its autocorrelation function. It could thus be stated that the acf for an AR has the same basic shape as the pacf for an MA, and the acf for an MA has the same shape as the pacf for an AR.

6.6 ARMA Processes

By combining the AR(p) and MA(q) models, an ARMA(p, q) model is obtained. Such a model states that the current value of some series y depends linearly on its own previous values plus a combination of current and previous values of a white noise error term. The model could be written

$$\phi(L)y_t = \mu + \theta(L)u_t \tag{6.122}$$

where

$$\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p \quad \text{and}$$

$$\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$$

or

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \theta_1 u_{t-} + \theta_2 u_{t-2} + \cdots + \theta_q u_{t-q} + u_t \tag{6.123}$$

with

$$\mathrm{E}(u_t) = 0; E\left(u_t^2\right) = \sigma^2; E(u_t u_s) = 0, t \neq s$$

The characteristics of an ARMA process will be a combination of those from the autoregressive (AR) and moving average (MA) parts. Note that the pacf is particularly useful in this context. The acf alone can distinguish between a pure autoregressive and a pure moving average process. However, an ARMA process will have a geometrically declining acf, as will a pure AR process. So, the pacf is useful for distinguishing between an AR(p) process and an ARMA(p, q) process – the former will have a geometrically declining autocorrelation function, but a partial autocorrelation function which cuts off to zero after p lags, while the latter will have both autocorrelation and partial autocorrelation functions which decline geometrically.

We can now summarise the defining characteristics of AR, MA and ARMA processes.

An autoregressive process has:

- a geometrically decaying acf
- a number of non-zero points of pacf = AR order.

A moving average process has:

- number of non-zero points of acf = MA order
- a geometrically decaying pacf.

A combination autoregressive moving average process has:

- a geometrically decaying acf
- ageometricallydecayingpacf.

In fact, the mean of an ARMA series is given by

$$E(y_t) = \frac{\mu}{1 - \phi_1 - \phi_2 - \cdots - \phi_p} \tag{6.124}$$

The autocorrelation function will display combinations of behaviour derived from the AR and MA parts, but for lags beyond q, the acf will simply be identical to the individual AR(p) model, so that the AR part will dominate in the long term. Deriving the acf and pacf for an ARMA process requires no new algebra, but is tedious and hence is left as an exercise for interested readers.

6.6.1 Sample acf and pacf Plots for Standard Processes

Figures 6.2–6.8 give some examples of typical processes from the ARMA family with their characteristic autocorrelation and partial autocorrelation functions. The acf and pacf are not produced analytically from the relevant formulae for a model of that type, but rather are estimated using 100,000 simulated observations with disturbances drawn from a normal distribution. Each figure also has 5% (two-sided) rejection bands represented by dotted lines. These are based on $(\pm 1.96/\sqrt{100000}) = \pm 0.0062$, calculated in the same way as given above. Notice how, in each case, the acf and pacf are identical for the first lag.

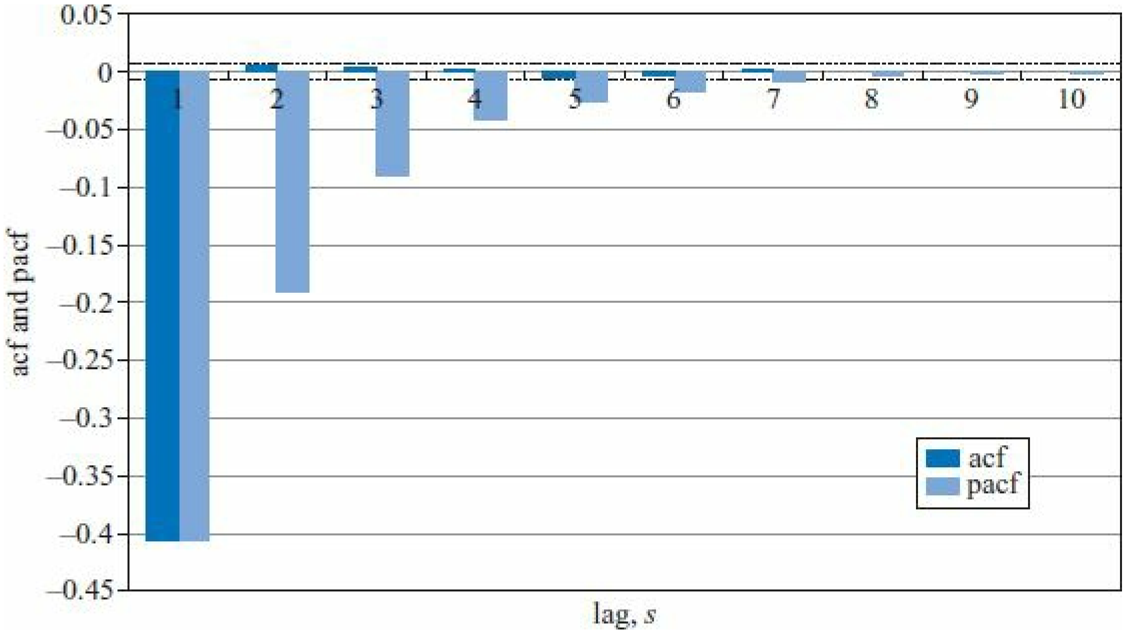


Figure 6.2 Sample autocorrelation and partial autocorrelation functions for an MA(1) model: $y_t = -0.5u_{t-1} + u_t$

In Figure 6.2, the MA(1) has an acf that is significant for only lag 1, while the pacf declines geometrically, and is significant until lag 7. The acf at lag 1 and all of the pacfs are negative as a result of the negative coefficient in the MA generating process.

Again, the structures of the acf and pacf in Figure 6.3 are as anticipated. The first two autocorrelation coefficients only are significant, while the partial autocorrelation coefficients are geometrically declining. Note also that, since the second coefficient on the lagged error term in the MA is negative, the acf and pacf alternate between positive and negative. In the case of the pacf, we term this alternating and declining function a ‘damped sine wave’ or ‘damped sinusoid’.

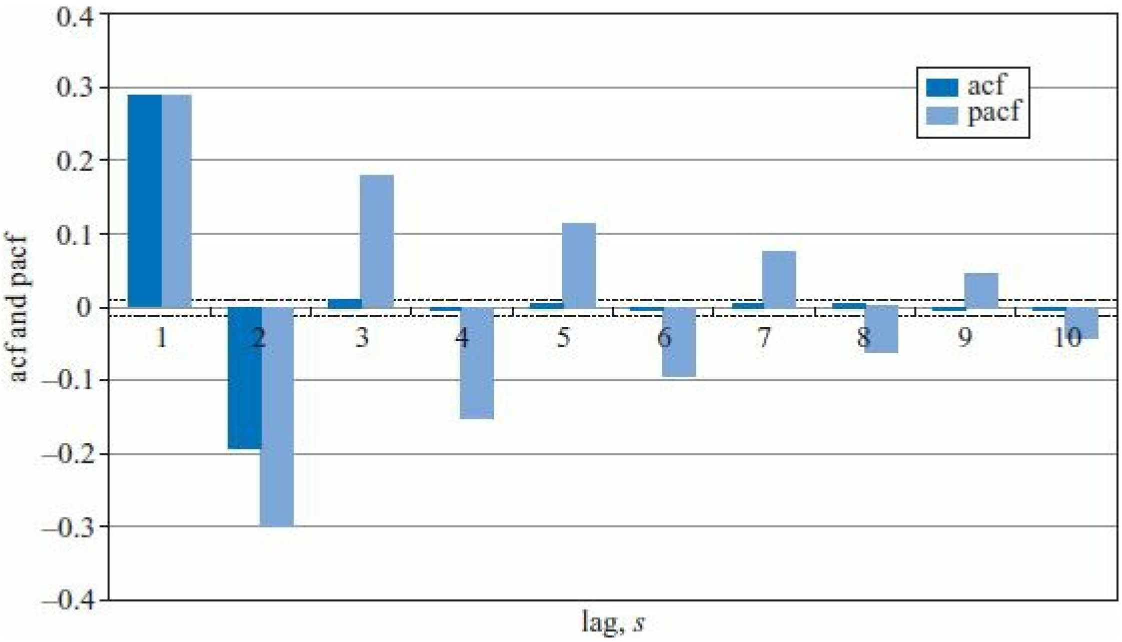


Figure 6.3 Sample autocorrelation and partial autocorrelation functions for an MA(2) model: $y_t = 0.5u_{t-1} - 0.25u_{t-2} + u_t$

For the autoregressive model of order 1 with a fairly high coefficient – i.e., relatively close to 1 – the autocorrelation function would be expected to die away relatively slowly, and this is exactly what is observed here in Figure 6.4. Again, as expected for an AR(1), only the first pacf coefficient is significant, while all others are virtually zero and are not significant.

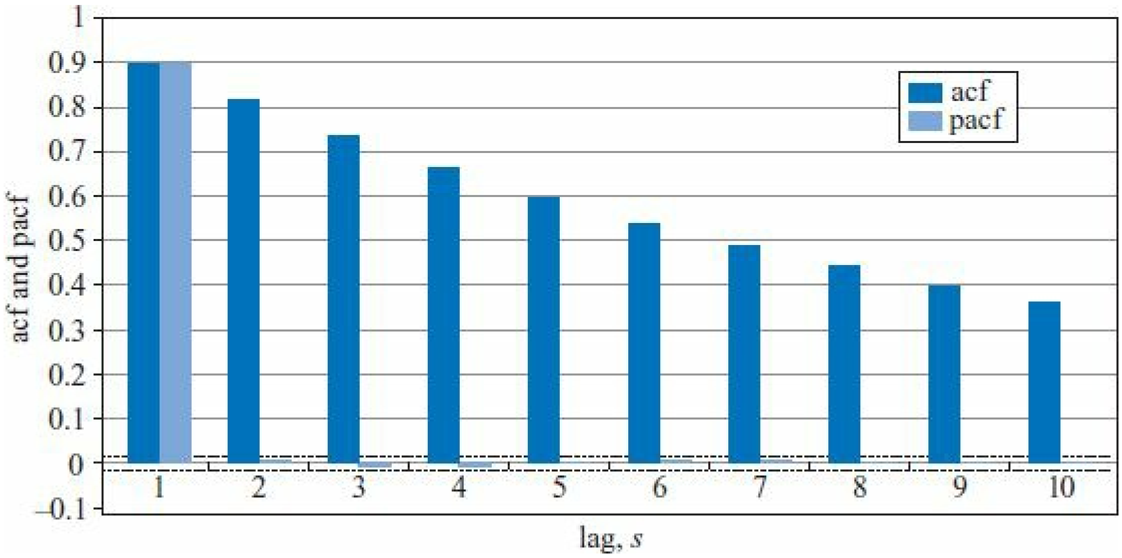


Figure 6.4 Sample autocorrelation and partial autocorrelation functions for a slowly decaying AR(1) model: $y_t = 0.9y_{t-1} + u_t$

Figure 6.5 plots an AR(1), which was generated using identical error terms, but a much smaller autoregressive coefficient. In this case, the autocorrelation function dies away much more quickly than in the previous example, and in fact becomes insignificant after around five lags.

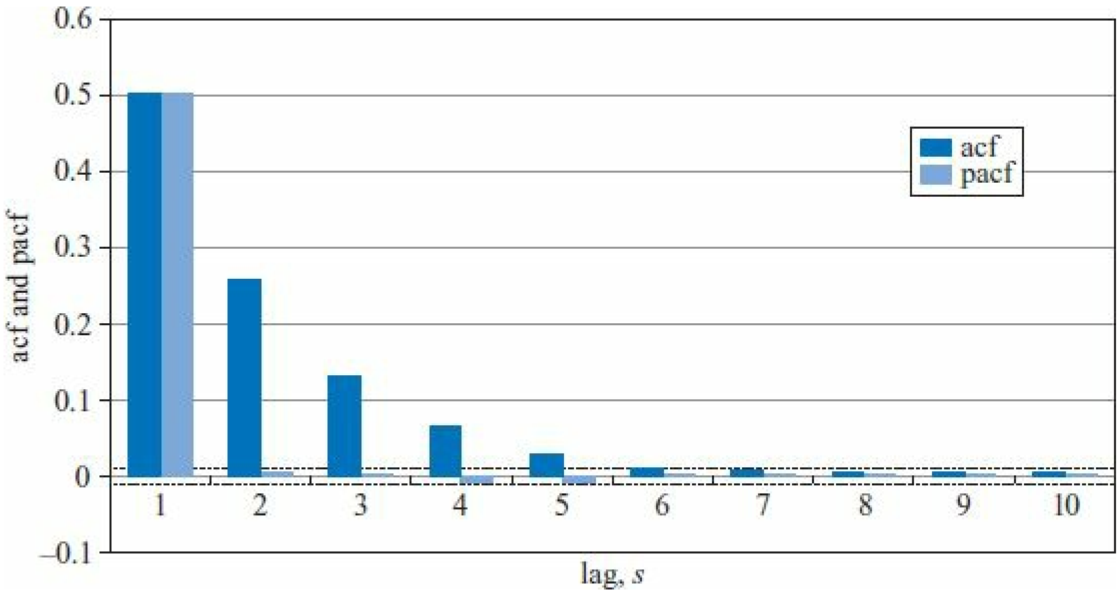


Figure 6.5 Sample autocorrelation and partial autocorrelation functions for a more rapidly decaying AR(1) model: $y_t = 0.5y_{t-1} + u_t$

Figure 6.6 shows the acf and pacf for an identical AR(1) process to that used for Figure 6.5, except that the autoregressive coefficient is now negative. This results in a damped sinusoidal pattern for the acf, which again becomes insignificant after around lag 5. Recalling that the autocorrelation coefficient for this AR(1) at lag s is equal to $(-0.5)^s$, this will be positive for even s, and negative for odd s. Only the first pacf coefficient is significant (and negative).

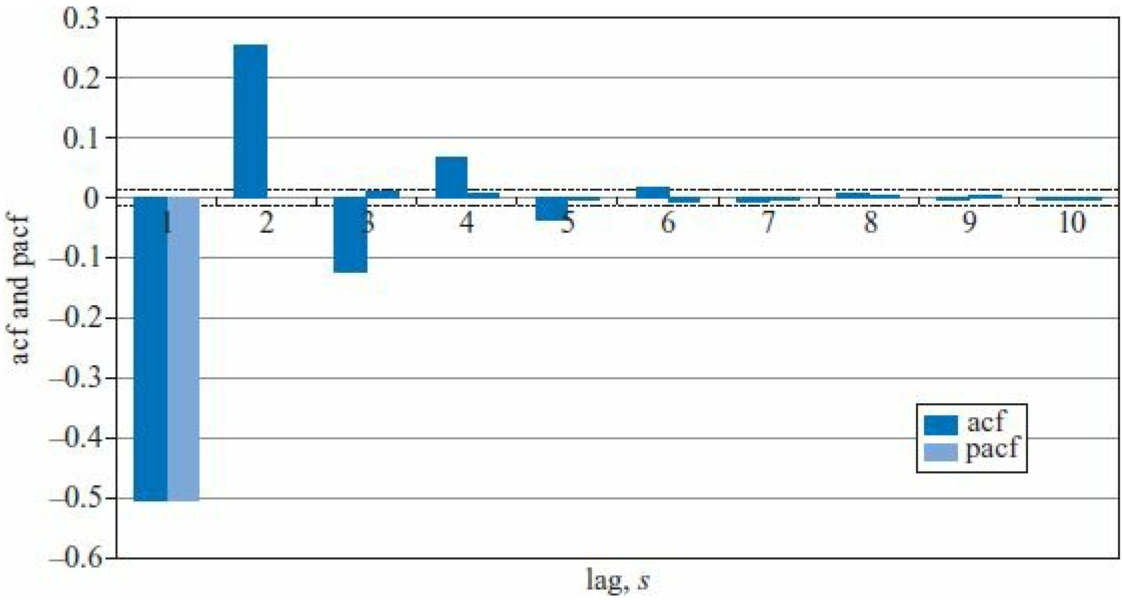


Figure 6.6 Sample autocorrelation and partial autocorrelation functions for a more rapidly decaying AR(1) model with negative coefficient: $y_t = -0.5y_{t-1} + u_t$

Figure 6.7 plots the acf and pacf for a non-stationary series (see Chapter 8 for an extensive discussion) that has a unit coefficient on the lagged dependent variable. The result is that shocks to *y* never die away, and persist indefinitely in the system. Consequently, the acf function remains relatively flat at unity, even up to lag 10. In fact, even by lag 10, the autocorrelation coefficient has fallen only to 0.9989. Note also that on some occasions, the acf does die away, rather than looking like Figure 6.7, even for such a non-stationary process, owing to its inherent instability combined with finite computer precision. The pacf, however, is significant only for lag 1, correctly suggesting that an autoregressive model with no moving average term is most appropriate.

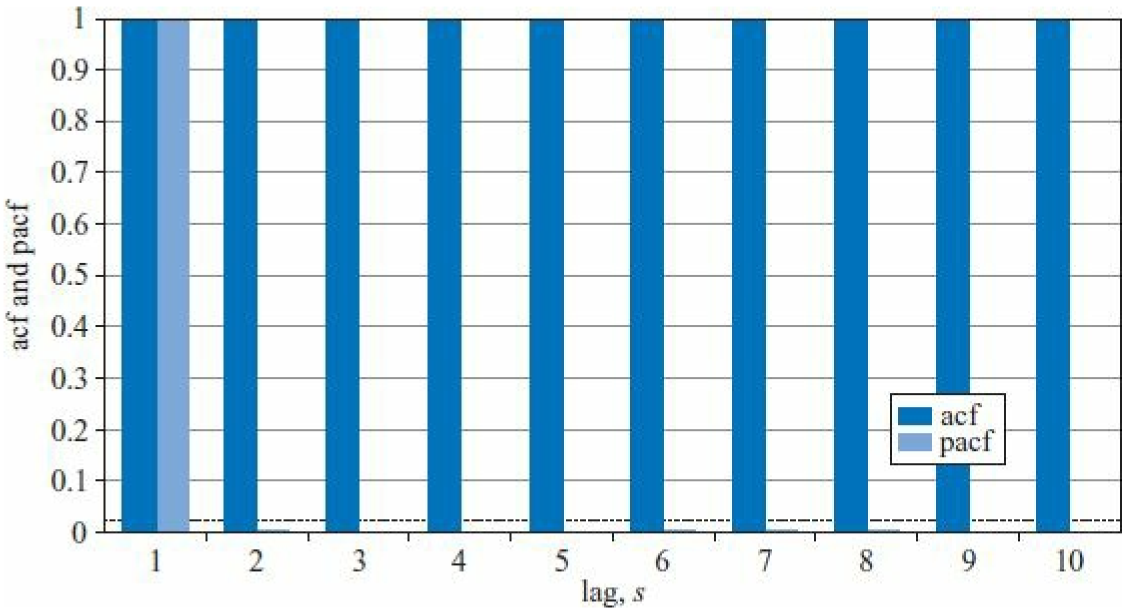


Figure 6.7 Sample autocorrelation and partial autocorrelation functions for a non-stationary model (i.e., a unit coefficient): $y_t = y_{t-1} + u_t$

Finally, Figure 6.8 plots the acf and pacf for a mixed ARMA process. As one would expect of such a process, both the acf and the pacf decline geometrically – the acf as a result of the AR part and the pacf as a result of the MA part. The coefficients on the AR and MA are, however, sufficiently small that both acf and pacf coefficients have become insignificant by lag 6.

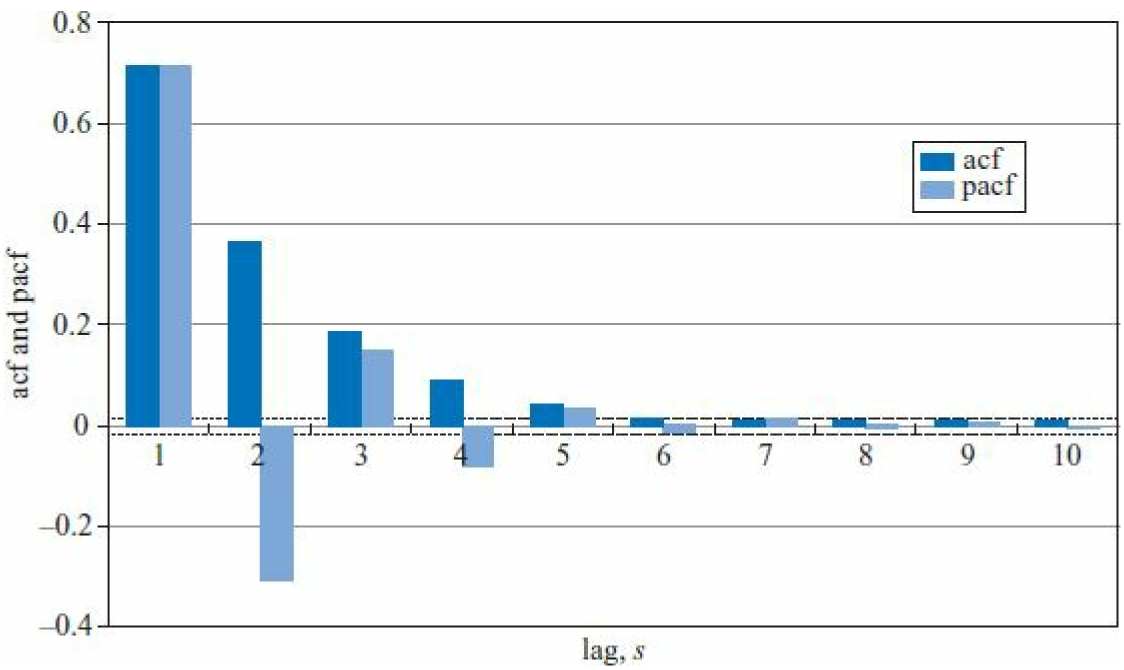


Figure 6.8 Sample autocorrelation and partial autocorrelation functions

for an ARMA(1, 1) model: $y_t = 0.5y_{t-1} + 0.5u_{t-1} + u_t$

6.7 Building ARMA Models: The Box–Jenkins Approach

Although the existence of ARMA models predates them, Box and Jenkins (1976) were the first to approach the task of estimating an ARMA model in a systematic manner. Their approach was a practical and pragmatic one, involving three steps:

(1) Identification

(2) Estimation

(3) Diagnostic checking.

These steps are now explained in greater detail.

Step 1

This involves *determining the order of the model required* to capture the dynamic features of the data. Graphical procedures are used (plotting the data over time and plotting the acf and pacf) to determine the most appropriate specification.

Step 2

This involves *estimation of the parameters of the model* specified in step 1. This can be done using least squares or another technique, known as maximum likelihood, depending on the model.

Step 3

This involves *model checking* – i.e., determining whether the model specified and estimated is adequate. Box and Jenkins suggest two methods: overfitting and residual diagnostics. *Overfitting* involves deliberately fitting a larger model than that required to capture the dynamics of the data as identified in stage 1. If the model specified at step 1 is adequate, any extra terms added to the ARMA model would be insignificant. *Residual diagnostics* imply checking the residuals for evidence of linear dependence which, if present, would suggest that the model originally specified was inadequate to capture the features of the data. The acf, pacf or Ljung–Box tests could be used.

It is worth noting that ‘diagnostic testing’ in the Box–Jenkins world essentially involves only autocorrelation tests rather than the whole barrage of tests outlined in Chapter 5. Also, such approaches to determining the adequacy of the model could only reveal a model that is underparameterised (‘too small’) and would not reveal a model that is overparameterised (‘too big’).

Examining whether the residuals are free from autocorrelation is much more commonly used than overfitting, and this may partly have arisen since for ARMA models, it can give rise to common factors in the overfitted model that make estimation of this model difficult and the statistical tests ill behaved. For example, if the true model is an ARMA(1,1) and we deliberately then fit an ARMA(2,2) there will be a common factor so that not all of the parameters in the latter model can be identified. This problem does not arise with pure AR or MA models, only with mixed processes.

It is usually the objective to form a *parsimonious model*, which is one that describes all of the features of data of interest using as few parameters (i.e., as simple a model) as possible. A parsimonious model is desirable because:

- The residual sum of squares is *proportional* to the number of degrees of freedom. A model which contains irrelevant lags of the variable or of the error term (and therefore unnecessary parameters) will usually lead to increased coefficient standard errors, implying that it will be more difficult to find significant relationships in the data. Whether an increase in the number of variables (i.e., a reduction in the number of degrees of freedom) will actually cause the estimated parameter standard errors to rise or fall will obviously depend on how much the RSS falls, and on the relative sizes of T and k. If T is very large relative to k, then the decrease in RSS is likely to outweigh the reduction in $T - k$ so that the standard errors fall. Hence ‘large’ models with many parameters are more often chosen when the sample size is large.
- Models that are profligate might be inclined to fit to data specific features, which would not be replicated out-of-sample. This means that the models may appear to fit the data very well, with perhaps a high value of R^2, but would give very inaccurate forecasts. Another interpretation of this concept, borrowed from physics, is that of the

distinction between ‘signal’ and ‘noise’. The idea is to fit a model which *captures the signal* (the important features of the data, or the underlying trends or patterns), but which does not try to fit a spurious model to the noise (the completely random aspect of the series).

6.7.1 Information Criteria for ARMA Model Selection

The identification stage would now typically not be done using graphical plots of the acf and pacf. The reason is that when ‘messy’ real data are used, they unfortunately rarely exhibit the simple patterns of Figures 6.2–6.8. This makes the acf and pacf very hard to interpret, and thus it is difficult to specify a model for the data. Another technique, which removes some of the subjectivity involved in interpreting the acf and pacf, is to use what are known as *information criteria*. Information criteria embody two factors: a term which is a function of the residual sum of squares (*RSS*), and some penalty for the loss of degrees of freedom from adding extra parameters. So, adding a new variable or an additional lag to a model will have two competing effects on the information criteria: the residual sum of squares will fall but the value of the penalty term will increase.

The object is to choose the number of parameters which minimises the value of the information criteria. So, adding an extra term will reduce the value of the criteria only if the fall in the residual sum of squares is sufficient to more than outweigh the increased value of the penalty term. There are several different criteria, which vary according to how stiff the penalty term is. The three most popular information criteria are Akaike’s (1974) information criterion (*AIC*), Schwarz’s (1978) Bayesian information criterion (*SBIC*) and the Hannan–Quinn criterion (*HQIC*). Algebraically, these are expressed, respectively, as

$$AIC = \ln(\hat{\sigma}^2) + \frac{2k}{T} \quad \textbf{(6.125)}$$

$$SBIC = \ln(\hat{\sigma}^2) + \frac{k}{T}\ln T \quad \textbf{(6.126)}$$

$$HQIC = \ln(\hat{\sigma}^2) + \frac{2k}{T}\ln(\ln \quad \textbf{(6.127)}$$

where $\hat{\sigma}^2$ is the residual variance (also equivalent to the residual sum of squares divided by the number of observations, T), $k = p + q + 1$ is the total number of parameters estimated and T is the sample size. The information criteria are actually minimised subject to $p \leq \bar{p}, q \leq \bar{q}$, i.e., an upper limit is specified on the number of moving average $(\bar{q})$ and/or autoregressive $(\bar{p})$ terms that will be considered.

It is worth noting that *SBIC* embodies a much stiffer penalty term than *AIC*, while *HQIC* is somewhere in between. The adjusted R^2 measure can also be viewed as an information criterion, although it is a very soft one, which would typically select the largest models of all.

6.7.2 Which Criterion Should be Preferred if they Suggest Different Model Orders?

SBIC is strongly consistent (but inefficient) and *AIC* is not consistent, but is generally more efficient. In other words, *SBIC* will asymptotically deliver the correct model order, while *AIC* will deliver on average too large a model, even with an infinite amount of data. On the other hand, the average variation in selected model orders from different samples within a given population will be greater in the context of *SBIC* than *AIC*. Overall, then, no criterion is definitely superior to others.

6.7.3 ARIMA Modelling

ARIMA modelling, as distinct from ARMA modelling, has the additional letter ‘I’ in the acronym, standing for ‘integrated’. An *integrated autoregressive process* is one whose characteristic equation has a root on the unit circle. Typically researchers difference the variable as necessary and then build an ARMA model on those differenced variables. An ARMA(p, q) model in the variable differenced d times is equivalent to an ARIMA(p, d, q) model on the original data – see Chapter 8 for further details. For the remainder of this chapter, it is assumed that the data used in model construction are stationary, or have been suitably transformed to make them stationary. Thus only ARMA models will be considered further.

6.8 Examples of Time-Series Modelling in Finance

6.8.1 Covered and Uncovered Interest Parity

The determination of the price of one currency in terms of another (i.e., the exchange rate) has received a great deal of empirical examination in the international finance literature. Of these, three hypotheses in particular are studied – covered interest parity (CIP), uncovered interest parity (UIP) and purchasing power parity (PPP). The first two of these will be considered as illustrative examples in this chapter, while PPP will be discussed in Chapter 8. All three relations are relevant for students of finance, for violation of one or more of the parities may offer the potential for arbitrage, or at least will offer further insights into how financial markets operate. All are discussed briefly here; for a more comprehensive treatment, see Cuthbertson and Nitzsche (2004) or the many references therein.

6.8.2 Covered Interest Parity

Stated in its simplest terms, CIP implies that, if financial markets are efficient, it should not be possible to make a riskless profit by borrowing at a risk-free rate of interest in a domestic currency, switching the funds borrowed into another (foreign) currency, investing them there at a risk-free rate and locking in a forward sale to guarantee the rate of exchange back to the domestic currency. Thus, if CIP holds, it is possible to write

$$f_t - s_t = (r - r^*)_t \quad \textbf{(6.128)}$$

where f_t and s_t are the log of the forward and spot prices of the domestic in terms of the foreign currency at time t (i.e., the exchange rates), r is the domestic interest rate and r^* is the foreign interest rate. This is an equilibrium condition which must hold otherwise there would exist riskless arbitrage opportunities, and the existence of such arbitrage would ensure that any deviation from the condition cannot hold indefinitely. It is worth noting that, underlying CIP are the assumptions that the risk-free rates are truly risk-free – that is, there is no possibility for default risk. It is also assumed that there are no transactions costs, such as broker’s fees, bid–ask spreads, stamp duty, etc., and that there are no capital controls, so that funds can be moved without restriction from one currency to another.

6.8.3 Uncovered Interest Parity

UIP takes CIP and adds to it a further condition known as ‘forward rate unbiasedness’ (FRU). Forward rate unbiasedness states that the forward rate of foreign exchange should be an unbiased predictor of the future value of the spot rate. If this condition does not hold, again in theory riskless arbitrage opportunities could exist. UIP, in essence, states that the expected change in the exchange rate should be equal to the interest rate differential between that available risk-free in each of the currencies. Algebraically, this may be stated as

$$s^e_{t+1} - s_t = (r - r^*)_t \tag{6.129}$$

where the notation is as above with s and s^e being the spot exchange rate, r and r^* the interest rates and s^e_{t+1} is the expectation, made at time t of the spot exchange rate that will prevail at time $t + 1$.

The literature testing CIP and UIP is huge with literally hundreds of published papers. Tests of CIP unsurprisingly (for it is a pure arbitrage condition) tend not to reject the hypothesis that the condition holds. Taylor (1987, 1989) has conducted extensive examinations of CIP, and concluded that there were historical periods when arbitrage was profitable, particularly during periods where the exchange rates were under management.

Relatively simple tests of UIP and FRU take equations of the form (6.129) and add intuitively relevant additional terms. If UIP holds, these additional terms should be insignificant. Ito (1988) tests UIP for the yen/dollar exchange rate with the three-month forward rate for January 1973 until February 1985. The sample period is split into three as a consequence of perceived structural breaks in the series. Strict controls on capital movements were in force in Japan until 1977, when some were relaxed and finally removed in 1980. A Chow test confirms Ito's intuition and suggests that the three sample periods should be analysed separately. Two separate regressions are estimated for each of the three sample sub-periods

$$s_{t+3} - f_{t,3} = a + b_1(s_t - f_{t-3,3}) + b_2(s_{t-1} - f_{t-4,3}) + u_t \tag{6.130}$$

where s_{t+3} is the spot interest rate prevailing at time $t + 3$, $f_{t,3}$ is the forward rate for three periods ahead available at time t, and so on, and u_t is an error term. A natural joint hypothesis to test is H_0: $a = 0$ and $b_1 = 0$ and $b_2 = 0$. This hypothesis represents the restriction that the deviation of the forward rate from the realised rate should have a mean value insignificantly different from zero ($a = 0$) and it should be independent of any information available at time t ($b_1 = 0$ and $b_2 = 0$). All three of these conditions must be fulfilled for UIP to hold. The second equation that Ito tests is

$$s_{t+3} - f_{t,3} = a + b(s_t - f_{t,3}) + v_t \tag{6.131}$$

where v_t is an error term and the hypothesis of interest in this case is H_0: $a = 0$ and $b = 0$.

Equation (6.130) tests whether past forecast errors have information useful for predicting the difference between the actual exchange rate at time $t + 3$, and the value of it that was predicted by the forward rate. Equation (6.131) tests whether the forward premium has any predictive power for the difference between the actual exchange rate at time $t + 3$, and the value of it that was predicted by the forward rate. The results for the three sample periods are presented in Ito's Table 3, and are adapted and reported here in Table 6.1.

Table 6.1 Uncovered interest parity test results

Sample period	1973M1–1977M3	1977M4–1980M12	1981M1–1985M2
Panel A: Estimates and hypothesis tests for			
	$S_{t+3} - f_{t,3} = a + b_1(s_t - f_{t-3,3}) + b_2(s_{t-1} - f_{t-4,3}) + u_t$		
Estimate of a	0.0099	0.0031	0.027
Estimate of b_1	0.0200	0.240	0.077
Estimate of b_2	−0.370	0.160	−0.210
Joint test $\chi^2(3)$	23.388	5.248	6.022
p-value for joint test	0.000	0.155	0.111
Panel B: Estimates and hypothesis tests for			
$S_{t+3} - f_{t,3} = a + b(s_t - f_{t,3}) + v_t$			
Estimate of a	0.00	−0.05	−0.89
Estimate of b	0.09	4.18	2.93
Joint test $\chi^2(2)$	31.92	22.06	5.39
p-value for joint test	0.00	0.00	0.07

Source: Ito (1988). Reprinted with permission from MIT Press Journals.

The main conclusion is that UIP clearly failed to hold throughout the period of strictest controls, but there is less and less evidence against UIP as controls were relaxed.

6.9 Exponential Smoothing

Exponential smoothing is another modelling technique (not based on the ARIMA approach) that uses only a linear combination of the previous values of a series for modelling it and for generating forecasts of its future values. Given that only previous values of the series of interest are used, the only question remaining is how much weight should be attached to each of the previous observations. Recent observations would be expected to have the most power in helping to forecast future values of a series. If this is accepted, a model that places more weight on recent observations than those further in the past would be desirable. On the other hand, observations a long way in the past may still contain some information useful for forecasting future values of a series, which would not be the case under a centred moving average. An exponential smoothing model will achieve this, by imposing a geometrically declining weighting scheme on the lagged values of a series. The equation for the model is

$$S_t = \alpha y_t + (1 - \alpha)S_{t-1} \tag{6.132}$$

where α is the smoothing constant, with $0 < \alpha < 1$, y_t is the current realised value, S_t is the current smoothed value.

Since $\alpha + (1 - \alpha) = 1$, S_t is modelled as a weighted average of the current observation y_t and the previous smoothed value. The model above can be rewritten to express the exponential weighting scheme more clearly. By lagging equation (6.132) by one period, the following expression is obtained

$$S_{t-1} = \alpha y_{t-1} + (1 - \alpha)S_{t-2} \tag{6.133}$$

and lagging again

$$S_{t-2} = \alpha y_{t-2} + (1 - \alpha)S_{t-3} \tag{6.134}$$

Substituting into (6.132) for S_{t-1} from equation (6.133)

$$S_t = \alpha y_t + (1 - \alpha)(\alpha y_{t-1} + (1 - \alpha)S_{t-2}) \tag{6.135}$$

$$S_t = \alpha y_t + (1 - \alpha)\alpha y_{t-1} + (1 - \alpha)^2 S_{t-2} \tag{6.136}$$

Substituting into (6.136) for S_{t-2} from equation (6.134)

$$S_t = \alpha y_t + (1 - \alpha)\alpha y_{t-1} + (1 - \alpha)^2(\alpha y_{t-2} + (1 - \alpha)S_{t-3}) \tag{6.137}$$

$$S_t = \alpha y_t + (1 - \alpha)\alpha y_{t-1} + (1 - \alpha)^2\alpha y_{t-2} + (1 - \alpha)^3 S_{t-3} \tag{6.138}$$

T successive substitutions of this kind would lead to

$$S_t = \left(\sum_{i=0}^{T} \alpha(1-\alpha)^i y_{t-i} \right) + (1-\alpha)^{T+1} S_{t-1-T} \quad \textbf{(6.139)}$$

Since $\alpha > 0$, the effect of each observation declines geometrically as the variable moves another observation forward in time. In the limit as $T \to \infty$, the second term tends to zero, so that the current smoothed value is a geometrically weighted infinite sum of the previous realisations.

The forecasts from an exponential smoothing model are simply set to the current smoothed value, for any number of steps ahead, s

$$f_{t,s} = S_t, \quad s = 1, 2, 3, \ldots \quad \textbf{(6.140)}$$

The exponential smoothing model can be seen as a special case of a Box–Jenkins model, an ARIMA(0,1,1), with MA coefficient $(1 - \alpha)$ – see Granger and Newbold (1986, p. 174).

The technique above is known as single or simple exponential smoothing, and it can be modified to allow for trends (Holt’s method) or to allow for seasonality (Winter’s method) in the underlying variable. These augmented models are not pursued further in this text since there is a much better way to model the trends (using a unit root process – see Chapter 8) and the seasonalities (see Chapter 10) of the form that are typically present in financial data.

Exponential smoothing has several advantages over the slightly more complex ARMA class of models discussed above. First, exponential smoothing is obviously very simple to use. There is no decision to be made on how many parameters to estimate (assuming only single exponential smoothing is considered). Thus it is easy to update the model if a new realisation becomes available.

Among the disadvantages of exponential smoothing is the fact that it is overly simplistic and inflexible. Exponential smoothing models can be viewed as but one model from the ARIMA family, which may not necessarily be optimal for capturing any linear dependence in the data. Also, the forecasts from an exponential smoothing model do not converge on the long-term mean of the variable as the horizon increases. The upshot is that long-term forecasts are overly affected by recent events in the history of the series under investigation and will therefore be sub-optimal.

6.10 Forecasting in Econometrics

Although the words 'forecasting' and 'prediction' are sometimes given different meanings in some studies, in this text the words will be used synonymously. In this context, prediction or forecasting simply means an attempt to *determine the values that a series is likely to take*. Of course, forecasts might also usefully be made in a cross-sectional environment. Although the discussion below refers to time-series data, some of the arguments will carry over to the cross-sectional context.

Determining the forecasting accuracy of a model is an important test of its adequacy. Some econometricians would go as far as to suggest that the statistical adequacy of a model in terms of whether it violates the CLRM assumptions or whether it contains insignificant parameters, is largely irrelevant if the model produces accurate forecasts. The following sub-sections of the book discuss why forecasts are made, how they are made from several important classes of models, how to evaluate the forecasts, and so on.

6.10.1 Why Forecast?

Forecasts are made essentially because they are useful! Financial decisions often involve a long-term commitment of resources, the returns to which will depend upon what happens in the future. In this context, the decisions made today will reflect forecasts of the future state of the world, and the more accurate those forecasts are, the more utility (or money!) is likely to be gained from acting on them.

Some examples in finance of where forecasts from econometric models might be useful include:

- Forecasting tomorrow's return on a particular *share*
- Forecasting the *price of a house* given its characteristics
- Forecasting the *riskiness of a portfolio* over the next year
- Forecasting the *volatility of bond returns*
- Forecasting the *correlation between US and UK stock market movements* tomorrow
- Forecasting the likely number of *defaults* on a portfolio of home loans.

Again, it is evident that forecasting can apply either in a cross-sectional or a time-series context. It is useful to distinguish between two approaches to forecasting:

- *Econometric* (*structural*) *forecasting* – relates a dependent variable to one or more independent variables. Such models often work well in the long run, since a long-run relationship between variables often arises from no-arbitrage or market efficiency conditions. Examples of such forecasts would include return predictions derived from arbitrage pricing models, or long-term exchange rate prediction based on purchasing power parity or uncovered interest parity theory.
- *Time series forecasting* – involves trying to forecast the future values of a series given its previous values and/or previous values of an error term.

The distinction between the two types is somewhat blurred – for example, it is not clear where vector autoregressive (VAR) models (see Chapter 7 for an extensive overview) fit into this classification.

It is also worth distinguishing between point and interval forecasts. *Point* forecasts predict a single value for the variable of interest, while *interval* forecasts provide a range of values in which the future value of the variable is expected to lie with a given level of confidence.

6.10.2 The Difference Between In-Sample and Out-of-Sample Forecasts

In-sample forecasts are those generated for the same set of data that was used to estimate the model's parameters. One would expect the 'forecasts' of a model to be relatively good in-sample, for this reason. Therefore, a sensible approach to model evaluation through an examination of forecast accuracy is not to use all of the observations in estimating the model parameters, but rather to hold some observations back. The latter sample, sometimes known as a *holdout sample*, would be used to construct out-of-sample forecasts.

To give an illustration of this distinction, suppose that some monthly FTSE returns for 120 months (January 1990–December 1999) are available. It would be possible to use all of them to build the model (and generate only in-sample forecasts), or some observations could be kept back, as shown in Figure 6.9.

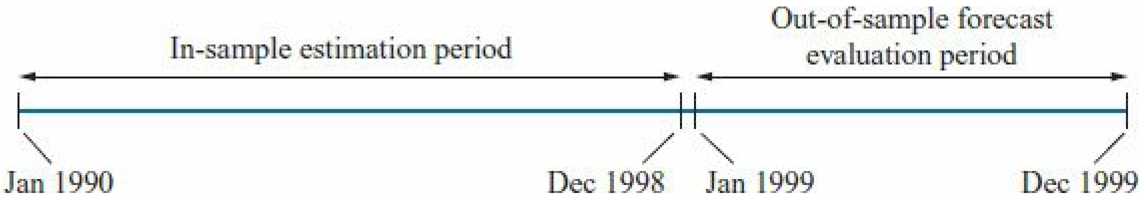


Figure 6.9 Use of in-sample and out-of-sample periods for analysis

What would be done in this case would be to use data from 1990*M*1 until 1998*M*12 to estimate the model parameters, and then the observations for 1999 would be forecast from the estimated parameters. Of course, where each of the in-sample and out-of-sample periods should start and finish is somewhat arbitrary and at the discretion of the researcher. One could then compare how close the forecasts for the 1999 months were relative to their actual values that are in the holdout sample. This procedure would represent a better test of the model than an examination of the in-sample fit of the model since the information from 1999*M*1 onwards has not been used when estimating the model parameters.

6.10.3 Some More Terminology: One-Step-Ahead versus Multi-Step-Ahead Forecasts and Rolling versus Recursive Samples

A *one-step-ahead forecast* is a forecast generated for the next observation only, whereas *multi-step-ahead forecasts* are those generated for 1, 2, 3, …, s steps ahead, so that the forecasting horizon is for the next s periods. Whether one-step- or multi-step-ahead forecasts are of interest will be determined by the forecasting horizon of interest to the researcher.

Suppose that the monthly FTSE data are used as described in the example above. If the in-sample estimation period stops in December 1998, then up to twelve-step-ahead forecasts could be produced, giving twelve predictions that can be compared with the actual values of the series. Comparing the actual and forecast values in this way is not ideal, for the forecasting horizon is varying from one to twelve steps ahead. It might be the case, for example, that the model produces very good forecasts for short horizons (say, one or two steps), but that it produces inaccurate forecasts further ahead. It would not be possible to evaluate whether this was in fact the case or not since only a single one-step-ahead forecast, a single two-step-ahead forecast, and so on, are available. An evaluation of the forecasts would require a considerably larger holdout sample.

A useful way around this problem is to use a *recursive or rolling window*, which generates a series of forecasts for a given number of steps ahead. A recursive forecasting model would be one where the initial estimation date is fixed, but additional observations are added one at a time to the estimation period. A rolling window, on the other hand, is one where the length of the in-sample period used to estimate the model is fixed, so that the start date and end date successively increase by one observation. Suppose now that only one-, two-, and three-step-ahead forecasts are of interest. They could be produced using the following recursive and rolling window approaches:

Objective: to produce	**Data used to estimate model parameters**	
1-, 2-, 3-step-ahead forecasts for:	**Rolling window**	**Recursive window**
1999*M*1, *M*2, *M*3	1990*M*1–1998*M*12	1990*M*1–1998*M*12
1999*M*2, *M*3, *M*4	1990*M*2–1999*M*1	1990*M*1–1999*M*1
1999*M*3, *M*4, *M*5	1990*M*3–1999*M*2	1990*M*1–1999*M*2
1999*M*4, *M*5, *M*6	1990*M*4–1999*M*3	1990*M*1–1999*M*3
1999*M*5, *M*6, *M*7	1990*M*5–1999*M*4	1990*M*1–1999*M*4
1999*M*6, *M*7, *M*8	1990*M*6–1999*M*5	1990*M*1–1999*M*5
1999*M*7, *M*8, *M*9	1990*M*7–1999*M*6	1990*M*1–1999*M*6
1999*M*8, *M*9, *M*10	1990*M*8–1999*M*7	1990*M*1–1999*M*7
1999*M*9, *M*10, *M*11	1990*M*9–1999*M*8	1990*M*1–1999*M*8
1999*M*10, *M*11, *M*12	1990*M*10–1999*M*9	1990*M*1–1999*M*9

The sample length for the rolling windows above is always set at 108 observations, while the number of observations used to estimate the parameters in the recursive case increases as we move down the table and through the sample.

6.10.4 Forecasting with Time-Series versus Structural Models

To understand how to construct forecasts, the idea of *conditional expectations* is required. A conditional expectation would be expressed as

$$E(y_{t+1}|\Omega_t)$$

This expression states that the expected value of *y* is taken for time *t* + 1, conditional upon, or given, (|) all information available up to and including time *t* (Ω_t). Contrast this with the unconditional expectation of *y*, which is the expected value of *y* without any reference to time, i.e., the unconditional mean of *y*. The conditional expectations operator is used to generate forecasts of the series.

How this conditional expectation is evaluated will of course depend on the model under consideration. Several families of models for forecasting will be developed in this and subsequent chapters.

A first point to note is that by definition the optimal forecast for a zero mean white noise process is zero

$$\mathrm{E}(u_{t+s}|\Omega_t) = 0 \; \forall \, s > 0 \tag{6.141}$$

The two simplest forecasting 'methods' that can be employed in almost every situation are shown in Box 6.3.

BOX 6.3 Naive forecasting methods

(1) Assume no change so that the forecast, *f*, of the value of *y*, *s* steps into the future is the current value of *y*

$$E(y_{t+s}|\Omega_t) = y_t \tag{6.142}$$

Such a forecast would be optimal if y_t followed a random walk process.

(2) In the absence of a full model, forecasts can be generated using the long-term average of the series. Forecasts using the unconditional mean would be more useful than 'no change' forecasts for any series that is 'mean-reverting' (i.e., stationary).

Time series models are generally better suited to the production of time series forecasts than structural models. For an illustration of this, consider the following linear regression model

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \cdots + \beta_k x_{kt} + u_t \tag{6.143}$$

To forecast y, the conditional expectation of its future value is required. We take conditional expectations of both sides of equation (6.143), and note that strictly conditional expectations should be added to all variables on the RHS of equations (6.144) and (6.145)

$$E(y_t | \Omega_{t-1}) = E(\beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \cdots + \beta_k x_{kt} + u_t) \quad \textbf{(6.144)}$$

The parameters can be taken through the expectations operator, since this is a population regression function and therefore they are assumed known. The following expression would be obtained

$$E(y_t | \Omega_{t-1}) = \beta_1 + \beta_2 E(x_{2t}) + \beta_3 E(x_{3t}) + \cdots + \beta_k E(x_{kt}) \quad \textbf{(6.145)}$$

But there is a problem: what are $E(x_{2t})$, etc.? Remembering that information is available only until time $t - 1$, the values of these variables are unknown. It may be possible to forecast them, but this would require another set of forecasting models for every explanatory variable. To the extent that forecasting the explanatory variables may be as difficult, or even more difficult, than forecasting the explained variable, this equation has achieved nothing! In the absence of a set of forecasts for the explanatory variables, one might think of using $\bar{x}_2$, etc., i.e., the mean values of the explanatory variables, giving

$$E(y_t) = \beta_1 + \beta_2 \bar{x}_2 + \beta_3 \bar{x}_3 + \cdots + \beta_k \bar{x}_k = \bar{y}\,! \quad \textbf{(6.146)}$$

Thus, if the mean values of the explanatory variables are used as inputs to the model, all that will be obtained as a forecast is the average value of y. Forecasting using pure time series models is relatively common, since it avoids this problem.

6.10.5 Forecasting with ARMA Models

Forecasting using ARMA models is a fairly simple exercise in calculating conditional expectations. Although any consistent and logical notation could be used, the following conventions will be adopted in this book. Let $f_{t,s}$ denote a forecast made using an ARMA(p,q) model at time t for s steps into the future for some series y. The forecasts are generated by what is known as a forecast function, typically of the form

$$f_{t,s} = \sum_{i=1}^{p} a_i f_{t,s-i} + \sum_{j=1}^{q} b_j u_{t+s-j} \quad \textbf{(6.147)}$$

where $f_{t,s} = y_{t+s}$, $s \leq 0$; $u_{t+s} = 0$, $s > 0 = u_{t+s}$, $s \leq 0$

and a_i and b_i are the autoregressive and moving average coefficients, respectively.

A demonstration of how one generates forecasts for separate AR and MA processes, leading to the general equation (6.147) above, will now be given.

6.10.6 Forecasting the Future Value of an MA(q) Process

A moving average process has a memory only of length q, and this limits the sensible forecasting horizon. For example, suppose that an MA(3) model has been estimated

$$y_t = \mu + \theta_1 u_{t-1} + \theta_2 u_{t-2} + \theta_3 u_{t-3} + u_t \quad \textbf{(6.148)}$$

Since parameter constancy over time is assumed, if this relationship holds for the series y at time t, it is also assumed to hold for y at time $t + 1$, $t + 2$, …, so 1 can be added to each of the time subscripts in equation (6.148), and 2 added to each of the time subscripts, and then 3, and so on, to arrive at the following

$$y_{t+1} = \mu + \theta_1 u_t + \theta_2 u_{t-1} + \theta_3 u_{t-2} + u_{t+1} \quad \textbf{(6.149)}$$

$$y_{t+2} = \mu + \theta_1 u_{t+1} + \theta_2 u_t + \theta_3 u_{t-1} + u_{t+2} \quad \textbf{(6.150)}$$

$$y_{t+3} = \mu + \theta_1 u_{t+2} + \theta_2 u_{t+1} + \theta_3 u_t + u_{t+3} \quad \textbf{(6.151)}$$

Suppose that all information up to and including that at time t is available and that forecasts for 1, 2, …, s steps ahead – i.e., forecasts for y at times $t + 1$, $t + 2$, …, $t + s$ are wanted. y_t, y_{t-1}, …, and u_t, u_{t-1}, are known, so producing the forecasts is just a matter of taking the conditional expectation of equation (6.149)

$$f_{t,1} = E(y_{t+1|t}) = E(\mu + \theta_1 u_t + \theta_2 u_{t-1} + \theta_3 u_{t-2} + u_{t+1}|\Omega_t) \quad \textbf{(6.152)}$$

where $E(y_{t+1|t})$ is a short-hand notation for $E(y_{t+1}|\Omega_t)$

$$f_{t,1} = E(y_{t+1|t}) = \mu + \theta_1 u_t + \theta_2 u_{t-1} + \theta_3 u_{t-2} \quad \textbf{(6.153)}$$

Thus the forecast for *y*, one step ahead, made at time *t*, is given by this linear combination of the disturbance terms. Note that it would not be appropriate to set the values of these disturbance terms to their unconditional mean of zero. This arises because it is the *conditional expectation* of their values that is of interest. Given that all information is known up to and including that at time *t* is available, the values of the error terms up to time *t* are known. But u_{t+1} is not known at time *t* and therefore $E(u_{t+1|t}) = 0$, and so on.

The forecast for two steps ahead is formed by taking the conditional expectation of equation (6.150)

$$f_{t,2} = E(y_{t+2|t}) = E(\mu + \theta_1 u_{t+1} + \theta_2 u_t + \theta_3 u_{t-1} + u_{t+2} \mid \Omega_t) \quad \textbf{(6.154)}$$

$$f_{t,2} = E(y_{t+2|t}) = \mu + \theta_2 u_t + \theta_3 u_{t-1} \quad \textbf{(6.155)}$$

In the case above, u_{t+2} is not known since information is available only to time *t*, so $E(u_{t+2})$ is set to zero. Continuing and applying the same rules to generate 3-, 4-, …, *s*-step-ahead forecasts

$$f_{t,3} = E(y_{t+3|t}) = E(\mu + \theta_1 u_{t+2} + \theta_2 u_{t+1} + \theta_3 u_t + u_{t+3} \mid \Omega_t) \quad \textbf{(6.156)}$$

$$f_{t,3} = E(y_{t+3|t}) = \mu + \theta_3 u_t \quad \textbf{(6.157)}$$

$$f_{t,4} = E(y_{t+4|t}) = \mu \quad \textbf{(6.158)}$$

$$f_{t,s} = E(y_{t+s|t}) = \mu \ \forall\ s \geq 4 \quad \textbf{(6.159)}$$

As the MA(3) process has a memory of only three periods, all forecasts four or more steps ahead collapse to the intercept. Obviously, if there had been no constant term in the model, the forecasts four or more steps ahead for an MA(3) would be zero.

6.10.7 Forecasting the Future Value of an AR(p) Process

Unlike a moving average process, an autoregressive process has infinite memory. To illustrate, suppose that an AR(2) model has been estimated

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + u_t \quad \textbf{(6.160)}$$

Again, by appealing to the assumption of parameter stability, this equation will hold for times $t + 1$, $t + 2$, and so on

$$y_{t+1} = \mu + \phi_1 y_t + \phi_2 y_{t-1} + u_{t+1} \quad \textbf{(6.161)}$$

$$y_{t+2} = \mu + \phi_1 y_{t+1} + \phi_2 y_t + u_{t+2} \quad \textbf{(6.162)}$$
$$y_{t+3} = \mu + \phi_1 y_{t+2} + \phi_2 y_{t+1} + u_{t+3} \quad \textbf{(6.163)}$$

Producing the one-step-ahead forecast is easy, since all of the information required is known at time t. Applying the expectations operator to equation (6.161), and setting E(u_{t+1}) to zero would lead to

$$f_{t,1} = E(y_{t+1|t}) = E(\mu + \phi_1 y_t + \phi_2 y_{t-1} + u_{t+1} \mid \Omega_t) \quad \textbf{(6.164)}$$
$$f_{t,1} = E(y_{t+1|t}) = \mu + \phi_1 E(y_t \mid \Omega_t) + \phi_2 E(y_{t-1} \mid \Omega_t) \quad \textbf{(6.165)}$$
$$f_{t,1} = E(y_{t+1|t}) = \mu + \phi_1 y_t + \phi_2 y_{t-1} \quad \textbf{(6.166)}$$

Applying the same procedure in order to generate a two-step-ahead forecast

$$f_{t,2} = E(y_{t+2|t}) = E(\mu + \phi_1 y_{t+1} + \phi_2 y_t + u_{t+2} \mid \Omega_t) \quad \textbf{(6.167)}$$
$$f_{t,2} = E(y_{t+2|t}) = \mu + \phi_1 E(y_{t+1} \mid \Omega_t) + \phi_2 E(y_t \mid \Omega_t) \quad \textbf{(6.168)}$$

The case above is now slightly more tricky, since E(y_{t+1}) is not known, although this in fact is the one-step-ahead forecast, so that equation (6.168) becomes

$$f_{t,2} = E(y_{t+2|t}) = \mu + \phi_1 f_{t,1} + \phi_2 y_t \quad \textbf{(6.169)}$$

Similarly, for three, four, …and s steps ahead, the forecasts will be, respectively, given by

$$f_{t,3} = E(y_{t+3|t}) = E(\mu + \phi_1 y_{t+2} + \phi_2 y_{t+1} + u_{t+3} \mid \Omega_t) \quad \textbf{(6.170)}$$
$$f_{t,3} = E(y_{t+3|t}) = \mu + \phi_1 E(y_{t+2} \mid \Omega_t) + \phi_2 E(y_{t+1} \mid \Omega_t) \quad \textbf{(6.171)}$$
$$f_{t,3} = E(y_{t+3|t}) = \mu + \phi_1 f_{t,2} + \phi_2 f_{t,1} \quad \textbf{(6.172)}$$
$$f_{t,4} = \mu + \phi_1 f_{t,3} + \phi_2 f_{t,2}, \quad \textbf{(6.173)}$$

etc. so

$$f_{t,s} = \mu + \phi_1 f_{t,s-1} + \phi_2 f_{t,s-2} \quad \textbf{(6.174)}$$

Thus the s-step-ahead forecast for an AR(2) process is given by the intercept + the coefficient on the one-period lag multiplied by the time $s - 1$ forecast + the coefficient on the two-period lag multiplied by the $s - 2$ forecast.

ARMA(p,q) forecasts can easily be generated in the same way by applying the rules for their component parts, and using the general formula given by equation (6.147).

6.10.8 Determining Whether a Forecast is Accurate or Not

For example, suppose that tomorrow's return on the FTSE is predicted to be 0.2, and that the outcome is actually −0.4. Is this an accurate forecast? Clearly, one cannot determine whether a forecasting model is good or not based upon only one forecast and one realisation. Thus in practice, forecasts would usually be produced for the whole of the out-of-sample period, which would then be compared with the actual values, and the difference between them aggregated in some way. The forecast error for observation i is defined as the difference between the actual value for observation i and the forecast made for it. The forecast error, defined in this way, will be positive (negative) if the forecast was too low (high). Therefore, it is not possible simply to sum the forecast errors, since the positive and negative errors will cancel one another out. Thus, before the forecast errors are aggregated, they are usually squared or the absolute value taken, which renders them all positive. To see how the aggregation works, consider the example in Table 6.2, where forecasts are made for a series up to five steps ahead, and are then compared with the actual realisations (with all calculations rounded to three decimal places).

Table 6.2 Forecast error aggregation

Steps ahead	Forecast	Actual	Squared error	Absolute error
1	0.20	−0.40	$(0.20 - -0.40)^2 = 0.360$	$\|0.20 - -0.40\| = 0.600$
2	0.15	0.20	$(0.15-0.20)^2 = 0.002$	$\|0.15-0.20\| = 0.050$
3	0.10	0.10	$(0.10-0.10)^2 = 0.000$	$\|0.10-0.10\| = 0.000$
4	0.06	−0.10	$(0.06 - -0.10)^2 = 0.026$	$\|0.06 - -0.10\| = 0.160$
5	0.04	−0.05	$(0.04 - -0.05)^2 = 0.008$	$\|0.06 - -0.10\| = 0.160$

The mean squared error (*MSE*) and mean absolute error (*MAE*) are now calculated by taking the average of the fourth and fifth columns, respectively

$$\text{MSE} = (0.360 + 0.002 + 0.000 + 0.026 + 0.008)/5 = 0.079 \tag{6.175}$$

$$\text{MAE} = (0.600 + 0.050 + 0.000 + 0.160 + 0.090)/5 = 0.180 \tag{6.176}$$

Taken individually, little can be gleaned from considering the size of the *MSE* or *MAE*, for the statistic is unbounded from above (like the residual sum of squares or *RSS*). Instead, the *MSE* or *MAE* from one model would be compared with those of other models for the same data and forecast period, and the model(s) with the lowest value of the error measure would be argued to be the most accurate.

MSE provides a quadratic loss function, and so may be particularly useful in situations where large forecast errors are disproportionately more serious than smaller errors. This may, however, also be viewed as a disadvantage if large errors are not disproportionately more serious, although the same critique could also, of course, be applied to the whole least squares methodology. Indeed Dielman (1986) goes as far as to say that when there are outliers present, least absolute values should be used to determine model parameters rather than least squares. Makridakis (1993, p. 528) argues that mean absolute percentage error (*MAPE*) is ‘a relative measure that incorporates the best characteristics among the various accuracy criteria’. Once again, denoting *s*-step-ahead forecasts of a variable made at time *t* as $f_{t,s}$ and the actual value of the variable at time *t* as y_t, then the MSE can be defined as

$$\text{MSE} = \frac{1}{T - (T_1 - 1)} \sum_{t=T_1}^{T} (y_{t+s} - f_{t,s})^2 \tag{6.177}$$

where *T* is the total sample size (in-sample + out-of-sample), and T_1 is the first outof-sample forecast observation. Thus in-sample model estimation initially runs from observation 1 to (T_1−1), and observations T_1 to *T* are available for out-of-sample estimation, i.e., a total holdout sample of $T - (T_1 - 1)$.

MAE measures the average absolute forecast error, and is given by

$$\text{MAE} = \frac{1}{T - (T_1 - 1)} \sum_{t=T_1}^{T} |y_{t+s} - f_{t,s}| \quad \textbf{(6.178)}$$

Adjusted MAPE (*AMAPE*) or symmetric *MAPE* corrects for the problem of asymmetry between the actual and forecast values

$$\text{AMAPE} = \frac{100}{T - (T_1 - 1)} \sum_{t=T_1}^{T} \left| \frac{y_{t+s} - f_{t,s}}{y_{t+s} + f_{t,s}} \right| \quad \textbf{(6.179)}$$

The symmetry in equation (6.179) arises since the forecast error is divided by twice the average of the actual and forecast values. So, for example, *AMAPE* will be the same whether the forecast is 0.5 and the actual value is 0.3, or the actual value is 0.5 and the forecast is 0.3. The same is not true of the standard *MAPE* formula, where the denominator is simply y_{t+s}, so that whether y_t or $f_{t,s}$ is larger will affect the result

$$\text{MAPE} = \frac{100}{T - (T_1 - 1)} \sum_{t=T_1}^{T} \left| \frac{y_{t+s} - f_{t,s}}{y_{t+s}} \right| \quad \textbf{(6.180)}$$

MAPE also has the attractive additional property compared to *MSE* that it can be interpreted as a percentage error, and furthermore, its value is bounded from below by 0.

Unfortunately, it is not possible to use the adjustment if the series and the forecasts can take on opposite signs (as they could in the context of returns forecasts, for example). This is due to the fact that the prediction and the actual value may, purely by coincidence, take on values that are almost equal and opposite, thus almost cancelling each other out in the denominator. This leads to extremely large and erratic values of *AMAPE*. In such an instance, it is not possible to use *MAPE* as a criterion either. Consider the following example: say we forecast a value of $f_{t,s} = 3$, but the out-turn is that $y_{t+s} = 0.0001$. The addition to total *MSE* from this one observation is given by

$$\frac{1}{391} \times (0.0001 - 3)^2 = 0.0230 \quad \textbf{(6.181)}$$

This value for the forecast is large, but perfectly feasible since in many cases it will be well within the range of the data. But the addition to total *MAPE* from just this single observation is given by

$$\frac{100}{391}\left|\frac{0.0001-3}{0.0001}\right| = 7670 \qquad \textbf{(6.182)}$$

MAPE has the advantage that for a random walk in the log levels (i.e., a zero forecast), the criterion will take the value one (or 100 if we multiply the formula by 100 to get a percentage, as was the case for the equation above). So if a forecasting model gives a *MAPE* smaller than one (or 100), it is superior to the random walk model. In fact the criterion is also not reliable if the series can take on absolute values less than one. This point may seem somewhat obvious, but it is clearly important for the choice of forecast evaluation criteria.

Another criterion which is popular is Theil's *U*-statistic (1966). The metric is defined as follows

$$U = \frac{\sqrt{\sum_{t=T_1}^{T}\left(\frac{y_{t+s}-f_{t,s}}{y_{t+s}}\right)^2}}{\sqrt{\sum_{t=T_1}^{T}\left(\frac{y_{t+s}-fb_{t,s}}{y_{t+s}}\right)^2}} \qquad \textbf{(6.183)}$$

where $fb_{t,s}$ is the forecast obtained from a benchmark model (typically a simple model such as a naive or random walk). A *U*-statistic of one implies that the model under consideration and the benchmark model are equally (in)accurate, while a value of less than one implies that the model is superior to the benchmark, and vice versa for $U > 1$. Although the measure is clearly useful, as Makridakis and Hibon (1995) argue, it is not without problems since if $fb_{t,s}$ is the same as y_{t+s}, U will be infinite since the denominator will be zero. The value of Uwill also be influenced by outliers in a similar vein to *MSE* and has little intuitive meaning.[2]

6.10.9 Statistical versus Financial or Economic Loss Functions

Many econometric forecasting studies evaluate the models' success using statistical loss functions such as those described above. However, it is not necessarily the case that models classed as accurate because they have small mean squared forecast errors are useful in practical situations. To give one specific illustration, it has been shown (Gerlow, Irwin and Liu, 1993) that the accuracy of forecasts according to traditional statistical criteria may give little guide to the potential profitability of employing those forecasts in a market trading strategy. So models that perform poorly on statistical grounds may still yield a profit if used for trading, and vice versa.

On the other hand, models that can accurately forecast the sign of future returns, or can predict turning points in a series have been found to be more profitable (Leitch and Tanner, 1991). Two possible indicators of the ability of a model to predict direction changes irrespective of their magnitude are those suggested by Pesaran and Timmerman (1992) and by Refenes (1995). The relevant formulae to compute these measures are, respectively,

$$\% \text{ correct sign predictions} = \frac{1}{T-(T_1-1)} \sum_{t=T_1}^{T} z_{t+s} \quad \textbf{(6.184)}$$

$$\text{where } z_{t+s} = 1 \quad \text{if } (y_{t+s} f_{t,s}) > 0$$

$$z_{t+s} = 0 \quad \textit{otherwise}$$

and

$$\% \text{ correct direction change predictions} = \frac{1}{T-(T_1-1)} \sum_{t=T_1}^{T} z_{t+s} \quad \textbf{(6.185)}$$

$$\text{where } z_{t+s} = 1 \quad \text{if } (y_{t+s} - y_t)(f_{t,s} - y_t) > 0$$

$$z_{t+s} = 0 \quad \textit{otherwise}$$

Thus, in each case, the criteria give the proportion of correctly predicted signs and directional changes for some given lead time *s*, respectively.

Considering how strongly each of the three criteria outlined above (*MSE*, *MAE* and proportion of correct sign predictions) penalises large errors relative to small ones, the criteria can be ordered as follows

$$\text{Penalises large errors least} \rightarrow \text{penalises large errors most heavily}$$

$$\text{Sign prediction} \rightarrow MAE \rightarrow MSE$$

MSE penalises large errors disproportionately more heavily than small errors, *MAE* penalises large errors proportionately equally as heavily as small errors, while the sign prediction criterion does not penalise large errors any more than small errors.

6.10.10 Finance Theory and Time-Series Analysis

An example of ARIMA model identification, estimation and forecasting in the context of commodity prices is given by Chu (1978). He finds ARIMA models useful compared with structural models for short-term forecasting, but also finds that they are less accurate over longer horizons. It also observed that ARIMA models have limited capacity to forecast unusual movements in prices.

Chu (1978) argues that, although ARIMA models may appear to be completely lacking in theoretical motivation, and interpretation, this may not necessarily be the case. He cites several papers and offers an additional example to suggest that ARIMA specifications quite often arise naturally as reduced form equations (see Chapter 7) corresponding to some underlying structural relationships. In such a case, not only would ARIMA models be convenient and easy to estimate, they could also be well grounded in financial or economic theory after all.

KEY CONCEPTS

The key terms to be able to define and explain from this chapter are

- ARIMA models
- invertible MA
- autocorrelation function
- Box–Jenkins methodology
- exponential smoothing
- rolling window
- multi-step forecast
- mean absolute percentage error
- Ljung–Box test
- Wold's decomposition theorem
- partial autocorrelation function
- information criteria
- recursive window
- out-of-sample
- mean squared error

SELF-STUDY QUESTIONS

1. What are the differences between autoregressive and moving average models?
2. Why might ARMA models be considered particularly useful for

financial time series? Explain, without using any equations or mathematical notation, the difference between AR, MA and ARMA processes.

3. Consider the following three models that a researcher suggests might be a reasonable model of stock market prices

$$y_t = y_{t-1} + u_t$$
$$y_t = 0.5y_{t-1} + u_t$$
$$y_t = 0.8u_{t-1} + u_t$$

(a) What classes of models are these examples of?

(b) What would the autocorrelation function for each of these processes look like? (You do not need to calculate the acf, simply consider what shape it might have given the class of model from which it is drawn.)

(c) Which model is more likely to represent stock market prices from a theoretical perspective, and why? If any of the three models truly represented the way stock market prices move, which could potentially be used to make money by forecasting future values of the series?

(d) By making a series of successive substitutions or from your knowledge of the behaviour of these types of processes, consider the extent of persistence of shocks in the series in each case.

4. (a) Describe the steps that Box and Jenkins (1976) suggested should be involved in constructing an ARMA model.

(b) What particular aspect of this methodology has been the subject of criticism and why?

(c) Describe an alternative procedure that could be used for this aspect.

5. You obtain the following estimates for an AR(2) model of some returns data

$$y_t = 0.803y_{t-1} + 0.682y_{t-2} + u_t$$

where u_t is a white noise error process. By examining the characteristic equation, check the estimated model for stationarity.

6. A researcher is trying to determine the appropriate order of an ARMA model to describe some actual data, with 200 observations available. She has the following figures for the log of the estimated residual variance (i.e., $\ln(\hat{\sigma}^2)$) for various candidate models. She has assumed that an order greater than (3,3) should not be necessary to model the dynamics of the data. What is the ‘optimal’ model order?

ARMA(*p*,*q*) *model order*	$\ln(\hat{\sigma}^2)$
(0,0)	0.932
(1,0)	0.864
(0,1)	0.902
(1,1)	0.836
(2,1)	0.801
(1,2)	0.821
(2,2)	0.789
(3,2)	0.773
(2,3)	0.782
(3,3)	0.764

7. How could you determine whether the order you suggested for Question 6 was in fact appropriate?

8. ‘Given that the objective of any econometric modelling exercise is to find the model that most closely ‘fits’ the data, then adding more lags to an ARMA model will almost invariably lead to a better fit. Therefore a large model is best because it will fit the data more closely.’

Comment on the validity (or otherwise) of this statement.

9. (a) You obtain the following sample autocorrelations and partial autocorrelations for a sample of 100 observations from actual data:

Lag	1	2	3	4	5	6	7
acf	0.420	0.104	0.032	−0.206	−0.138	0.042	−0.
pacf	0.632	0.381	0.268	0.199	0.205	0.101	0.

Can you identify the most appropriate time-series process for this data?

(b) Use the Ljung–Box Q^* test to determine whether the first three autocorrelation coefficients taken together are jointly significantly different from zero.

10. You have estimated the following ARMA(1,1) model for some time-series data

$$y_t = 0.036 + 0.69y_{t-1} + 0.42u_{t-1} + u_t$$

Suppose that you have data for time to $t-1$, i.e., you know that $y_{t-1} = 3.4$, and $\hat{u}_{t-1} = -1.3$

(a) Obtain forecasts for the series *y* for times *t*, *t* + 1, and *t* + 2 using the estimated ARMA model.

(b) If the actual values for the series turned out to be −0.032, 0.961, 0.203 for *t*, *t* + 1, *t* + 2, calculate the (out-of-sample) mean squared error.

(c) A colleague suggests that a simple exponential smoothing model might be more useful for forecasting the series. The estimated value of the smoothing constant is 0.15, with the most recently available smoothed value, S_{t-1} being 0.0305. Obtain forecasts for the series *y* for times *t*, *t* +1, and *t* +2 using this model.

(d) Given your answers to parts (a) to (c) of the question, determine whether Box–Jenkins or exponential smoothing models give the most accurate forecasts in this application.

11. (a) Explain what stylised shapes would be expected for the autocorrelation and partial autocorrelation functions for the following stochastic processes:

- white noise
- an AR(2)
- an MA(1)
- an ARMA (2,1).

(b) Consider the following ARMA process.

$$y_t = 0.21 + 1.32y_{t-1} + 0.58u_{t-1} + u_t$$

Determine whether the MA part of the process is invertible.

(c) Produce one-, two-, three- and four-step-ahead forecasts for the process given in part (b) of the question.

(d) Outline two criteria that are available for evaluating the forecasts produced in part (c) of the question, highlighting the differing characteristics of each.

(e) What procedure might be used to estimate the parameters of an ARMA model? Explain, briefly, how such a procedure operates, and why OLS is not appropriate.

12. (a) Briefly explain any difference you perceive between the characteristics of macroeconomic and financial data. Which of these features suggest the use of different econometric tools for each class of data?

(b) Consider the following autocorrelation and partial autocorrelation coefficients estimated using 500 observations for a weakly stationary series, y_t :

Lag	acf	pacf
1	0.307	0.307
2	−0.013	0.264
3	0.086	0.147
4	0.031	0.086
5	−0.197	0.049

Using a simple 'rule of thumb', determine which, if any, of the acf and pacf coefficients are significant at the 5% level. Use both the Box–Pierce and Ljung–Box statistics to test the joint null hypothesis that the first five autocorrelation coefficients are jointly zero.

(c) What process would you tentatively suggest could represent the most appropriate model for the series in part (b)? Explain your answer.

(d) Two researchers are asked to estimate an ARMA model for a daily USD/GBP exchange rate return series, denoted x_t. Researcher *A* uses Schwarz's criterion for determining the appropriate model order and arrives at an ARMA(0,1). Researcher *B* uses Akaike's information criterion which deems an ARMA(2,0) to be optimal. The estimated models are

$$A : \hat{x}_t = 0.38 + 0.10u_{t-1}$$

$$B : \hat{x}_t = 0.63 + 0.17x_{t-1} - 0.09x_{t-2}$$

where u_t is an error term.

You are given the following data for time until day z (i.e., $t = z$)

$$x_z = 0.31,\ x_{z-1} = 0.02,\ x_{z-2} = -0.16$$

$$u_z = -0.02,\ u_{z-1} = 0.13,\ u_{z-2} = 0.19$$

Produce forecasts for the next four days (i.e., for times $z + 1$, $z + 2$, $z + 3$, $z + 4$) from both models.

(e) Outline two methods proposed by Box and Jenkins (1976) for determining the adequacy of the models proposed in part (d).

(f) Suppose that the actual values of the series x on days $z + 1$, $z + 2$, $z + 3$, $z + 4$ turned out to be 0.62, 0.19, −0.32, 0.72, respectively. Determine which researcher's model produced the most accurate forecasts.

1 Note that the τ_s will not follow an exact geometric sequence, but rather the absolute value of the τ_s is bounded by a geometric series. This means that the autocorrelation function does not have to be monotonically decreasing and may change sign.

2 Note that the Theil's *U*-formula reported by EViews and some other software packages is slightly different.

7

Multivariate Models

LEARNING OUTCOMES

In this chapter, you will learn how to

- Compare and contrast single equation and systems-based approaches to building models
- Discuss the cause, consequence and solution to simultaneous equations bias
- Derive the reduced form equations from a structural model
- Describe several methods for estimating simultaneous equations models
- Explain the relative advantages and disadvantages of VAR modelling
- Determine whether an equation from a system is identified
- Estimate optimal lag lengths, impulse responses and variance decompositions
- Conduct Granger causality tests

7.1 Motivations

All of the structural models that have been considered thus far have been single equations models of the form

$$y = X\beta + u \qquad \textbf{(7.1)}$$

One of the assumptions of the classical linear regression model (CLRM) is that the explanatory variables are *non-stochastic*, or fixed in repeated samples. There are various ways of stating this condition, some of which are slightly more or less strict, but all of which have the same broad implication. It could also be stated that all of the variables contained in the X matrix are assumed to be *exogenous* – that is, their values are determined outside that equation. This is a rather simplistic working definition of exogeneity, although several alternatives are possible; this issue will be revisited later in the chapter. Another way to state this is that the model is 'conditioned on' the variables in X.

As stated in Chapter 3, the X matrix is assumed not to have a probability distribution. Note also that causality in this model runs from X to y, and not vice versa, i.e., that changes in the values of the explanatory variables cause changes in the values of y, but that changes in the value of y will not impact upon the explanatory variables. On the other hand, y is an *endogenous* variable – that is, its value is determined by equation (7.1).

The purpose of the first part of this chapter is to investigate one of the important circumstances under which the assumption presented above will be violated. The impact on the OLS estimator of such a violation will then be considered.

To illustrate a situation in which such a phenomenon may arise, consider the following two equations that describe a possible model for the total aggregate (country-wide) supply of new houses (or any other physical asset).

$$Q_{dt} = \alpha + \beta P_t + \gamma S_t + u_t \tag{7.2}$$

$$Q_{st} = \lambda + \mu P_t + \kappa T_t + v_t \tag{7.3}$$

$$Q_{dt} = Q_{st} \tag{7.4}$$

where

Q_{dt} = quantity of new houses demanded at time t

Q_{st} = quantity of new houses supplied (built) at time t

P_t = (average) price of new houses prevailing at time t

S_t = price of a substitute (e.g., older houses)

T_t = some variable embodying the state of housebuilding technology, u_t and v_t are error terms.

Equation (7.2) is an equation for modelling the demand for new houses, and equation (7.3) models the supply of new houses. Equation (7.4) is an equilibrium condition for there to be no excess demand (people willing and able to buy new houses but cannot) and no excess supply (constructed houses that remain empty owing to lack of demand).

Assuming that the market always clears, that is, that the market is always in equilibrium, and dropping the time subscripts for simplicity, equations (7.2)–(7.4) can be written

$$Q = \alpha + \beta P + \gamma S + u \quad \textbf{(7.5)}$$

$$Q = \lambda + \mu P + \kappa T + v \quad \textbf{(7.6)}$$

Equations (7.5) and (7.6) together comprise a simultaneous structural form of the model, or a set of structural equations. These are the equations incorporating the variables that economic or financial theory suggests should be related to one another in a relationship of this form. The point is that price and quantity are determined simultaneously (price affects quantity and quantity affects price). Thus, in order to sell more houses, everything else equal, the builder will have to lower the price. Equally, in order to obtain a higher price for each house, the builder should construct and expect to sell fewer houses. *P* and *Q* are endogenous variables, while *S* and *T* are exogenous.

A set of reduced form equations corresponding to equations (7.5) and (7.6) can be obtained by solving equations (7.5) and (7.6) for *P* and for *Q* (separately). There will be a reduced form equation for each endogenous variable in the system.

Solving for *Q*

$$\alpha + \beta P + \gamma S + u = \lambda + \mu P + \kappa T + v \quad \textbf{(7.7)}$$

Solving for *P*

$$\frac{Q}{\beta} - \frac{\alpha}{\beta} - \frac{\gamma S}{\beta} - \frac{u}{\beta} = \frac{Q}{\mu} - \frac{\lambda}{\mu} - \frac{\kappa T}{\mu} - \frac{v}{\mu} \quad \textbf{(7.8)}$$

Rearranging equation (7.7)

$$\beta P - \mu P = \lambda - \alpha + \kappa T - \gamma S + v - u \quad \textbf{(7.9)}$$

$$(\beta - \mu)P = (\lambda - \alpha) + \kappa T - \gamma S + (v - u) \quad \textbf{(7.10)}$$

$$P = \frac{\lambda - \alpha}{\beta - \mu} + \frac{\kappa}{\beta - \mu}T - \frac{\gamma}{\beta - \mu}S + \frac{v - u}{\beta - \mu} \quad \textbf{(7.11)}$$

Multiplying equation (7.8) through by $\beta\mu$ and rearranging

$$\mu Q - \mu\alpha - \mu\gamma S - \mu u = \beta Q - \beta\lambda - \beta\kappa T - \beta v \quad \textbf{(7.12)}$$

$$\mu Q - \beta Q = \mu\alpha - \beta\lambda - \beta\kappa T + \mu\gamma S + \mu u - \beta v \quad \textbf{(7.13)}$$

$$(\mu - \beta)Q = (\mu\alpha - \beta\lambda) - \beta\kappa T + \mu\gamma S + (\mu u - \beta v) \quad \textbf{(7.14)}$$

$$Q = \frac{\mu\alpha - \beta\lambda}{\mu - \beta} - \frac{\beta\kappa}{\mu - \beta}T + \frac{\mu\gamma}{\mu - \beta}S + \frac{\mu u - \beta v}{\mu - \beta} \quad \textbf{(7.15)}$$

Equations (7.11) and (7.15) are the reduced form equations for P and Q. They are the equations that result from solving the simultaneous structural equations given by equations (7.5) and (7.6). Notice that these reduced form equations have only exogenous variables on the RHS.

7.2 Simultaneous Equations Bias

It would not be possible to estimate equations (7.5) and (7.6) validly using OLS, as they are clearly related to one another since they both contain P and Q, and OLS would require them to be estimated separately. But what would have happened if a researcher had estimated them separately using OLS? Both equations depend on P. One of the CLRM assumptions was that X and u are independent (where X is a matrix containing all the variables on the RHS of the equation), and given also the assumption that $E(u) = 0$, then $E(X'u) = 0$, i.e., the errors are uncorrelated with the explanatory variables. But it is clear from equation (7.11) that P is related to the errors in equations (7.5) and (7.6) – i.e., it is *stochastic*. So this assumption has been violated.

What would be the consequences for the OLS estimator, $\hat{\beta}$ if the simultaneity were ignored? Recall that

$$\hat{\beta} = (X'X)^{-1}X'y \quad \textbf{(7.16)}$$

and that

$$y = X\beta + u \tag{7.17}$$

Replacing y in (7.16) with the RHS of equation (7.17)

$$\hat{\beta} = (X'X)^{-1}X'(X\beta + u) \tag{7.18}$$

so that

$$\hat{\beta} = (X'X)^{-1}X'X\beta + (X'X)^{-1}X'u \tag{7.19}$$

$$\hat{\beta} = \beta + (X'X)^{-1}X'u \tag{7.20}$$

Taking expectations,

$$E(\hat{\beta}) = E(\beta) + E((X'X)^{-1}X'u) \tag{7.21}$$

$$E(\hat{\beta}) = \beta + E((X'X)^{-1}X'u) \tag{7.22}$$

If the Xs are non-stochastic (i.e., if the assumption had not been violated), $E[(X'X)^{-1}X'u] = (X'X)^{-1}X'E[u] = 0$, which would be the case in a single equation system, so that $E(\hat{\beta}) = \beta$ in (7.22). The implication is that the OLS estimator, $\hat{\beta}$, would be unbiased.

But, if the equation is part of a system, then $E[(X'X)^{-1}X'u] \neq 0$, in general, so that the last term in equation (7.22) will not drop out, and so it can be concluded that application of OLS to structural equations which are part of a simultaneous system will lead to biased coefficient estimates. This is known as *simultaneity bias* or *simultaneous equations bias*.

Is the OLS estimator still consistent, even though it is biased? No, in fact, the estimator is inconsistent as well, so that the coefficient estimates would still be biased even if an infinite amount of data were available, although proving this would require a level of algebra beyond the scope of this book.

7.3 So how can Simultaneous Equations Models be Validly Estimated?

Taking equations (7.11) and (7.15), i.e., the reduced form equations, they can be rewritten as

$$P = \pi_{10} + \pi_{11}T + \pi_{12}S + \varepsilon_1$$

$$(7.23)$$

$$Q = \pi_{20} + \pi_{21}T + \pi_{22}S + \varepsilon_2 \tag{7.24}$$

where the π coefficients in the reduced form are simply combinations of the original coefficients, so that

$$\pi_{10} = \frac{\lambda - \alpha}{\beta - \mu}, \quad \pi_{11} = \frac{\kappa}{\beta - \mu}, \quad \pi_{12} = \frac{-\gamma}{\beta - \mu}, \quad \varepsilon_1 = \frac{v - u}{\beta - \mu},$$

$$\pi_{20} = \frac{\mu\alpha - \beta\lambda}{\mu - \beta}, \quad \pi_{21} = \frac{-\beta\kappa}{\mu - \beta}, \quad \pi_{22} = \frac{\mu\gamma}{\mu - \beta}, \quad \varepsilon_2 = \frac{\mu u - \beta v}{\mu - \beta}$$

Equations (7.23) and (7.24) can be estimated using OLS since all the RHS variables are exogenous, so the usual requirements for consistency and unbiasedness of the OLS estimator will hold (provided that there are no other misspecifications). Estimates of the π_{ij} coefficients would thus be obtained. But, the values of the π coefficients are probably not of much interest; what was wanted were the original parameters in the structural equations – α, β, γ, λ, μ, κ. The latter are the parameters whose values determine how the variables are related to one another according to financial or economic theory.

7.4 Can the Original Coefficients be Retrieved from the *π s*?

The short answer to this question is 'sometimes', depending upon whether the equations are identified. *Identification* is the issue of whether there is enough information in the reduced form equations to enable the structural form coefficients to be calculated. Consider the following demand and supply equations

$$Q = \alpha + \beta P \quad \text{Supply equation} \tag{7.25}$$

$$Q = \lambda + \mu P \quad \text{Demand equation} \tag{7.26}$$

It is impossible to tell which equation is which, so that if one simply observed some quantities of a good sold and the price at which they were sold, it would not be possible to obtain the estimates of α, β, λ and μ. This arises since there is insufficient information from the equations to estimate four parameters. Only two parameters could be estimated here, although each would be some combination of demand and supply parameters, and so neither would be of any use. In this case, it would be stated that both equations are *unidentified* (or not identified or underidentified). Notice that this problem would not have arisen with equations (7.5) and (7.6) since they have different exogenous variables.

7.4.1 What Determines Whether an Equation is Identified or Not?

Any one of three possible situations could arise, as shown in Box 7.1. How can it be determined whether an equation is identified or not? Broadly, the answer to this question depends upon how many and which variables are present in each structural equation. There are two conditions that could be examined to determine whether a given equation from a system is identified – the *order condition* and the *rank condition*

- The *order condition* – is a necessary but not sufficient condition for an equation to be identified. That is, even if the order condition is satisfied, the equation might not be identified.
- The *rank condition* – is a necessary and sufficient condition for identification. The structural equations are specified in a matrix form and the rank of a coefficient matrix of all of the variables excluded from a particular equation is examined. An examination of the rank condition requires some technical algebra beyond the scope of this text.

BOX 7.1 Determining whether an equation is identified

(1) An equation is *unidentified*, such as equations (7.25) or (7.26). In the case of an unidentified equation, structural coefficients cannot be obtained from the reduced form estimates by any means.

(2) An equation is *exactly identified* (*just identified*), such as equations (7.5) or (7.6). In the case of a just identified equation, unique structural form coefficient estimates can be obtained by substitution from the reduced form equations.

(3) If an equation is *overidentified*, more than one set of structural coefficients can be obtained from the reduced form. An example of this will be presented later in this chapter.

Even though the order condition is not sufficient to ensure identification of an equation from a system, the rank condition will not be considered further here. For relatively simple systems of equations, the two rules would lead to the same conclusions. Also, in fact, most systems of equations in economics and finance are overidentified, so that underidentification is not a big issue in practice.

7.4.2 Statement of the Order Condition

There are a number of different ways of stating the order condition; that employed here is an intuitive one (taken from Ramanathan, 1995, p. 666, and slightly modified):

Let G denote the number of structural equations. An equation is just identified if the number of variables excluded from an equation is G–1, where 'excluded' means the number of all endogenous and exogenous variables that are not present in this particular equation. If more than G– 1 are absent, it is overidentified. If less than G–1 are absent, it is not identified.

One obvious implication of this rule is that equations in a system can have differing degrees of identification, as illustrated by Example 7.1.

EXAMPLE 7.1

In the following system of equations, the Ys are endogenous, while the Xs are exogenous (with time subscripts suppressed). Determine whether each equation is overidentified, underidentified, or just identified.

$$Y_1 = \alpha_0 + \alpha_1 Y_2 + \alpha_3 Y_3 + \alpha_4 X_1 + \alpha_5 X_2 + u_1 \quad \textbf{(7.27)}$$

$$Y_2 = \beta_0 + \beta_1 Y_3 + \beta_2 X_1 + u_2 \quad \textbf{(7.28)}$$

$$Y_3 = \gamma_0 + \gamma_1 Y_2 + \gamma_2 X_2 + u_3 \quad \textbf{(7.29)}$$

In this case, there are G = 3 equations and 3 endogenous variables. Thus, if the number of excluded variables is exactly 2, the equation is just identified. If the number of excluded variables is more than 2, the equation is overidentified. If the number of excluded variables is less than 2, the equation is not identified.

The variables that appear in one or more of the three equations are Y_1, Y_2, Y_3, X_1, X_2. Applying the order condition to equations (7.27)–(7.29):

- Equation (7.27): contains all variables, with none excluded, so that it is not identified
- Equation (7.28): has variables Y_1 and X_2 excluded, and so is just identified
- Equation (7.29): has variables Y_1 and X_1 excluded, and so is also just identified

7.5 Simultaneous Equations in Finance

There are of course numerous situations in finance where a simultaneous equations framework is more relevant than a single equation model. Two illustrations from the market microstructure literature are presented later in this chapter, while another, drawn from the banking literature, will be discussed now.

There has recently been much debate internationally, but especially in the UK, concerning the effectiveness of competitive forces in the banking industry. Governments and regulators express concern at the increasing concentration in the industry, as evidenced by successive waves of merger activity, and at the enormous profits that many banks made in the late 1990s and early twenty-first century. They argue that such profits result from a lack of effective competition. However, many (most notably, of course, the banks themselves!) suggest that such profits are not the result of excessive concentration or anti-competitive practices, but rather partly arise owing to recent world prosperity at that phase of the business cycle (the ‘profits won’t last’ argument) and partly owing to massive cost-cutting by the banks, given recent technological improvements. These debates have fuelled a resurgent interest in models of banking profitability and banking competition. One such model is employed by Shaffer and DiSalvo (1994) in the context of two banks operating in south central Pennsylvania. The model is given by

$$\ln q_{it} = a_0 + a_1 \ln P_{it} + a_2 \ln P_{jt} + a_3 \ln Y_t + a_4 \ln Z_t + a_5 t + u_{i1t} \quad \textbf{(7.30)}$$

$$\ln TR_{it} = b_0 + b_1 \ln q_{it} + \sum_{k=1}^{3} b_{k+1} \ln w_{ikt} + u_{i2t} \tag{7.31}$$

where i = 1, 2 are the two banks, q is bank output, P_t is the price of the output at time t, Y_t is a measure of aggregate income at time t, Z_t is the price of a substitute for bank activity at time t, the variable t represents a time trend, TR_{it} is the total revenue of bank i at time t, w_{ikt} are the prices of input k (k = 1, 2, 3 for labour, bank deposits and physical capital) for bank i at time t and the u are unobservable error terms. The coefficient estimates are not presented here, but suffice to say that a simultaneous framework, with the resulting model estimated separately using annual time-series data for each bank, is necessary. Output is a function of price on the RHS of equation (7.30), while in equation (7.31), total revenue, which is a function of output on the RHS, is obviously related to price. Therefore, OLS is again an inappropriate estimation technique. Both of the equations in this system are overidentified, since there are only two equations, and the income, the substitute for banking activity and the trend terms are missing from equation (7.31), whereas the three input prices are missing from equation (7.30).

7.6 A Definition of Exogeneity

Leamer (1985) defines a variable x as exogenous if the conditional distribution of y given x does not change with modifications of the process generating x. Although several slightly different definitions exist, it is possible to classify two forms of exogeneity – predeterminedness and strict exogeneity

- A *predetermined* variable is one that is independent of the contemporaneous and future errors in that equation
- A *strictly exogenous* variable is one that is independent of all contemporaneous, future and past errors in that equation.

7.6.1 Tests for Exogeneity

How can a researcher tell whether variables really need to be treated as endogenous or not? In other words, financial theory might suggest that there should be a twoway relationship between two or more variables, but how can it be tested whether a simultaneous equations model is necessary in practice?

EXAMPLE 7.2

Consider again equations (7.27)–(7.29). Equation (7.27) contains Y_2 and Y_3 – but are separate equations required for them, or could the variables Y_2 and Y_3 be treated as exogenous variables (in which case, they would be called X_3 and X_4!)? This can be formally investigated using a Hausman test, which is calculated as shown in Box 7.2.

BOX 7.2 Conducting a Hausman test for exogeneity

(1) Obtain the reduced form equations corresponding to equations (7.27)–(7.29).

The reduced form equations are obtained as follows. Note that there is not a unique path to finding these solutions and several different routes would eventually arrive at the same reduced form equations.

Substituting in equation (7.28) for Y_3 from equation (7.29):

$$Y_2 = \beta_0 + \beta_1(\gamma_0 + \gamma_1 Y_2 + \gamma_2 X_2 + u_3) + \beta_2 X_1 + u_2 \quad \textbf{(7.32)}$$

$$Y_2 = \beta_0 + \beta_1\gamma_0 + \beta_1\gamma_1 Y_2 + \beta_1\gamma_2 X_2 + \beta_1 u_3 + \beta_2 X_1 + u_2 \quad \textbf{(7.33)}$$

$$Y_2(1 - \beta_1\gamma_1) = (\beta_0 + \beta_1\gamma_0) + (\beta_2 X_1 + \beta_1\gamma_2 X_2) + (u_2 + \beta_1 u_3) \quad \textbf{(7.34)}$$

$$Y_2 = \frac{(\beta_0 + \beta_1\gamma_0)}{(1 - \beta_1\gamma_1)} + \frac{(\beta_2 X_1 + \beta_1\gamma_2 X_2)}{(1 - \beta_1\gamma_1)} + \frac{(u_2 + \beta_1 u_3)}{(1 - \beta_1\gamma_1)} \quad \textbf{(7.35)}$$

Equation (7.35) is the reduced form equation for Y_2, since there are no endogenous variables on the RHS. Substituting in equation (7.27) for Y_3 from equation (7.29)

$$Y_1 = \alpha_0 + \alpha_1 Y_2 + \alpha_3(\gamma_0 + \gamma_1 Y_2 + \gamma_2 X_2 + u_3) + \alpha_4 X_1 + \alpha_5 X_2 + u_1 \quad \textbf{(7.36)}$$

$$Y_1 = \alpha_0 + \alpha_1 Y_2 + \alpha_3\gamma_0 + \alpha_3\gamma_1 Y_2 + \alpha_3\gamma_2 X_2 + \alpha_3 u_3 + \alpha_4 X_1 + \alpha_5 X_2 + u_1 \quad \textbf{(7.37)}$$

$$Y_1 = (\alpha_0 + \alpha_3\gamma_0) + (\alpha_1 + \alpha_3\gamma_1)Y_2 + \alpha_4 X_1 + (\alpha_3\gamma_2 + \alpha_5)X_2 + (u_1 + \alpha_3 u_3) \quad \textbf{(7.38)}$$

Substituting in equation (7.38) for Y_2 from equation (7.35)

$$Y_1 = (\alpha_0 + \alpha_3\gamma_0) + (\alpha_1 + \alpha_3\gamma_1)\left(\frac{(\beta_0 + \beta_1\gamma_0)}{(1 - \beta_1\gamma_1)} + \frac{\beta_2 X_1 + \beta_1\gamma_2 X_2}{(1 - \beta_1\gamma_1)} + \frac{(u_2 + \beta_1 u_3)}{(1 - \beta_1\gamma_1)}\right) + \alpha_4 X_1 + (\alpha_3\gamma_2 + \alpha_5)X_2 + (u_1 + \alpha_3 u_3) \quad \textbf{(7.39)}$$

$$Y_1 = \left(\alpha_0 + \alpha_3\gamma_0 + (\alpha_1 + \alpha_3\gamma_1)\frac{(\beta_0 + \beta_1\gamma_0)}{(1 - \beta_1\gamma_1)}\right) + \frac{(\alpha_1 + \alpha_3\gamma_1)(\beta_2 X_1 + \beta_1\gamma_2 X_2)}{(1 - \beta_1\gamma_1)} + \frac{(\alpha_1 + \alpha_3\gamma_1)(u_2 + \beta_1 u_3)}{(1 - \beta_1\gamma_1)} + \alpha_4 X_1 + (\alpha_3\gamma_2 + \alpha_5)X_2 + (u_1 + \alpha_3 u_3) \quad \textbf{(7.40)}$$

$$Y_1 = \left(\alpha_0 + \alpha_3\gamma_0 + (\alpha_1 + \alpha_3\gamma_1)\frac{(\beta_0 + \beta_1\gamma_0)}{(1 - \beta_1\gamma_1)}\right) + \left(\frac{(\alpha_1 + \alpha_3\gamma_1)\beta_2}{(1 - \beta_1\gamma_1)} + \alpha_4\right)X_1 + \left(\frac{\beta_1\gamma_2(\alpha_1 + \alpha_3\gamma_1)}{(1 - \beta_1\gamma_1)} + (\alpha_3\gamma_2 + \alpha_5)\right)X_2 + \left(\frac{(\alpha_1 + \alpha_3\gamma_1)(u_2 + \beta_1 u_3)}{(1 - \beta_1\gamma_1)} + (u_1 + \alpha_3 u_3)\right) \quad \textbf{(7.41)}$$

Equation (7.41) is the reduced form equation for Y_1. Finally, to obtain the reduced form equation for Y_3, substitute in equation (7.29) for Y_2 from equation (7.35) and simplifying

$$Y_3 = \left(\gamma_0 + \frac{\gamma_1(\beta_0 + \beta_1\gamma_0)}{(1 - \beta_1\gamma_1)}\right) + \frac{(\gamma_1\beta_2 X_1 + \gamma_2 X_2)}{(1 - \beta_1\gamma_1)} + \left(\frac{\gamma_1(u_2 + \beta_1 u_3)}{(1 - \beta_1\gamma_1)} + u_3\right) \quad \textbf{(7.42)}$$

So, the reduced form equations corresponding to equations (7.27)–(7.29) are, respectively, given by equations (7.41), (7.35) and (7.42). These three equations can also be expressed using π_{ij} for the coefficients, as discussed above

$$Y_1 = \pi_{10} + \pi_{11}X_1 + \pi_{12}X_2 + v_1 \quad \textbf{(7.43)}$$

$$Y_2 = \pi_{20} + \pi_{21}X_1 + \pi_{22}X_2 + v_2 \quad \textbf{(7.44)}$$

$$Y_3 = \pi_{30} + \pi_{31}X_1 + \pi_{32}X_2 + v_3 \quad \textbf{(7.45)}$$

Estimate the reduced form equations (7.43)–(7.45) using OLS, and obtain the fitted values, $\hat{Y}_1^1$, $\hat{Y}_2^1$, $\hat{Y}_3^1$, where the superfluous superscript[1] denotes the fitted values from the reduced form estimation.

(2) Run the regression corresponding to equation (7.27) – i.e., the structural form equation, at this stage ignoring any possible simultaneity.

(3) Run the regression equation (7.27) again, but now also including the fitted values from the reduced form equations, $\hat{Y}_2^1$, $\hat{Y}_3^1$, as additional regressors

$$Y_1 = \alpha_0 + \alpha_1 Y_2 + \alpha_3 Y_3 + \alpha_4 X_1 + \alpha_5 X_2 + \lambda_2 \hat{Y}_2^1 + \lambda_3 \hat{Y}_3^1 + \varepsilon_1 \quad \textbf{(7.46)}$$

(4) Use an F-test to test the joint restriction that $\lambda_2 = 0$, and $\lambda_3 = 0$. If the null hypothesis is rejected, Y_2 and Y_3 should be treated as endogenous. If λ_2 and λ_3 are significantly different from zero, there is extra important information for modelling Y_1 from the reduced form equations. On the other hand, if the null is not rejected, Y_2 and Y_3 can be treated as exogenous for Y_1, and there is no useful additional information available for Y_1 from modelling Y_2 and Y_3 as endogenous variables.

Steps (2)–(4) would then be repeated for equations (7.28) and (7.29).

7.7 Triangular Systems

Consider the following system of equations, with time subscripts omitted for simplicity

$$Y_1 = \beta_{10} + \gamma_{11}X_1 + \gamma_{12}X_2 + u_1 \quad \textbf{(7.47)}$$

$$Y_2 = \beta_{20} + \beta_{21}Y_1 + \gamma_{21}X_1 + \gamma_{22}X_2 + u_2 \quad \textbf{(7.48)}$$

$$Y_3 = \beta_{30} + \beta_{31} Y_1 + \beta_{32} Y_2 + \gamma_{31} X_1 + \gamma_{32} X_2 + u_3 \quad \textbf{(7.49)}$$

Assume that the error terms from each of the three equations are not correlated with each other. Can the equations be estimated individually using OLS? At first blush, an appropriate answer to this question might appear to be, 'No, because this is a simultaneous equations system'. But consider the following:

- Equation (7.47): contains no endogenous variables, so X_1 and X_2 are not correlated with u_1. So OLS can be used on equation (7.47).
- Equation (7.48): contains endogenous Y_1 together with exogenous X_1 and X_2. OLS can be used on equation (7.48) if all the RHS variables in equation (7.48) are uncorrelated with that equation's error term. In fact, Y_1 is not correlated with u_2 because there is no Y_2 term in equation (7.47). So OLS can be used on equation (7.48).
- Equation (7.49): contains both Y_1 and Y_2; these are required to be uncorrelated with u_3. By similar arguments to the above, equations (7.47) and (7.48) do not contain Y_3. So OLS can be used on equation (7.49).

This is known as a *recursive or triangular system*, which is really a special case – a set of equations that looks like a simultaneous equations system, but isn't. In fact, there is not a simultaneity problem here, since the dependence is not bi-directional, for each equation it all goes one way.

7.8 Estimation Procedures for Simultaneous Equations Systems

Each equation that is part of a recursive system can be estimated separately using OLS. But in practice, not many systems of equations will be recursive, so a direct way to address the estimation of equations that are from a true simultaneous system must be sought. In fact, there are potentially many methods that can be used, three of which – indirect least squares, two-stage least squares and instrumental variables – will be detailed here. Each of these will be discussed below.

7.8.1 Indirect Least Squares (ILS)

Although it is not possible to use OLS directly on the structural equations, it is possible to validly apply OLS to the reduced form equations. If the system is just identified, ILS involves estimating the reduced form equations using OLS, and then using them to substitute back to obtain the structural parameters. ILS is intuitive to understand in principle; however, it is not widely applied because

(1) *Solving back to get the structural parameters can be tedious*. For a large system, the equations may be set up in a matrix form, and to solve them may therefore require the inversion of a large matrix.

(2) *Most simultaneous equations systems are overidentified*, and ILS can be used to obtain coefficients only for just identified equations. For overidentified systems, ILS would not yield unique structural form estimates.

ILS estimators are consistent and asymptotically efficient, but in general they are biased, so that in finite samples ILS will deliver biased structural form estimates. In a nutshell, the bias arises from the fact that the structural form coefficients under ILS estimation are transformations of the reduced form coefficients. When expectations are taken to test for unbiasedness, it is in general not the case that the expected value of a (non-linear) combination of reduced form coefficients will be equal to the combination of their expected values (see Gujarati, 2003, for a proof).

7.8.2 Estimation of Just Identified and Overidentified Systems using 2SLS

This technique is applicable for the estimation of overidentified systems, where ILS cannot be used. In fact, it can also be employed for estimating the coefficients of just identified systems, in which case the method would yield asymptotically equivalent estimates to those obtained from ILS.

Two-stage least squares (2SLS or TSLS) is done in two stages

- *Stage 1* Obtain and estimate the reduced form equations using OLS. Save the fitted values for the dependent variables.
- *Stage 2* Estimate the structural equations using OLS, but replace any RHS endogenous variables with their stage 1 fitted values.

EXAMPLE 7.3

Suppose that equations (7.27)–(7.29) are required. 2SLS would involve the following two steps:

- *Stage 1* Estimate the reduced form equations (7.43)–(7.45) individually by OLS and obtain the fitted values, and denote them $\hat{Y}_1^1, \hat{Y}_2^1, \hat{Y}_3^1$, where the superfluous superscript[1] indicates that these are the fitted values from the first stage.
- *Stage 2* Replace the RHS endogenous variables with their stage 1 estimated values

$$Y_1 = \alpha_0 + \alpha_1 \hat{Y}_2^1 + \alpha_3 \hat{Y}_3^1 + \alpha_4 X_1 + \alpha_5 X_2 + u_1 \quad \textbf{(7.50)}$$

$$Y_2 = \beta_0 + \beta_1 \hat{Y}_3^1 + \beta_2 X_1 + u_2 \quad \textbf{(7.51)}$$

$$Y_3 = \gamma_0 + \gamma_1 \hat{Y}_2^1 + u_3 \quad \textbf{(7.52)}$$

where $\hat{Y}_2^1$ and $\hat{Y}_3^1$ are the fitted values from the reduced form estimation. Now $\hat{Y}_2^1$ and $\hat{Y}_3^1$ will not be correlated with u_1, $\hat{Y}_3^1$ will not be correlated with u_2, and $\hat{Y}_2^1$ will not be correlated with u_3. The simultaneity problem has therefore been removed. It is worth noting that the 2SLS estimator is consistent, but not unbiased.

In a simultaneous equations framework, it is still of concern whether the usual assumptions of the CLRM are valid or not, although some of the test statistics require modifications to be applicable in the systems context. Most econometrics packages will automatically make any required changes. To illustrate one potential consequence of the violation of the CLRM assumptions, if the disturbances in the structural equations are autocorrelated, the 2SLS estimator is not even consistent.

The standard error estimates also need to be modified compared with their OLS counterparts (again, econometrics software will usually do this automatically), but once this has been done, the usual *t*-tests can be used to test hypotheses about the structural form coefficients. This modification arises as a result of the use of the reduced form fitted values on the RHS rather than actual variables, which implies that a modification to the error variance is required.

7.8.3 Instrumental Variables

Broadly, the method of instrumental variables (IV) is another technique for parameter estimation that can be validly used in the context of a simultaneous equations system. Recall that the reason that OLS cannot be used directly on the structural equations is that the endogenous variables are correlated with the errors.

One solution to this would be not to use Y_2 or Y_3, but rather to use some other variables instead. These other variables should be (highly) correlated with Y_2 and Y_3, but not correlated with the errors – such variables would be known as *instruments*. Suppose that suitable instruments for Y_2 and Y_3, were found and denoted z_2 and z_3, respectively. The instruments are not used in the structural equations directly, but rather, regressions of the following form are run

$$Y_2 = \lambda_1 + \lambda_2 z_2 + \varepsilon_1 \tag{7.53}$$

$$Y_3 = \lambda_3 + \lambda_4 z_3 + \varepsilon_2 \tag{7.54}$$

Obtain the fitted values from equations (7.53) and (7.54), $\hat{Y}_2^1$ and $\hat{Y}_3^1$, and replace Y_2 and Y_3 with these in the structural equation. It is typical to use more than one instrument per endogenous variable. If the instruments are the variables in the reduced form equations, then IV is equivalent to 2SLS, so that the latter can be viewed as a special case of the former.

7.8.4 What Happens if IV or 2SLS are Used Unnecessarily?

In other words, suppose that one attempted to estimate a simultaneous system when the variables specified as endogenous were in fact independent of one another. The consequences are similar to those of including irrelevant variables in a singleequation OLS model. That is, the coefficient estimates will still be consistent, but will be inefficient compared to those that just used OLS directly.

7.8.5 Other Estimation Techniques

There are, of course, many other estimation techniques available for systems of equations, including three-stage least squares (3SLS), full information maximum likelihood (FIML) and limited information maximum likelihood (LIML). Three-stage least squares provides a third step in the estimation process that allows for non-zero covariances between the error terms in the structural equations. It is asymptotically more efficient than 2SLS since the latter ignores any information that may be available concerning the error covariances (and also any additional information that may be contained in the endogenous variables of other equations). Full information maximum likelihood involves estimating all of the equations in the system simultaneously using maximum likelihood (see Chapter 8 for a discussion of the principles of maximum likelihood estimation). Thus under FIML, all of the parameters in all equations are treated jointly, and an appropriate likelihood function is formed and maximised. Finally, limited information maximum likelihood involves estimating each equation separately by maximum likelihood. LIML and 2SLS are asymptotically equivalent. For further technical details on each of these procedures, see Greene (2002, Chapter 15).

The following section presents an application of the simultaneous equations approach in finance to the joint modelling of bid–ask spreads and trading activity in the S&P100 index options market. Two related applications of this technique that are also worth examining are by Wang, Yau and Baptiste (1997) and by Wang and Yau (2000). The former employs a bivariate system to model trading volume and bid–ask spreads and they show using a Hausman test that the two are indeed simultaneously related and so must both be treated as endogenous variables and are modelled using 2SLS. The latter paper employs a trivariate system to model trading volume, spreads and intra-day volatility.

7.9 An Application of a Simultaneous Equations Approach to Modelling Bid–Ask Spreads and Trading Activity

7.9.1 Introduction

One of the most rapidly growing areas of empirical research in finance is the study of market microstructure. This research is involved with issues such as price formation in financial markets, how the structure of the market may affect the way it operates, determinants of the bid–ask spread, and so on. One application of simultaneous equations methods in the market microstructure literature is a study by George and Longstaff (1993). Among other issues, this paper considers the questions

- Is trading activity related to the size of the bid–ask spread?
- How do spreads vary across options, and how is this related to the volume of contracts traded? ‘Across options’ in this case means for different maturities and strike prices for an option on a given

underlying asset.

This chapter will now examine the George and Longstaff models, results and conclusions.

7.9.2 The Data

The data employed by George and Longstaff (1993) comprise options prices on the S&P100 index, observed on all trading days during 1989. The S&P100 index has been traded on the Chicago Board Options Exchange (CBOE) since 1983 on a continuous open-outcry auction basis. The option price as used in the paper is defined as the average of the bid and the ask. The average bid and ask prices are calculated for each option during the time 2.00p.m.–2.15p.m. (US Central Standard Time) to avoid time-of-day effects, such as differences in behaviour at the open and the close of the market. The following are then dropped from the sample for that day to avoid any effects resulting from stale prices

- Any options that do not have bid and ask quotes reported during the fifteen minutes
- Any options with fewer than ten trades during the day.

This procedure results in a total of 2,456 observations. A 'pooled' regression is conducted since the data have both time-series and cross-sectional dimensions. That is, the data are measured every trading day and across options with different strikes and maturities, and the data are stacked in a single column for analysis.

7.9.3 How Might the Option Price/Trading Volume and the Bid–Ask Spread be Related?

George and Longstaff argue that the bid–ask spread will be determined by the interaction of market forces. Since there are many market makers trading the S&P100 contract on the CBOE, the bid–ask spread will be set to just cover marginal costs. There are three components of the costs associated with being a market maker. These are administrative costs, inventory holding costs and 'risk costs'. George and Longstaff (1993) consider three possibilities for how the bid–ask spread might be determined

- *Market makers equalise spreads across options* This is likely to be

the case if order-processing (administrative) costs make up the majority of costs associated with being a market maker. This could be the case since the CBOE charges market makers the same fee for each option traded. In fact, for every contract (100 options) traded, a CBOE fee of 9 cents and an Options Clearing Corporation (OCC) fee of 10 cents is levied on the firm that clears the trade.
- *The spread might be a constant proportion of the option value* This would be the case if the majority of the market maker's cost is in inventory holding costs, since the more expensive options will cost more to hold and hence the spread would be set wider.
- *Market makers might equalise marginal costs across options irrespective of trading volume* This would occur if the riskiness of an unwanted position were the most important cost facing market makers. Market makers typically do not hold a particular view on the direction of the market – they simply try to make money by buying and selling. Hence, they would like to be able to offload any unwanted (long or short) positions quickly. But trading is not continuous, and in fact the average time between trades in 1989 was approximately five minutes. The longer market-makers hold an option, the higher the risk they face since the higher the probability that there will be a large adverse price movement. Thus options with low trading volumes would command higher spreads since it is more likely that the market-maker would be holding these options for longer.

In a non-quantitative exploratory analysis, George and Longstaff (1993) find that, comparing across contracts with different maturities, the bid–ask spread does indeed increase with maturity (as the option with longer maturity is worth more) and with 'moneyness' (that is, an option that is deeper in the money has a higher spread than one which is less in the money). This is seen to be true for both call and put options.

7.9.4 The Influence of Tick-Size Rules on Spreads

The CBOE limits the *tick size* (the minimum granularity of price quotes), which will of course place a lower limit on the size of the spread. The tick sizes are

- \$1/8 for options worth \$3 or more
- \$1/16 for options worth less than \$3.

7.9.5 The Models and Results

The intuition that the bid–ask spread and trading volume may be simultaneously related arises since a wider spread implies that trading is relatively more expensive so that marginal investors would withdraw from the market. On the other hand, market-makers face additional risk if the level of trading activity falls, and hence they may be expected to respond by increasing their fee (the spread). The models developed seek to simultaneously determine the size of the bid–ask spread and the time between trades.

For the calls, the model is

$$CBA_i = \alpha_0 + \alpha_1 CDUM_i + \alpha_2 C_i + \alpha_3 CL_i + \alpha_4 T_i + \alpha_5 CR_i + e_i \tag{7.55}$$

$$CL_i = \gamma_0 + \gamma_1 CBA_i + \gamma_2 T_i + \gamma_3 T_i^2 + \gamma_4 M_i^2 + v_i \tag{7.56}$$

And symmetrically for the puts:

$$PBA_i = \beta_0 + \beta_1 PDUM_i + \beta_2 P_i + \beta_3 PL_i + \beta_4 T_i + \beta_5 PR_i + u_i \tag{7.57}$$

$$PL_i = \delta_0 + \delta_1 PBA_i + \delta_2 T_i + \delta_3 T_i^2 + \delta_4 M_i^2 + w_i \tag{7.58}$$

where CBA_i and PBA_i are the call bid–ask spread and the put bid–ask spread for option i, respectively

C_i and P_i are the call price and put price for option i, respectively

CL_i and PL_i are the times between trades for the call and put option i, respectively

$CDUM_i$ and $PDUM_i$ are dummy variables to allow for the minimum tick size

$$= 0 \quad \text{if } C_i \text{ or } P_i < \$3$$

$$= 1 \quad \text{if } C_i \text{ or } P_i \geq \$3$$

T is the time to maturity

T^2 allows for a non-linear relationship between time to maturity and the spread M^2 is the square of moneyness, which is employed in quadratic form since at-the-money options have a higher trading volume, while out-of-the-money and in-the-money options both have lower trading activity

CR_i and PR_i are measures of risk for the call and put options, respectively, given by the square of their deltas.

Equations (7.55) and (7.56), and then separately (7.57) and (7.58), are estimated using 2SLS. The results are given here in Tables 7.1 and 7.2.

Table 7.1 Call bid–ask spread and trading volume regression

$CBA_i = \alpha_0 + \alpha_1 CDUM_i + \alpha_2 C_i + \alpha_3 CL_i + \alpha_4 T_i + \alpha_5 CR_i + e_i$						(7.55)
$CL_i = \gamma_0 + \gamma_1 CBA_i + \gamma_2 T_i + \gamma_3 T_i^2 + \gamma_4 M_i^2 + v_i$						(7.56)
α_0	α_1	α_2	α_3	α_4	α_5	Adj. R^2
0.08362	0.06114	0.01679	0.00902	−0.00228	−0.15378	0.688
(16.80)	(8.63)	(15.49)	(14.01)	(−12.31)	(−12.52)	
γ_0	γ_1	γ_2	γ_3	γ_4	Adj. R^2	
−3.8542	46.592	−0.12412	0.00406	0.00866	0.618	
(−10.50)	(30.49)	(−6.01)	(14.43)	(4.76)		

Note: *t*-ratios in parentheses.

Source: George and Longstaff (1993). Reprinted with the permission of School of Business Administration, University of Washington.

Table 7.2 Put bid–ask spread and trading volume regression

$PBA_i = \beta_0 + \beta_1 PDUM_i + \beta_2 P_i + \beta_3 PL_i + \beta_4 T_i + \beta_5 PR_i + u_i$						(7.57)
$PL_i = \delta_0 + \delta_1 PBA_i + \delta_2 T_i + \delta_3 T_i^2 + \delta_4 M_i^2 + w_i$						(7.58)
β_0	β_1	β_2	β_3	β_4	β_5	Adj.R^2
0.05707	0.03258	0.01726	0.00839	−0.00120	−0.08662	0.675
(15.19)	(5.35)	(15.90)	(12.56)	(−7.13)	(−7.15)	
δ_0	δ_1	δ_2	δ_3	δ_4	Adj. R^2	
−2.8932	46.460	−0.15151	0.00339	0.01347	0.517	
(−8.42)	(34.06)	(−7.74)	(12.90)	(10.86)		

Note: *t*-ratios in parentheses.

Source: George and Longstaff (1993). Reprinted with the permission of School of Business Administration, University of Washington.

The adjusted $R^2 \approx 0.6$ for all four equations, indicating that the variables selected do a good job of explaining the spread and the time between trades. George and Longstaff (1993) argue that strategic market maker behaviour, which cannot be easily modelled, is important in influencing the spread and that this precludes a higher adjusted R^2.

A next step in examining the empirical plausibility of the estimates is to consider the sizes, signs and significances of the coefficients. In the call and put spread regressions, respectively, α_1 and β_1 measure the tick size constraint on the spread – both are statistically significant and positive. α_2 and β_2 measure the effect of the option price on the spread. As expected, both of these coefficients are again significant and positive since these are inventory or holding costs. The coefficient value of approximately 0.017 implies that a one dollar increase in the price of the option will on average lead to a 1.7 cent increase in the spread. α_3 and β_3 measure the effect of trading activity on the spread. Recalling that an inverse trading activity variable is used in the regressions, again, the coefficients have their correct sign. That is, as the time between trades increases (that is, as trading activity falls), the bid–ask spread widens. Furthermore, although the coefficient values are small, they are statistically significant. In the put spread regression, for example, the coefficient of approximately 0.009 implies that, even if the time between trades widened from one minute to one hour, the spread would increase by only 54 cents. α_4 and β_4 measure the effect of time to maturity on the spread; both are negative and statistically significant. The authors argue that this may arise as market making is a more risky activity for near-maturity options. A possible alternative explanation, which they dismiss after further investigation, is that the early exercise possibility becomes more likely for very short-dated options since the loss of time value would be negligible. Finally, α_5 and β_5 measure the effect of risk on the spread; in both the call and put spread regressions, these coefficients are negative and highly statistically significant. This seems an odd result, which the authors struggle to justify, for it seems to suggest that more risky options will command lower spreads.

Turning attention now to the trading activity regressions, γ_1 and δ_1 measure the effect of the spread size on call and put trading activity, respectively. Both are positive and statistically significant, indicating that a rise in the spread will increase the time between trades. The coefficients are such that a one *cent* increase in the spread would lead to an increase in the average time between call and put trades of nearly half a minute. γ_2 and δ_2 give the effect of an increase in time to maturity, while γ_3 and δ_3 are coefficients attached to the square of time to maturity. For both the call and put regressions, the coefficient on the level of time to maturity is negative and significant, while that on the square is positive and significant. As time to maturity increases, the squared term would dominate, and one could therefore conclude that the time between trades will show a U-shaped relationship with time to maturity. Finally, γ_4 and δ_4 give the effect of an increase in the square of moneyness (i.e., the effect of an option going deeper into the money or deeper out of the money) on the time between trades. For both the call and put regressions, the coefficients are statistically significant and positive, showing that as the option moves further from the money in either direction, the time between trades rises. This is consistent with the authors' supposition that trade is most active in at-the-money options, and less active in both out-of-the-money and in-the-money options.

7.9.6 Conclusions

The value of the bid–ask spread on S&P100 index options and the time between trades (a measure of market liquidity) can be usefully modelled in a simultaneous system with exogenous variables such as the options' deltas, time to maturity, moneyness, etc.

This study represents a nice example of the use of a simultaneous equations system, but, in this author's view, it can be criticised on several grounds. First, there are no diagnostic tests performed. Second, clearly the equations are all overidentified, but it is not obvious how the over-identifying restrictions have been generated. Did they arise from consideration of financial theory? For example, why do the *CL* and *PL* equations not contain the *CR* and *PR* variables? Why do the *CBA* and *PBA* equations not contain moneyness or squared maturity variables? The authors could also have tested for endogeneity of *CBA* and *CL*. Finally, the wrong sign on the highly statistically significant squared deltas is puzzling.

7.10 Vector Autoregressive Models

Vector autoregressive models (VARs) were popularised in econometrics by Sims (1980) as a natural generalisation of univariate autoregressive models discussed in Chapter 6. A VAR is a systems regression model (i.e., there is more than one dependent variable) that can be considered a kind of hybrid between the univariate time series models considered in Chapter 6 and the simultaneous equations models developed previously in this chapter. VARs have often been advocated as an alternative to large-scale simultaneous equations structural models.

The simplest case that can be entertained is a bivariate VAR, where there are only two variables, y_{1t} and y_{2t}, each of whose current values depend on different combinations of the previous k values of both variables, and error terms

$$y_{1t} = \beta_{10} + \beta_{11} y_{1t-1} + \cdots + \beta_{1k} y_{1t-k} + \alpha_{11} y_{2t-1} + \cdots + \alpha_{1k} y_{2t-k} + u_{1t} \quad \textbf{(7.59)}$$

$$y_{2t} = \beta_{20} + \beta_{21} y_{2t-1} + \cdots + \beta_{2k} y_{2t-k} + \alpha_{21} y_{1t-1} + \cdots + \alpha_{2k} y_{1t-k} + u_{2t} \quad \textbf{(7.60)}$$

where u_{it} is a white noise disturbance term with $E(u_{it}) = 0$, ($i = 1, 2$).

Although, for simplicity, we usually assume that $E(u_{1t}\ u_{2t}) = 0$ so that the disturbances are uncorrelated across equations, it is common and more realistic to allow them to be contemporaneously correlated, so Cov($u_{1t}\ u_{2t}$ $= \sigma_{12}$).

As should already be evident, an important feature of the VAR model is its flexibility and the ease of generalisation. For example, the model could be extended to encompass moving average errors, which would be a multivariate version of an ARMA model, known as a VARMA. Instead of having only two variables, y_{1t} and y_{2t}, the system could also be expanded to include g variables, $y_{1t}, y_{2t}, y_{3t}, \ldots, y_{gt}$, each of which has an equation.

Another useful facet of VAR models is the compactness with which the notation can be expressed. For example, consider the case from above where $k = 1$, so that each variable depends only upon the immediately previous values of y_{1t} and y_{2t}, plus an error term. This could be written as

$$y_{1t} = \beta_{10} + \beta_{11} y_{1t-1} + \alpha_{11} y_{2t-1} + u_{1t} \quad \textbf{(7.61)}$$

$$y_{2t} = \beta_{20} + \beta_{21} y_{2t-1} + \alpha_{21} y_{1t-1} + u_{2t} \quad \textbf{(7.62)}$$

or

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} \beta_{10} \\ \beta_{20} \end{pmatrix} + \begin{pmatrix} \beta_{11} & \alpha_{11} \\ \alpha_{21} & \beta_{21} \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} + \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} \quad \textbf{(7.63)}$$

or even more compactly as

$$\underset{g \times 1}{y_t} = \underset{g \times 1}{\beta_0} + \underset{g \times g\, g \times 1}{\beta_1 y_{t-1}} + \underset{g \times 1}{u_t} \quad \textbf{(7.64)}$$

In equation (7.64), there are g = 2 variables in the system. Extending the model to the case where there are k lags of each variable in each equation is also easily accomplished using this notation

$$\underset{g\times 1}{y_t} = \underset{g\times 1}{\beta_0} + \underset{g\times g}{\beta_1}\underset{g\times 1}{y_{t-1}} + \underset{g\times g}{\beta_2}\underset{g\times 1}{y_{t-2}} + \cdots + \underset{g\times g}{\beta_k}\underset{g\times 1}{y_{t-k}} + \underset{g\times 1}{u_t} \qquad \textbf{(7.65)}$$

The model could be further extended to the case where the model includes first difference terms and cointegrating relationships (a vector error correction model (VECM) – see Chapter 8).

7.10.1 Advantages of VAR Modelling

VAR models have several advantages compared with univariate time series models or simultaneous equations structural models

- The researcher does not need to specify which variables are endogenous or exogenous – *all are endogenous*. This is a very important point, since a requirement for simultaneous equations structural models to be estimable is that all equations in the system are identified. Essentially, this requirement boils down to a condition that some variables are treated as exogenous and that the equations contain different RHS variables. Ideally, this restriction should arise naturally from financial or economic theory. However, in practice theory will be at best vague in its suggestions of which variables should be treated as exogenous. This leaves the researcher with a great deal of discretion concerning how to classify the variables. Since Hausman-type tests are often not employed in practice when they should be, the specification of certain variables as exogenous, required to form identifying restrictions, is likely in many cases to be invalid. Sims (1980) termed these identifying restrictions 'incredible'. VAR estimation, on the other hand, requires no such restrictions to be imposed.
- VARs allow the value of a variable to depend on more than just its own lags or combinations of white noise terms, so VARs are more flexible than univariate AR models; the latter can be viewed as a restricted case of VAR models. VAR models can therefore offer a very *rich structure*, implying that they may be able to capture more features of the data.
- Provided that there are no contemporaneous terms on the RHS of the equations and that the disturbances are uncorrelated across equations,

it is possible to *simply use OLS separately on each equation*. This arises from the fact that all variables on the RHS are pre-determined – that is, at time *t*, they are known. This implies that there is no possibility for feedback from any of the LHS variables to any of the RHS variables. Pre-determined variables include all exogenous variables and lagged values of the endogenous variables. If the disturbances are correlated, the equations could be jointly estimated using *maximum likelihood* (discussed at length in Chapter 9) based on the joint density of all the disturbances u_{it} assuming multivariate normality.
- The forecasts generated by VARs are often *better than 'traditional structural' models*. It has been argued in a number of articles (see, for example, Sims, 1980) that large-scale structural models performed badly in terms of their out-of-sample forecast accuracy. This could perhaps arise as a result of the ad hoc nature of the restrictions placed on the structural models to ensure identification discussed above. McNees (1986) shows that forecasts for some variables (e.g., the US unemployment rate and real gross national product (GNP), etc.) are produced more accurately using VARs than from several different structural specifications.

7.10.2 Problems with VARs

VAR models of course also have drawbacks and limitations relative to other model classes:

- VARs are *a-theoretical* (as are ARMA models), since they use little theoretical information about the relationships between the variables to guide the specification of the model. On the other hand, valid exclusion restrictions that ensure identification of equations from a simultaneous structural system will inform on the structure of the model. An upshot of this is that VARs are less amenable to theoretical analysis and therefore to policy prescriptions. There also exists an increased possibility under the VAR approach that a hapless researcher could obtain an essentially spurious relationship by mining the data. It is also often not clear how the VAR coefficient estimates should be interpreted.
- How should the appropriate *lag lengths* for the VAR be determined? There are several approaches available for dealing with this issue, which will be discussed below.

- *So many parameters*! If there are g equations, one for each of g variables and with k lags of each of the variables in each equation, ($g + kg^2$) parameters will have to be estimated. For example, if $g = 3$ and $k = 3$ there will be thirty parameters to estimate. For relatively small sample sizes, degrees of freedom will rapidly be used up, implying large standard errors and therefore wide confidence intervals for model coefficients.
- Should *all of the components of the VAR be stationary*? Obviously, if one wishes to use hypothesis tests, either singly or jointly, to examine the statistical significance of the coefficients, then it is essential that all of the components in the VAR are stationary. However, many proponents of the VAR approach recommend that differencing to induce stationarity should not be done. They would argue that the purpose of VAR estimation is purely to examine the relationships between the variables, and that differencing will throw information on any long-run relationships between the series away. It is also possible to combine levels and first differenced terms in a VECM – see Chapter 8.

7.10.3 Choosing the Optimal Lag Length for a VAR

Often, financial theory will have little to say on what is an appropriate lag length for a VAR and how long changes in the variables should take to work through the system. In such instances, there are broadly three methods that could be used to arrive at the optimal lag length: rules of thumb, cross-equation restrictions and information criteria.

7.10.4 Rules of Thumb for VAR Lag Length Selection

Similar to univariate AR(p) models, it might be possible to use the data frequency to decide the lag order and thus, for example, selecting $p = 5$ for daily data, $p = 4$ for quarterly data and so on. However, if the number of variables in the system is quite large, then a value of p this big, let alone the number that would by analogy be suggested for monthly data, would quickly become infeasible. It is also common to use an arbitrary fixed number of lags (typically, 1, 2, or 3) without further testing.

Two more scientific approaches to choosing the lag order are given in the following sub-sections, but before moving on to those, it is worth noting that a high value of p may be required if the number of variables g in the system is too small and excludes relevant influences on the included variables (possibly also including the wrong variables). Thus, in a sense, there is a trade-off between a larger p and a larger g. In such a situation, a better model is likely to arise from thinking creatively about additional variables to include in the model, even if these are hard to estimate, rather than increasing the lag length.

7.10.5 Cross-Equation Restrictions for VAR Lag Length Selection

A first (but incorrect) response to the question of how to determine the appropriate lag length would be to use the block F-tests highlighted in Section 7.12 on page 319. These, however, are not appropriate in this case as the F-test would be used separately for the set of lags in each equation, and what is required here is a procedure to test the coefficients on a set of lags on all variables for all equations in the VAR at the same time.

It is worth noting here that in the spirit of VAR estimation (as Sims, 1980, for example, thought that model specification should be conducted), the models should be as unrestricted as possible. A VAR with different lag lengths for each equation could be viewed as a restricted VAR. For example, consider a VAR with three lags of both variables in one equation and four lags of each variable in the other equation. This could be viewed as a restricted model where the coefficient on the fourth lags of each variable in the first equation have been set to zero.

An alternative approach would be to specify the same number of lags in each equation and to determine the model order as follows. Suppose that a VAR estimated using quarterly data has eight lags of the two variables in each equation, and it is desired to examine a restriction that the coefficients on lags five–eight are jointly zero. This can be done using a likelihood ratio test (see Chapter 9 for more general details concerning such tests). Denote the variance–covariance matrix of residuals (given by $\hat{u}\hat{u}'$), as $\hat{\Sigma}$. The likelihood ratio test for this joint hypothesis is given by

$$LR = T[\ln|\hat{\Sigma}_r| - \ln|\hat{\Sigma}_u|] \tag{7.66}$$

where $|\hat{\Sigma}_r|$ is the determinant of the variance–covariance matrix of the residuals for the restricted model (with four lags), $|\hat{\Sigma}_u|$ is the determinant of the variance–covariance matrix of residuals for the unrestricted VAR (with eight lags) and T is the sample size. The test statistic is asymptotically distributed as a χ^2 variate with degrees of freedom equal to the total number of restrictions. In the VAR case above, four lags of two variables are being restricted in each of the two equations = a total of 4 × 2 × 2 = 16 restrictions. In the general case of a VAR with g equations, to impose the restriction that the last q lags have zero coefficients, there would be g^2q restrictions altogether. Intuitively, the test is a multivariate equivalent to examining the extent to which the *RSS* rises when a restriction is imposed. If $\hat{\Sigma}_r$ and $\hat{\Sigma}_u$ are 'close together', the restriction is supported by the data.

7.10.6 Information Criteria for VAR Lag Length Selection

The likelihood ratio (LR) test explained above is intuitive and fairly easy to estimate, but has its limitations. Principally, one of the two VARs must be a special case of the other and, more seriously, only pairwise comparisons can be made. In the above example, if the most appropriate lag length had been seven or even ten, there is no way that this information could be gleaned from the LR test conducted. One could achieve this only by starting with a VAR(10), and successively testing one set of lags at a time.

A further disadvantage of the LR test approach is that the χ^2 test will strictly be valid asymptotically only under the assumption that the errors from each equation are normally distributed. This assumption is unlikely to be upheld for financial data. An alternative approach to selecting the appropriate VAR lag length would be to use an information criterion, as defined in Chapter 6 in the context of ARMA model selection. Information criteria require no such normality assumptions concerning the distributions of the errors. Instead, the criteria trade off a fall in the *RSS* of each equation as more lags are added, with an increase in the value of the penalty term. The univariate criteria could be applied separately to each equation but, again, it is usually deemed preferable to require the number of lags to be the same for each equation. This requires the use of multivariate versions of the information criteria, which can be defined as

$$MAIC = \ln\left|\hat{\Sigma}\right| + 2k'/T \tag{7.67}$$

$$MSBIC = \ln\left|\hat{\Sigma}\right| + \frac{k'}{T}\ln(T) \tag{7.68}$$

$$MHQIC = \ln\left|\hat{\Sigma}\right| + \frac{2k'}{T}\ln(\ln(T)) \tag{7.69}$$

where again $\hat{\Sigma}$ is the variance–covariance matrix of residuals, T is the number of observations and k' is the total number of regressors in all equations, which will be equal to $p^2k + p$ for p equations in the VAR system, each with k lags of the p variables, plus a constant term in each equation. As previously, the values of the information criteria are constructed for $0, 1, ..., \bar{k}$ lags (up to some pre-specified maximum $\bar{k}$), and the chosen number of lags is that number minimising the value of the given information criterion.

7.11 Does the VAR Include Contemporaneous Terms?

So far, it has been assumed that the VAR specified is of the form

$$y_{1t} = \beta_{10} + \beta_{11}y_{1t-1} + \alpha_{11}y_{2t-1} + u_{1t} \tag{7.70}$$

$$y_{2t} = \beta_{20} + \beta_{21}y_{2t-1} + \alpha_{21}y_{1t-1} + u_{2t} \tag{7.71}$$

so that there are no contemporaneous terms on the RHS of equation (7.70) or (7.71) – i.e., there is no term in y_{2t} on the RHS of the equation for y_{1t} and no term in y_{1t} on the RHS of the equation for y_{2t}. But what if the equations had a contemporaneous feedback term, as in the following case?

$$y_{1t} = \beta_{10} + \beta_{11}y_{1t-1} + \alpha_{11}y_{2t-1} + \alpha_{12}y_{2t} + u_{1t} \tag{7.72}$$

$$y_{2t} = \beta_{20} + \beta_{21}y_{2t-1} + \alpha_{21}y_{1t-1} + \alpha_{22}y_{1t} + u_{2t} \tag{7.73}$$

Equations (7.72) and (7.73) could also be written by stacking up the terms into matrices and vectors:

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} \beta_{10} \\ \beta_{20} \end{pmatrix} + \begin{pmatrix} \beta_{11} & \alpha_{11} \\ \alpha_{21} & \beta_{21} \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} + \begin{pmatrix} \alpha_{12} & 0 \\ 0 & \alpha_{22} \end{pmatrix} \begin{pmatrix} y_{2t} \\ y_{1t} \end{pmatrix} + \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} \tag{7.74}$$

This would be known as a *VAR in primitive form*, similar to the structural form for a simultaneous equations model. Some researchers have argued that the a-theoretical nature of reduced form VARs leaves them unstructured and their results difficult to interpret theoretically. They argue that the forms of VAR given previously are merely reduced forms of a more general structural VAR (such as equation (7.74)), with the latter being of more interest.

The contemporaneous terms from equation (7.74) can be taken over to the LHS and written as

$$\begin{pmatrix} 1 & -\alpha_{12} \\ -\alpha_{22} & 1 \end{pmatrix}\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} \beta_{10} \\ \beta_{20} \end{pmatrix} + \begin{pmatrix} \beta_{11} & \alpha_{11} \\ \alpha_{21} & \beta_{21} \end{pmatrix}\begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} + \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} \quad \textbf{(7.75)}$$

or

$$\mathrm{A}y_t = \beta_0 + \beta_1 y_{t-1} + u_t \quad \textbf{(7.76)}$$

If both sides of equation (7.76) are pre-multiplied by A^{-1}

$$y_t = \mathrm{A}^{-1}\beta_0 + \mathrm{A}^{-1}\beta_1 y_{t-1} + \mathrm{A}^{-1}u_t \quad \textbf{(7.77)}$$

or

$$y_t = \mathrm{A}_0 + \mathrm{A}_1 y_{t-1} + e_t \quad \textbf{(7.78)}$$

This is known as a *standard form VAR*, which is akin to the reduced form from a set of simultaneous equations. This VAR contains only pre-determined values on the RHS (i.e., variables whose values are known at time *t*), and so there is no contemporaneous feedback term. This VAR can therefore be estimated equation by equation using OLS.

Equation (7.74), the structural or primitive form VAR, is not identified, since identical pre-determined (lagged) variables appear on the RHS of both equations. In order to circumvent this problem, a restriction that one of the coefficients on the contemporaneous terms is zero must be imposed. In equation (7.74), either α_{12} or α_{22} must be set to zero to obtain a triangular set of VAR equations that can be validly estimated. The choice of which of these two restrictions to impose is ideally made on theoretical grounds. For example, if financial theory suggests that the current value of y_{1t} should affect the current value of y_{2t} but not the other way around, set $\alpha_{12} = 0$, and so on. Another possibility would be to run separate estimations, first imposing $\alpha_{12} = 0$ and then $\alpha_{22} = 0$, to determine whether the general features of the results are much changed. It is also very common to estimate only a reduced form VAR, which is of course perfectly valid provided that such a formulation is not at odds with the relationships between variables that financial theory says should hold.

One fundamental weakness of the VAR approach to modelling is that its a-theoretical nature and the large number of parameters involved make the estimated models difficult to interpret. In particular, some lagged variables may have coefficients which change sign across the lags, and this, together with the interconnectivity of the equations, could render it difficult to see what effect a given change in a variable would have upon the future values of the variables in the system. In order to partially alleviate this problem, three sets of statistics are usually constructed for an estimated VAR model: block significance tests, impulse responses and variance decompositions. How important an intuitively interpretable model is will of course depend on the purpose of constructing the model. Interpretability may not be an issue at all if the purpose of producing the VAR is to make forecasts – see Box 7.3.

BOX 7.3 Forecasting with VARs

One of the main advantages of the VAR approach to modelling and forecasting is that since only lagged variables are used on the RHS, forecasts of the future values of the dependent variables can be calculated iteratively using only information from within the system, very similar in approach to the manner in which forecasts are derived from an AR(p) model. We could term these *unconditional forecasts* since they are not constructed conditional on a particular set of assumed values.

However, conversely it may be useful to produce forecasts of the future values of some variables *conditional upon* known values of other variables in the system. For example, it may be the case that the values of some variables become known before the values of the others. If the known values of the former are employed, we would anticipate that the forecasts should be more accurate than if estimated values were used unnecessarily, thus throwing known information away.

Alternatively, conditional forecasts can be employed for counterfactual analysis based on examining the impact of certain scenarios. For example, given a trivariate VAR system incorporating monthly stock returns, inflation and gross domestic product (GDP), we could answer the question: ‘What is the likely impact on the stock market over the next 1–6 months of a 2-percentage point increase in inflation and a 1% rise in GDP?’

Usually, the forecasts from a VAR are evaluated separately equation-by-equation and then the forecasts can be compared with those of other approaches (such as a linear regression or AR(p) model) using the standard forecast error aggregation tools such as RMSE as discussed in Chapter 6. Indeed, it will often be the case that only the forecasts from one equation in the VAR are actually of interest.

7.12 Block Significance and Causality Tests

It is likely that, when a VAR includes many lags of variables, it will be difficult to see which sets of variables have significant effects on each dependent variable and which do not. In order to address this issue, tests are usually conducted that restrict all of the lags of a particular variable to zero. For illustration, consider the following bivariate VAR(3)

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} \alpha_{10} \\ \alpha_{20} \end{pmatrix} + \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} + \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \begin{pmatrix} y_{1t-2} \\ y_{2t-2} \end{pmatrix} + \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix} \begin{pmatrix} y_{1t-3} \\ y_{2t-3} \end{pmatrix} + \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} \quad \textbf{(7.79)}$$

This VAR could be written out to express the individual equations as

$$\begin{aligned} y_{1t} &= \alpha_{10} + \beta_{11} y_{1t-1} + \beta_{12} y_{2t-1} + \gamma_{11} y_{1t-2} + \gamma_{12} y_{2t-2} \\ &\quad + \delta_{11} y_{1t-3} + \delta_{12} y_{2t-3} + u_{1t} \\ y_{2t} &= \alpha_{20} + \beta_{21} y_{1t-1} + \beta_{22} y_{2t-1} + \gamma_{21} y_{1t-2} + \gamma_{22} y_{2t-2} \\ &\quad + \delta_{21} y_{1t-3} + \delta_{22} y_{2t-3} + u_{2t} \end{aligned} \quad \textbf{(7.80)}$$

One might be interested in testing the hypotheses and their implied restrictions on the parameter matrices given in Table 7.3.

Table 7.3 Granger causality tests and implied restrictions on VAR models

	Hypothesis	Implied restriction
1	Lags of y_{1t} do not explain current y_{2t}	$\beta_{21} = 0$ and $\gamma_{21} = 0$ and $\delta_{21} = 0$
2	Lags of y_{1t} do not explain current y_{1t}	$\beta_{11} = 0$ and $\gamma_{11} = 0$ and $\delta_{11} = 0$
3	Lags of y_{2t} do not explain current y_{1t}	$\beta_{12} = 0$ and $\gamma_{12} = 0$ and $\delta_{12} = 0$
4	Lags of y_{2t} do not explain current y_{2t}	$\beta_{22} = 0$ and $\gamma_{22} = 0$ and $\delta_{22} = 0$

Assuming that all of the variables in the VAR are stationary, the joint hypotheses can easily be tested within the *F*-test framework, since each individual set of restrictions involves parameters drawn from only one equation. The equations would be estimated separately using OLS to obtain the unrestricted *RSS*, then the restrictions imposed and the models re-estimated to obtain the restricted *RSS*. The *F*-statistic would then take the usual form described in Chapter 4. Thus, evaluation of the significance of variables in the context of a VAR almost invariably occurs on the basis of joint tests on all of the lags of a particular variable in an equation, rather than by examination of individual coefficient estimates.

In fact, the tests described above could also be referred to as causality tests. Tests of this form were described by Granger (1969) and a slight variant due to Sims (1972). Causality tests seek to answer simple questions of the type, 'Do changes in y_1 cause changes in y_2?' The argument follows that if y_1 causes y_2, lags of y_1 should be significant in the equation for y_2. If this is the case and not vice versa, it would be said that y_1 'Granger-causes' y_2 or that there exists unidirectional causality from y_1 to y_2. On the other hand, if y_2 causes y_1, lags of y_2 should be significant in the equation for y_1. If both sets of lags were significant, it would be said that there was 'bi-directional causality' or 'bi-directional feedback'. If y_1 is found to Granger-cause y_2, but not vice versa, it would be said that variable y_1 is strongly exogenous (in the equation for y_2). If neither set of lags are statistically significant in the equation for the other variable, it would be said that y_1 and y_2 are independent. Finally, the word 'causality' is somewhat of a misnomer, for Granger-causality really means only a correlation between the *current* value of one variable and the *past* values of others; it does not mean that movements of one variable cause movements of another.

7.12.1 Restricted VARs

As written above in, for example, equations (7.59) and (7.60), the VAR is completely unrestricted and as general as possible in the sense that the lags of every variable in the system enter into the equations for every variable. However, as discussed this can lead to very highly parameterised systems with surprisingly few degrees of freedom even when the number of time-series observations is quite large. Such systems may also produce relatively poor forecasts as a result.

It is possible that theory might suggest that certain lagged variables should not appear in certain equations, in which case a *restricted* or *unbalanced* VAR with fewer parameters could be formed. For example, if we were measuring interactions between small and large economies, it might be plausible to assume that whatever goes on in the small economy cannot affect the large one and thus we might set the lags of the former to zero in the equation(s) for the latter. Alternatively, it might be that the block significance tests outlined in the previous section suggested that particular sets of lags could be removed from one or more equations. However, it probably does not make sense to remove specific lags of individual variables from an equation while leaving other lags of the same variable in the equation (e.g., removing y_{2t-1} and y_{2t-3} from the equation for y_{1t} but leaving y_{2t-2}): either a given variable influences another variable or it does not.

An alternative way to reduce the number of parameters to estimate in a VAR is to use a Bayesian approach, where specific prior distributions are imposed on the VAR to leave fewer free parameters than originally. See Doan et al. (1984) or Giannone, Lenza and Primiceri (2014) and the references therein for further details.

7.13 VARs with Exogenous Variables

Consider the following specification for a VAR(1) where X_t is a vector of exogenous variables and B is a matrix of coefficients

$$y_t = A_0 + A_1 y_{t-1} + BX_t + e_t \tag{7.81}$$

The components of the vector X_t are known as exogenous variables since their values are determined outside of the VAR system – in other words, there are no equations in the VAR with any of the components of X_t as dependent variables. Such a model is sometimes termed a VARX, although it could be viewed as simply a very restricted VAR where there are equations for each of the exogenous variables, but with the coefficients on the RHS in those equations restricted to zero. Such a restriction may be considered desirable if theoretical considerations suggest it, although it is clearly not in the true spirit of VAR modelling, which is not to impose any restrictions on the model but rather to 'let the data decide'.

7.14 Impulse Responses and Variance Decompositions

Block F-tests and an examination of causality in a VAR will suggest which of the variables in the model have statistically significant impacts on the future values of each of the variables in the system. But F-test results will not, by construction, be able to explain the sign of the relationship or how long these effects require to take place. That is, F-test results will not reveal whether changes in the value of a given variable have a positive or negative effect on other variables in the system, or how long it would take for the effect of that variable to work through the system. Such information will, however, be given by an examination of the VAR's impulse responses and variance decompositions.

Impulse responses trace out the responsiveness of the dependent variables in the VAR to shocks to each of the variables. So, for each variable from each equation separately, a unit shock is applied to the error, and the effects upon the VAR system over time are noted. Effectively, the impulse responses are partial derivatives of the variables (y_{jt}, $j = 1, \ldots, g$) with respect to each error term $(u_{kt}, k = 1, \ldots, g)$: $\frac{\partial y_{jt}}{\partial u_{kt}}$.

In practice, one standard deviation shocks are often used rather than one unit, as it might be the case that a one unit shock is empirically implausible, but a one standard deviation shock will almost always be relevant.

If there are g variables in a system, a total of g^2 impulse responses could be generated. The way that this is achieved in practice is by expressing the VAR model as a VMA – that is, the vector autoregressive model is written as a vector moving average (in the same way as was done for univariate autoregressive models in Chapter 5). Provided that the system is stable, the shock should gradually die away.

To illustrate how impulse responses operate, consider the following bivariate VAR(1)

$$y_t = A_1 y_{t-1} + u_t \qquad \text{where } A_1 = \begin{bmatrix} 0.5 & 0.3 \\ 0.0 & 0.2 \end{bmatrix} \tag{7.82}$$

The VAR can also be written out using the elements of the matrices and vectors as

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.3 \\ 0.0 & 0.2 \end{bmatrix} \begin{bmatrix} y_{1t-1} \\ y_{2t-1} \end{bmatrix} + \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix} \tag{7.83}$$

Consider the effect at time $t = 0, 1, \ldots$, of a unit shock to y_{1t} at time $t = 0$, where the $T \times 1$ vector for y at time t is stacked up and written as simply $[y_t]$ for notational convenience

$$y_0 = \begin{bmatrix} y_{10} \\ y_{20} \end{bmatrix} = \begin{bmatrix} u_{10} \\ u_{20} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \textbf{(7.84)}$$

$$y_1 = A_1 y_0 = \begin{bmatrix} 0.5 & 0.3 \\ 0.0 & 0.2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \quad \textbf{(7.85)}$$

$$y_2 = A_1 y_1 = \begin{bmatrix} 0.5 & 0.3 \\ 0.0 & 0.2 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} \quad \textbf{(7.86)}$$

and so on. It would thus be possible to plot the impulse response functions of y_{1t} and y_{2t} to a unit shock in y_{1t}. Notice that the effect on y_{2t} is always zero, since the variable y_{1t-1} has a zero coefficient attached to it in the equation for y_{2t}.

Now consider the effect of a unit shock to y_{2t} at time $t = 0$

$$y_0 = \begin{bmatrix} u_{10} \\ u_{20} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \textbf{(7.87)}$$

$$y_1 = A_1 y_0 = \begin{bmatrix} 0.5 & 0.3 \\ 0.0 & 0.2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix} \quad \textbf{(7.88)}$$

$$y_2 = A_1 y_1 = \begin{bmatrix} 0.5 & 0.3 \\ 0.0 & 0.2 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.21 \\ 0.04 \end{bmatrix} \quad \textbf{(7.89)}$$

and so on. Although it is probably fairly easy to see what the effects of shocks to the variables will be in such a simple VAR, the same principles can be applied in the context of VARs containing more equations or more lags, where it is much more difficult to see by eye what are the interactions between the equations.

Variance decompositions offer a slightly different method for examining VAR system dynamics. They give the proportion of the movements in the dependent variables that are due to their 'own' shocks, versus shocks to the other variables. A shock to the *i*th variable will directly affect that variable of course, but it will also be transmitted to all of the other variables in the system through the dynamic structure of the VAR. Variance decompositions determine how much of the *s*-step-ahead forecast error variance of a given variable is explained by innovations to each explanatory variable for $s = 1, 2, \ldots$ In practice, it is usually observed that own series shocks explain most of the (forecast) error variance of the series in a VAR. To some extent, impulse responses and variance decompositions offer very similar information.

For calculating impulse responses and variance decompositions, the ordering of the variables is important. To see why this is the case, recall that the impulse responses refer to a unit shock to the errors of one VAR equation alone. This implies that the error terms of all other equations in the VAR system are held constant. However, this is not realistic since the error terms are likely to be correlated across equations to some extent. Thus, assuming that they are completely independent would lead to a misrepresentation of the system dynamics. In practice, the errors will have a common component that cannot be associated with a single variable alone.

The usual approach to this difficulty is to generate *orthogonalised impulse responses*. In the context of a bivariate VAR, the whole of the common component of the errors is attributed somewhat arbitrarily to the first variable in the VAR. In the general case where there are more than two variables in the VAR, the calculations are more complex but the interpretation is the same. Such a restriction in effect implies an 'ordering' of variables, so that the equation for y_{1t} would be estimated first and then that of y_{2t}, a bit like a recursive or triangular system.

Assuming a particular ordering is necessary to compute the impulse responses and variance decompositions. Ideally, financial theory should suggest an ordering (in other words, that movements in some variables are likely to follow, rather than precede, others). Failing this, the sensitivity of the results to changes in the ordering can be observed by assuming one ordering, and then exactly reversing it and recomputing the impulse responses and variance decompositions. It is also worth noting that the more highly correlated are the residuals from an estimated equation, the more the variable ordering will be important. But when the residuals are almost uncorrelated, the ordering of the variables will make little difference (see Lütkepohl, 1991, Chapter 2 for further details).

Runkle (1987) argues that both impulse responses and variance decompositions are notoriously difficult to interpret accurately. He argues that confidence bands around the impulse responses and variance decompositions should always be constructed. However, he further states that, even then, the confidence intervals are typically so wide that sharp inferences are impossible.

7.15 VAR Model Example: The Interaction Between Property Returns and the Macroeconomy

7.15.1 Background, Data and Variables

Brooks and Tsolacos (1999) employ a VAR methodology for investigating the interaction between the UK property market and various macroeconomic variables. Monthly data, in logarithmic form, are used for the period from December 1985 to January 1998. The selection of the variables for inclusion in the VAR model is governed by the time series that are commonly included in studies of stock return predictability. It is assumed that stock returns are related to macroeconomic and business conditions, and hence time series which may be able to capture both current and future directions in the broad economy and the business environment are used in the investigation.

Broadly, there are two ways to measure the value of property-based assets – *direct measures of property value* and *equity-based measures*. Direct property measures are based on periodic appraisals or valuations of the actual properties in a portfolio by surveyors, while equity-based measures evaluate the worth of properties indirectly by considering the values of stock market traded property companies. Both sources of data have their drawbacks. Appraisal-based value measures suffer from valuation biases and inaccuracies. Surveyors are typically prone to 'smooth' valuations over time, such that the measured returns are too low during property market booms and too high during periods of property price falls. Additionally, not every property in the portfolio that comprises the value measure is appraised during every period, resulting in some stale valuations entering the aggregate valuation, further increasing the degree of excess smoothness of the recorded property price series. Indirect property vehicles – property-related companies traded on stock exchanges – do not suffer from the above problems, but are excessively influenced by general stock market movements. It has been argued, for example, that over three-quarters of the variation over time in the value of stock exchange traded property companies can be attributed to general stock market-wide price movements. Therefore, the value of equity-based property series reflects much more the sentiment in the general stock market than the sentiment in the property market specifically.

Brooks and Tsolacos (1999) elect to use the equity-based FTSE Property Total Return Index to construct property returns. In order to purge the real estate return series of its general stock market influences, it is common to regress property returns on a general stock market index (in this case the FTA All-Share Index is used), saving the residuals. These residuals are expected to reflect only the variation in property returns, and thus become the property market return measure used in subsequent analysis, and are denoted PROPRES.

Hence, the variables included in the VAR are the property returns (with general stock market effects removed), the rate of unemployment, nominal interest rates, the spread between the long- and short-term interest rates, unanticipated inflation and the dividend yield. The motivations for including these particular variables in the VAR together with the property series, are as follows

- *The rate of unemployment* (denoted UNEM) is included to indicate general economic conditions. In US research, authors tend to use aggregate consumption, a variable that has been built into asset pricing models and examined as a determinant of stock returns. Data for this variable and for alternative variables such as GDP are not available on a monthly basis in the UK. Monthly data are available for industrial production series but other studies have not shown any evidence that industrial production affects real estate returns. As a result, this series was not considered as a potential causal variable.
- *Short-term nominal interest rates* (denoted SIR) are assumed to contain information about future economic conditions and to capture the state of investment opportunities. It was found in previous studies that short-term interest rates have a very significant negative influence on property stock returns.
- *Interest rate spreads* (denoted SPREAD), i.e., the yield curve, are usually measured as the difference in the returns between long-term Treasury Bonds (of maturity, say, ten or twenty years), and the one-month or three-month Treasury Bill rate. It has been argued that the yield curve has extra predictive power, beyond that contained in the short-term interest rate, and can help predict GDP up to four years ahead. It has also been suggested that the term structure also affects real estate market returns.
- *Inflation rate* influences are also considered important in the pricing of stocks. For example, it has been argued that unanticipated inflation could be a source of economic risk and as a result, a risk premium will also be added if the stock of firms has exposure to unanticipated inflation. The unanticipated inflation variable (denoted UNINFL) is defined as the difference between the realised inflation rate, computed

as the percentage change in the Retail Price Index (RPI), and an estimated series of expected inflation. The latter series was produced by fitting an ARMA model to the actual series and making a one-period(month)-ahead forecast, then rolling the sample forward one period, and re-estimating the parameters and making another one-step-ahead forecast, and so on.

- *Dividend yields* (denoted DIVY) have been widely used to model stock market returns, and also real estate property returns, based on the assumption that movements in the dividend yield series are related to long-term business conditions and that they capture some predictable components of returns.

All variables to be included in the VAR are required to be stationary in order to carry out joint significance tests on the lags of the variables. Hence, all variables are subjected to augmented Dickey–Fuller (ADF) tests (see Chapter 8). Evidence that the log of the RPI and the log of the unemployment rate both contain a unit root is observed. Therefore, the first differences of these variables are used in subsequent analysis. The remaining four variables led to rejection of the null hypothesis of a unit root in the log-levels, and hence these variables were not first differenced.

7.15.2 Methodology

A reduced form VAR is employed and therefore each equation can effectively be estimated using OLS. For a VAR to be unrestricted, it is required that the same number of lags of all of the variables is used in all equations. Therefore, in order to determine the appropriate lag lengths, the multivariate generalisation of Akaike's information criterion (*AIC*) is used.

Within the framework of the VAR system of equations, the significance of all the lags of each of the individual variables is examined jointly with an *F*-test. Since several lags of the variables are included in each of the equations of the system, the coefficients on individual lags may not appear significant for all lags, and may have signs and degrees of significance that vary with the lag length. However, *F*-tests will be able to establish whether all of the lags of a particular variable are jointly significant. In order to consider further the effect of the macro-economy on the real estate returns index, the impact multipliers (orthogonalised impulse responses) are also calculated for the estimated VAR model. Two standard error bands are calculated using the Monte Carlo integration approach employed by McCue and Kling (1994), and based on Doan (1984). The forecast error variance is also decomposed to determine the proportion of the movements in the real estate series that are a consequence of its own shocks rather than shocks to other variables.

7.15.3 Results

The number of lags that minimises the value of Akaike's information criterion is fourteen, consistent with the fifteen lags used by McCue and Kling (1994). There are thus (1 + 14 × 6) = 85 variables in each equation, implying fifty-nine degrees of freedom. *F*-tests for the null hypothesis that all of the lags of a given variable are jointly insignificant in a given equation are presented in Table 7.4.

Table 7.4 Marginal significance levels associated with joint F-tests

variable	Lags of variable					
	SIR	DIVY	SPREAD	UNEM	UNINFL	PRO
SIR	0.0000	0.0091	0.0242	0.0327	0.2126	0.000
DIVY	0.5025	0.0000	0.6212	0.4217	0.5654	0.403
SPREAD	0.2779	0.1328	0.0000	0.4372	0.6563	0.000
UNEM	0.3410	0.3026	0.1151	0.0000	0.0758	0.276
UNINFL	0.3057	0.5146	0.3420	0.4793	0.0004	0.388
PROPRES	0.5537	0.1614	0.5537	0.8922	0.7222	0.000

The test is that all fourteen lags have no explanatory power for that particular equation in the VAR.

Source: Brooks and Tsolacos (1999).

In contrast to a number of US studies which have used similar variables, it is found to be difficult to explain the variation in the UK real estate returns index using macroeconomic factors, as the last row of Table 7.4 shows. Of all the lagged variables in the real estate equation, only the lags of the real estate returns themselves are highly significant, and the dividend yield variable is significant only at the 20% level. No other variables have any significant explanatory power for the real estate returns. Therefore, based on the *F*-tests, an initial conclusion is that the variation in property returns, net of stock market influences, cannot be explained by any of the main macroeconomic or financial variables used in existing research. One possible explanation for this might be that, in the UK, these variables do not convey the information about the macro-economy and business conditions assumed to determine the intertemporal behaviour of property returns. It is possible that property returns may reflect property market influences, such as rents, yields or capitalisation rates, rather than macroeconomic or financial variables. However, again the use of monthly data limits the set of both macroeconomic and property market variables that can be used in the quantitative analysis of real estate returns in the UK.

It appears, however, that lagged values of the real estate variable have explanatory power for some other variables in the system. These results are shown in the last column of Table 7.4. The property sector appears to help in explaining variations in the term structure and short-term interest rates, and moreover since these variables are not significant in the property index equation, it is possible to state further that the property residual series Granger-causes the short-term interest rate and the term spread. This is a bizarre result. The fact that property returns are explained by own lagged values – i.e., that is there is interdependency between neighbouring data points (observations) – may reflect the way that property market information is produced and reflected in the property return indices.

Table 7.5 gives variance decompositions for the property returns index equation of the VAR for one, two, three, four, twelve and twenty-four steps ahead for the two variable orderings

Order I: PROPRES, DIVY, UNINFL, UNEM, SPREAD, SIR

Order II: SIR, SPREAD, UNEM, UNINFL, DIVY, PROPRES

Unfortunately, the ordering of the variables is important in the decomposition. Thus two orderings are applied, which are the exact opposite of one another, and the sensitivity of the result is considered. It is clear that by the two-year forecasting horizon, the variable ordering has become almost irrelevant in most cases. An interesting feature of the results is that shocks to the term spread and unexpected inflation together account for over 50% of the variation in the real estate series. The short-term interest rate and dividend yield shocks account for only 10–15% of the variance of the property index. One possible explanation for the difference in results between the *F*-tests and the variance decomposition is that the former is a causality test and the latter is effectively an exogeneity test. Hence the latter implies the stronger restriction that both current and lagged shocks to the explanatory variables do not influence the current value of the dependent variable of the property equation. Another way of stating this is that the term structure and unexpected inflation have a contemporaneous rather than a lagged effect on the property index, which implies insignificant *F*-test statistics but explanatory power in the variance decomposition. Therefore, although the *F*-tests did not establish any significant effects, the error variance decompositions show evidence of a contemporaneous relationship between PROPRES and both SPREAD and UNINFL. The lack of lagged effects could be taken to imply speedy adjustment of the market to changes in these variables.

Table 7.5 Variance decompositions for the property sector index residuals

Months ahead	Explained by innovations in									
	SIR		DIVY		SPREAD		UNEM		UNINFL	
	I	II	I	II	I	II	I	II	I	II
1	0.0	0.8	0.0	38.2	0.0	9.1	0.0	0.7	0.0	0.2
2	0.2	0.8	0.2	35.1	0.2	12.3	0.4	1.4	1.6	2.9
3	3.8	2.5	0.4	29.4	0.2	17.8	1.0	1.5	2.3	3.0
4	3.7	2.1	5.3	22.3	1.4	18.5	1.6	1.1	4.8	4.4
12	2.8	3.1	15.5	8.7	15.3	19.5	3.3	5.1	17.0	13.5
24	8.2	6.3	6.8	3.9	38.0	36.2	5.5	14.7	18.1	16.9

Source: Brooks and Tsolacos (1999).

Figures 7.1 and 7.2 give the impulse responses for PROPRES associated with separate unit shocks to unexpected inflation and the dividend yield, as examples (as stated above, a total of thirty-six impulse responses could be calculated since there are six variables in the system).

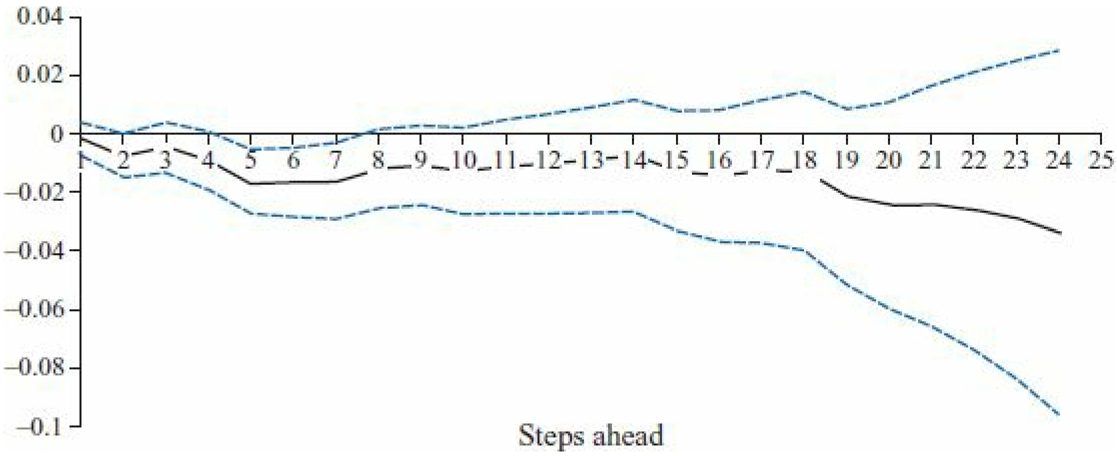


Figure 7.1 Impulse responses and standard error bands for innovations in unexpected inflation equation errors

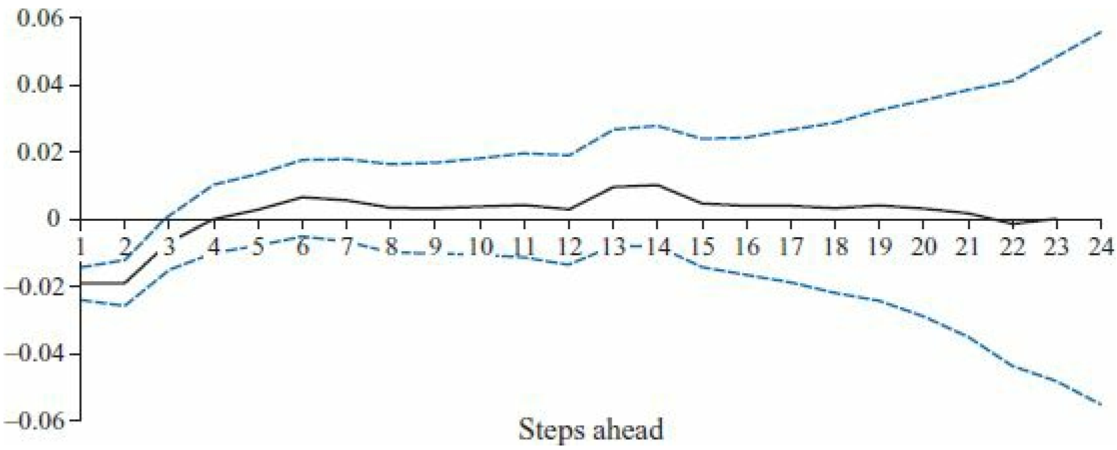


Figure 7.2 Impulse responses and standard error bands for innovations in the dividend yields

Considering the signs of the responses, innovations to unexpected inflation (Figure 7.1) always have a negative impact on the real estate index, since the impulse response is negative, and the effect of the shock does not die down, even after twenty-four months. Increasing stock dividend yields (Figure 7.2) have a negative impact for the first three periods, but beyond that, the shock appears to have worked its way out of the system.

7.15.4 Conclusions

The conclusion from the VAR methodology adopted in the Brooks and Tsolacos paper is that overall, UK real estate returns are difficult to explain on the basis of the information contained in the set of the variables used in existing studies based on non-UK data. The results are not strongly suggestive of any significant influences of these variables on the variation of the filtered property returns series. There is, however, some evidence that the interest rate term structure and unexpected inflation have a contemporaneous effect on property returns, in agreement with the results of a number of previous studies.

7.16 A Couple of Final Points on VARs

The VAR approach to model-building has become enormously popular over the past three decades, due in part to their simplicity but also to their flexibility. Two further ways in which the standard approach to VAR construction can be extended are

- The possibility of latent (hidden) variables can be accounted for by using a VAR in state space form, which can then be estimated via the Kalman filter (see Chapter 15 of this book for further details on the latter)
- Non-linear VARs can be constructed, incorporating Markov switching regimes or threshold dynamics (see Chapter 10 for a discussion of these models)

The free e-book by Ouliaris, Pagan and Restrepo (2016) contains further details on many aspects of VAR models and their implementation in EViews.

KEY CONCEPTS

The key terms to be able to define and explain from this chapter are

- endogenous variable
- simultaneous equations bias
- order condition
- Hausman test
- structural form
- indirect least squares
- vector autoregression
- impulse response
- exogenous variable
- identified

- rank condition
- reduced form
- instrumental variables
- two-stage least squares
- Granger causality
- variance decomposition

SELF-STUDY QUESTIONS

1. Consider the following simultaneous equations system

$$y_{1t} = \alpha_0 + \alpha_1 y_{2t} + \alpha_2 y_{3t} + \alpha_3 X_{1t} + \alpha_4 X_{2t} + u_{1t} \quad \textbf{(7.90)}$$

$$y_{2t} = \beta_0 + \beta_1 y_{3t} + \beta_2 X_{1t} + \beta_3 X_{3t} + u_{2t} \quad \textbf{(7.91)}$$

$$y_{3t} = \gamma_0 + \gamma_1 y_{1t} + \gamma_2 X_{2t} + \gamma_3 X_{3t} + u_{3t} \quad \textbf{(7.92)}$$

(a) Derive the reduced form equations corresponding to equations (7.90)–(7.92).

(b) What do you understand by the term ‘identification’? Describe a rule for determining whether a system of equations is identified. Apply this rule to equations (7.90)–(7.92). Does this rule guarantee that estimates of the structural parameters can be obtained?

(c) Which would you consider the more serious misspecification: treating exogenous variables as endogenous, or treating endogenous variables as exogenous? Explain your answer.

(d) Describe a method of obtaining the structural form coefficients corresponding to an overidentified system.

(e) Using EViews, estimate a VAR model for the interest rate series used in the principal components example of Chapter 4. Use a method for selecting the lag length in the VAR optimally. Determine whether certain maturities lead or lag others, by conducting Granger causality tests and plotting impulse responses and variance decompositions. Is there any evidence that new information is reflected more quickly in some maturities than others?

2. Consider the following system of two equations

$$y_{1t} = \alpha_0 + \alpha_1 y_{2t} + \alpha_2 X_{1t} + \alpha_3 X_{2t} + u_{1t} \quad \textbf{(7.93)}$$

$$y_{2t} = \beta_0 + \beta_1 y_{1t} + \beta_2 X_{1t} + u_{2t} \quad \textbf{(7.94)}$$

(a) Explain, with reference to these equations, the undesirable consequences that would arise if equations (7.93) and (7.94) were estimated separately using OLS.

(b) What would be the effect upon your answer to (a) if the variable y_{1t} had not appeared in equation (7.94)?

(c) State the order condition for determining whether an equation which is part of a system is identified. Use this condition to determine whether equations (7.93) or (7.94) or both or neither are identified.

(d) Explain whether indirect least squares (ILS) or two-stage least squares (2SLS) could be used to obtain the parameters of equation (7.93) and (7.94). Describe how each of these two procedures (ILS and 2SLS) are used to calculate the parameters of an equation. Compare and evaluate the usefulness of ILS, 2SLS and IV.

(e) Explain briefly the Hausman procedure for testing for exogeneity.

3. Explain, using an example if you consider it appropriate, what you understand by the equivalent terms ‘recursive equations’ and ‘triangular system’. Can a triangular system be validly estimated using OLS? Explain your answer.

4. Consider the following vector autoregressive model

$$y_t = \beta_0 + \sum_{i=1}^{k} \beta_i y_{t-i} + u_t \quad \textbf{(7.95)}$$

where y_t is a $p \times 1$ vector of variables determined by k lags of all p variables in the system, u_t is a $p\times 1$ vector of error terms, β_0 is a $p\times 1$ vector of constant term coefficients and β_i are $p \times p$ matrices of coefficients on the ith lag of y.

(a) If $p = 2$, and $k = 3$, write out all the equations of the VAR in full, carefully defining any new notation you use that is not given in the question.

(b) Why have VARs become popular for application in economics and finance, relative to structural models derived from some underlying theory?

(c) Discuss any weaknesses you perceive in the VAR approach to econometric modelling.

(d) Two researchers, using the same set of data but working independently, arrive at different lag lengths for the VAR equation (7.95). Describe and evaluate two methods for determining which of the lag lengths is more appropriate.

5. Define carefully the following terms

- Simultaneous equations system
- Exogenous variables
- Endogenous variables
- Structural form model
- Reduced form model.

8

Modelling Long-Run Relationships in Finance

LEARNING OUTCOMES

In this chapter, you will learn how to

- Highlight the problems that may occur if non-stationary data are used in their levels form
- Test for unit roots
- Examine whether systems of variables are cointegrated
- Estimate error correction and vector error correction models
- Explain the intuition behind Johansen's test for cointegration
- Describe how to test hypotheses in the Johansen framework

8.1 Stationarity and Unit Root Testing

8.1.1 Why are Tests for Non-Stationarity Necessary?

There are several reasons why the concept of non-stationarity is important and why it is essential that variables that are non-stationary be treated differently from those that are stationary. Two definitions of non-stationarity were presented at the start of Chapter 6. For the purpose of the analysis in this chapter, a stationary series can be defined as one with a *constant mean, constant variance* and *constant autocovariances* for each given lag. Therefore, the discussion in this chapter relates to the concept of weak stationarity. An examination of whether a series can be viewed as stationary or not is essential for the following reasons

- The stationarity or otherwise of a series can *strongly influence its* *behaviour and properties*. To offer one illustration, the word 'shock' is usually used to denote a change or an unexpected change in a variable or perhaps simply the value of the error term during a particular time period. For a stationary series, 'shocks' to the system will gradually die away. That is, a shock during time t will have a smaller effect in time $t + 1$, a smaller effect still in time $t + 2$, and so on. This can be contrasted with the case of non-stationary data, where the persistence of shocks will always be infinite, so that for a non-stationary series, the effect of a shock during time t will not have a smaller effect in time $t + 1$, and in time $t + 2$, etc.

- The use of non-stationary data can lead to *spurious regressions*. If two stationary variables are generated as independent random series, when one of those variables is regressed on the other, the t-ratio on the slope coefficient would be expected not to be significantly different from zero, and the value of R^2 would be expected to be very low. This seems obvious, for the variables are not related to one another. However, if two variables are trending over time, a regression of one on the other could have a high R^2 even if the two are totally unrelated. So, if standard regression techniques are applied to non-stationary data, the end result could be a regression that 'looks' good under standard measures (significant coefficient estimates and a high R^2), but which is really valueless. Such a model would be termed a 'spurious regression'.

  To give an illustration of this, two independent sets of non-stationary variables, y and x, were generated with sample size 500, one regressed on the other and the R^2 noted. This was repeated 1,000 times to obtain 1,000 R^2 values. A histogram of these values is given in Figure 8.1.

  As Figure 8.1 shows, although one would have expected the R^2 values for each regression to be close to zero, since the explained and explanatory variables in each case are independent of one another, in fact R^2 takes on values across the whole range. For one set of data, R^2 is bigger than 0.9, while it is bigger than 0.5 over 16% of the time!

- If the variables employed in a regression model are *not stationary*, then it can be proved that the standard assumptions for asymptotic analysis will not be valid. In other words, the usual 't-ratios' will not follow a t-distribution, and the F-statistic will not follow an F-distribution, and so on. Using the same simulated data as used to produce Figure 8.1, Figure 8.2 plots a histogram of the estimated t-

ratio on the slope coefficient for each set of data.

In general, if one variable is regressed on another unrelated variable, the *t*-ratio on the slope coefficient will follow a *t*-distribution. For a sample of size 500, this implies that 95% of the time, the *t*-ratio will lie between ±2. As Figure 8.2 shows quite dramatically, however, the standard *t*-ratio in a regression of non-stationary variables can take on enormously large values. In fact, in the above example, the *t*-ratio is bigger than 2 in absolute value over 98% of the time, when it should be bigger than 2 in absolute value only approximately 5% of the time! Clearly, it is therefore not possible to validly undertake hypothesis tests about the regression parameters if the data are non-stationary.

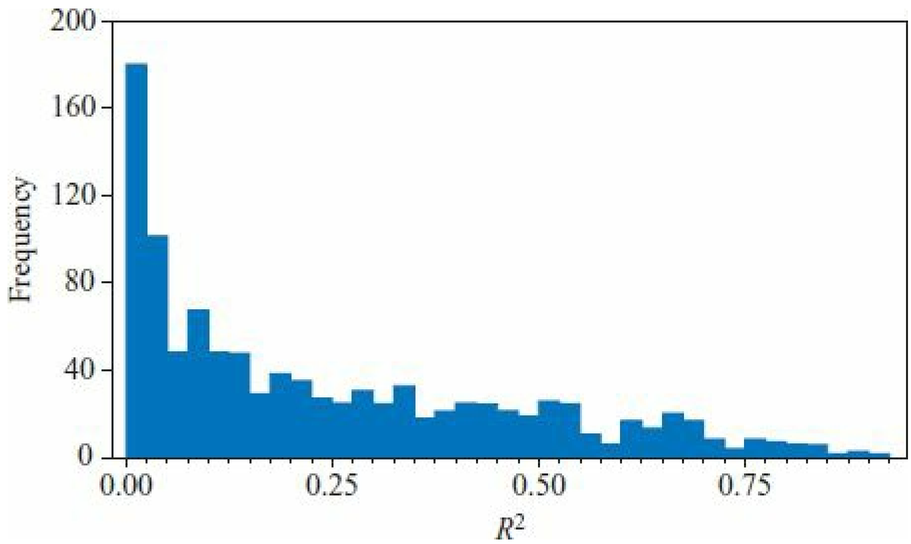


Figure 8.1 Value of R^2 for 1000 sets of regressions of a non-stationary variable on another independent non-stationary variable

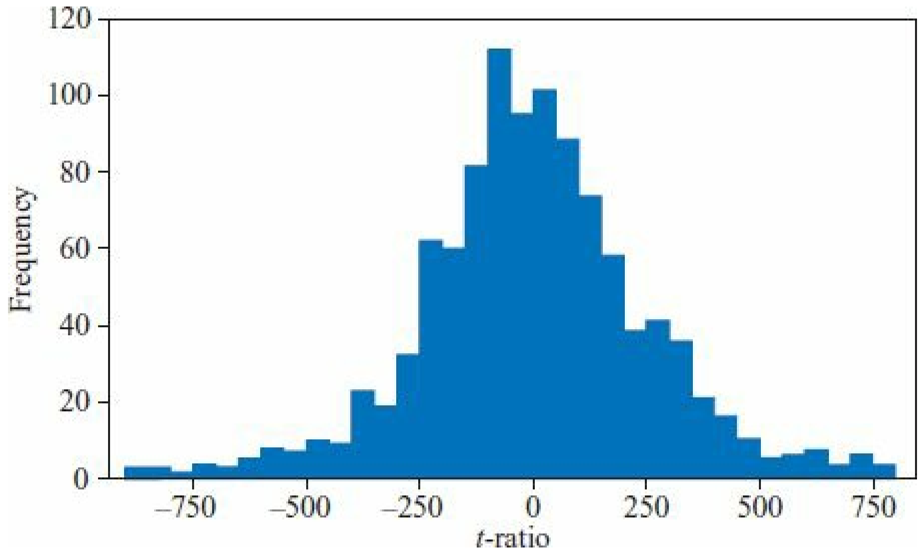


Figure 8.2 Value of *t*-ratio of slope coefficient for 1,000 sets of regressions of a non-stationary variable on another independent non-stationary variable

8.1.2 Two Types of Non-Stationarity

There are two models that have been frequently used to characterise the non-stationarity, the *random walk model with drift*

$$y_t = \mu + y_{t-1} + u_t \qquad \textbf{(8.1)}$$

and the *trend-stationary process* – so called because it is stationary around a linear trend

$$y_t = \alpha + \beta t + u_t \qquad \textbf{(8.2)}$$

where u_t is a white noise disturbance term in both cases.

Note that the model (8.1) could be generalised to the case where y_t is an explosive process

$$y_t = \mu + \phi y_{t-1} + u_t \qquad \textbf{(7.47)}$$

where $\phi > 1$. Typically, this case is ignored and $\phi = 1$ is used to characterise the non-stationarity because $\phi > 1$ does not describe many data series in economics and finance, but $\phi = 1$ has been found to describe accurately many financial and economic time series. Moreover, $\phi > 1$ has an intuitively unappealing property: shocks to the system are not only persistent through time, they are propagated so that a given shock will have an increasingly large influence. In other words, the effect of a shock during time t will have a larger effect in time $t + 1$, a larger effect still in time $t + 2$, and so on. To see this, consider the general case of an AR(1) with no drift

$$y_t = \phi y_{t-1} + u_t \tag{8.4}$$

Let ϕ take any value for now. Lagging equation (8.4) one and then two periods

$$y_{t-1} = \phi y_{t-2} + u_{t-1} \tag{8.5}$$

$$y_{t-2} = \phi y_{t-3} + u_{t-2} \tag{8.6}$$

Substituting into equation (8.4) from equation (8.5) for y_{t-1} yields

$$y_t = \phi(\phi y_{t-2} + u_{t-1}) + u_t \tag{8.7}$$

$$y_t = \phi^2 y_{t-2} + \phi u_{t-1} + u_t \tag{8.8}$$

Substituting again for y_{t-2} from equation (8.6)

$$y_t = \phi^2(\phi y_{t-3} + u_{t-2}) + \phi u_{t-1} + u_t \tag{8.9}$$

$$y_t = \phi^3 y_{t-3} + \phi^2 u_{t-2} + \phi u_{t-1} + u_t \tag{8.10}$$

T successive substitutions of this type lead to

$$y_t = \phi^{T+1} y_{t-(T+1)} + \phi u_{t-1} + \phi^2 u_{t-2} + \phi^3 u_{t-3} + \cdots + \phi^T u_{t-T} + u_t \tag{8.11}$$

There are three possible cases:

(1) $\phi < 1 \Rightarrow \phi^T \rightarrow 0$ as $T \rightarrow \infty$
So the shocks to the system gradually die away – this is the *stationary case*.

(2) $\phi = 1 \Rightarrow \phi^T = 1 \;\forall\; T$
So shocks persist in the system and never die away. The following is obtained

$$y_t = y_0 + \sum_{t=0}^{\infty} u_t \text{ as } T \rightarrow \infty \tag{8.12}$$

So the current value of y is just an infinite sum of past shocks plus some starting value of y_0. This is known as the *unit root case*, for the root of the characteristic equation would be unity.

(3) $\phi > 1$. Now given shocks become more influential as time goes on, since if $\phi > 1$, $\phi^3 > \phi^2 > \phi$, etc. This is the *explosive case* which, for the reasons listed above, will not be considered as a plausible description of the data.

Going back to the two characterisations of non-stationarity, the random walk with drift

$$y_t = \mu + y_{t-1} + u_t \tag{8.13}$$

and the trend-stationary process

$$y_t = \alpha + \beta t + u_t \tag{8.14}$$

The two will require different treatments to induce stationarity. The second case is known as *deterministic non-stationarity* and de-trending is required. In other words, if it is believed that only this class of non-stationarity is present, a regression of the form given in equation (8.14) would be run, and any subsequent estimation would be done on the residuals from equation (8.14), which would have had the linear trend removed.

The first case is known as stochastic non-stationarity, where there is a stochastic trend in the data. Letting $\Delta y_t = y_t - y_{t-1}$ and $Ly_t = y_{t-1}$ so that $(1 - L)\, y_t = y_t - Ly_t = y_t - y_{t-1}$. If equation (8.13) is taken and y_{t-1} subtracted from both sides

$$y_t - y_{t-1} = \mu + u_t \tag{8.15}$$

$$(1 - L)\, y_t = \mu + u_t \tag{8.16}$$

$$\Delta y_t = \mu + u_t \tag{8.17}$$

There now exists a new variable Δy_t, which will be stationary. It would be said that stationarity has been induced by ‘differencing once’. It should also be apparent from the representation given by equation (8.16) why y_t is also known as a *unit root process*: i.e., that the root of the characteristic equation $(1 - z) = 0$, will be unity.

Although trend-stationary and difference-stationary series are both ‘trending’ over time, the correct approach needs to be used in each case. If first differences of a trend-stationary series were taken, it would ‘remove’ the non-stationarity, but at the expense of introducing an MA(1) structure into the errors. To see this, consider the trend-stationary model

$$y_t = \alpha + \beta t + u_t \tag{8.18}$$

This model can be expressed for time $t - 1$, which would be obtained by removing 1 from all of the time subscripts in equation (8.18)

$$y_{t-1} = \alpha + \beta(t-1) + u_{t-1} \tag{8.19}$$

Subtracting equation (8.19) from equation (8.18) gives

$$\Delta y_t = \beta + u_t - u_{t-1} \tag{8.20}$$

Not only is this a moving average in the errors that has been created, it is a noninvertible MA (i.e., one that cannot be expressed as an autoregressive process). Thus the series, Δy_t would in this case have some very undesirable properties.

Conversely if one tried to de-trend a series which has stochastic trend, then the non-stationarity would not be removed. Clearly then, it is not always obvious which way to proceed. One possibility is to nest both cases in a more general model and to test that. For example, consider the model

$$\Delta y_t = \alpha_0 + \alpha_1 t + (\gamma - 1) y_{t-1} + u_t \tag{8.21}$$

Although again, of course the t-ratios in equation (8.21) will not follow a t-distribution and thus hypotheses about these parameters cannot be tested unless y is actually stationary in levels. Such a model could allow for both deterministic and stochastic non-stationarity. However, this book will now concentrate on the stochastic stationarity model since it is the model that has been found to best describe most non-stationary financial and economic time series. Consider again the simplest stochastic trend model

$$y_t = y_{t-1} + u_t \tag{8.22}$$

or

$$\Delta y_t = u_t \tag{8.23}$$

This concept can be generalised to consider the case where the series contains more than one 'unit root'. That is, the first difference operator, Δ, would need to be applied more than once to induce stationarity. This situation will be described later in this chapter.

Arguably the best way to understand the ideas discussed above is to consider some diagrams showing the typical properties of certain relevant types of processes. Figure 8.3 plots a white noise (pure random) process, while Figures 8.4 and 8.5 plot a random walk versus a random walk with drift and a deterministic trend process, respectively.

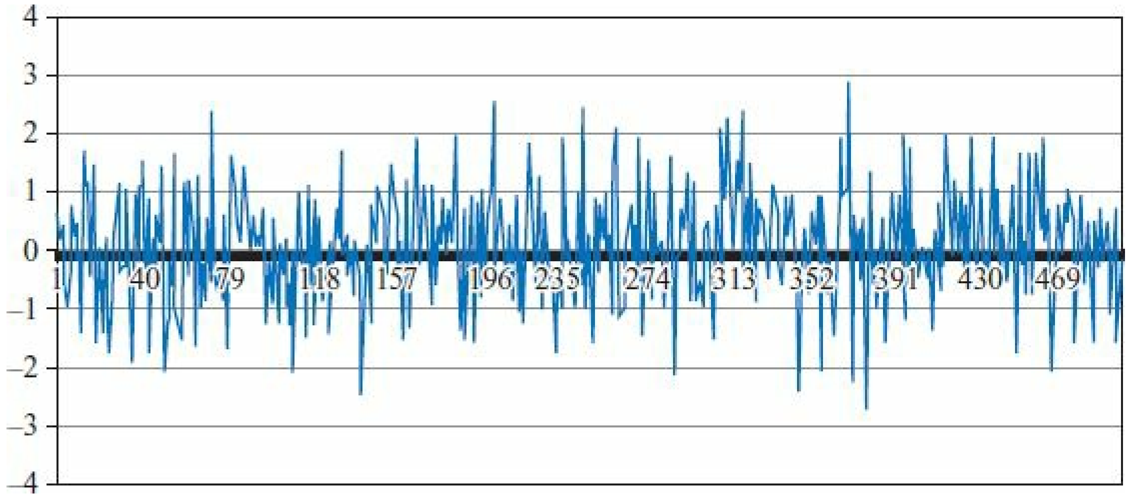


Figure 8.3 Example of a white noise process

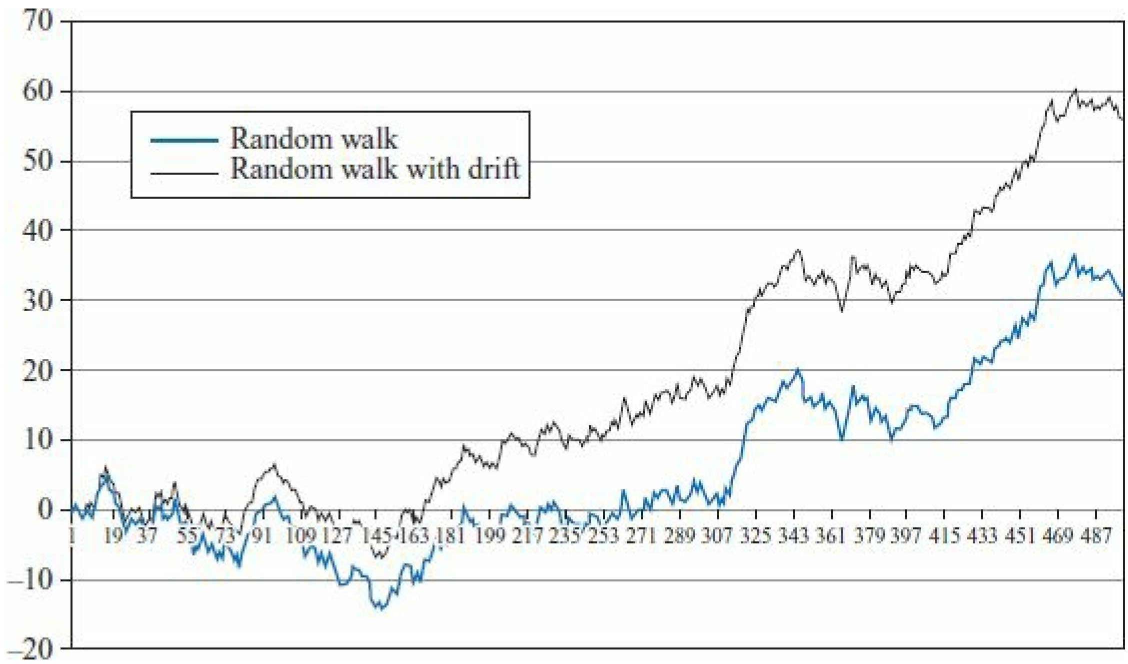


Figure 8.4 Time-series plot of a random walk versus a random walk with

drift

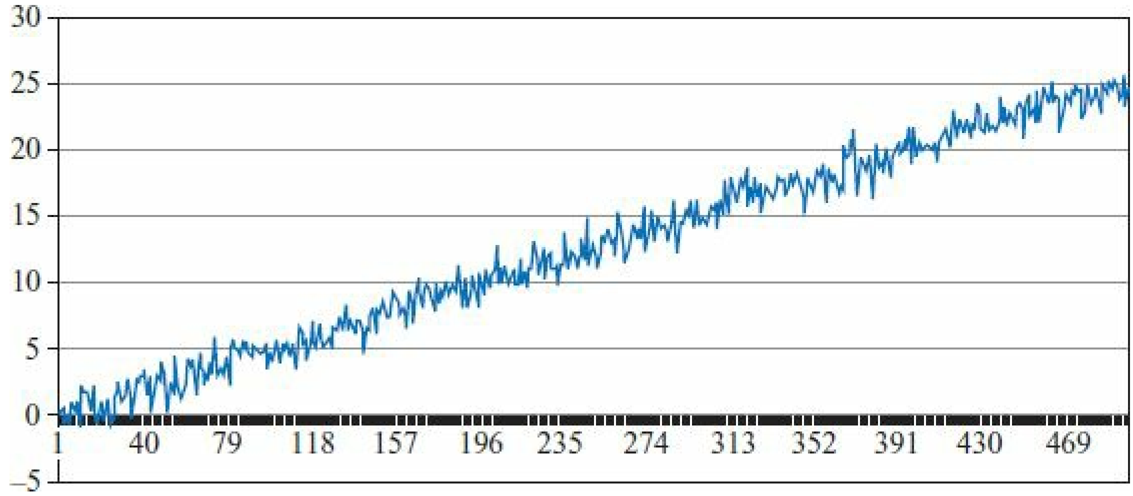


Figure 8.5 Time-series plot of a deterministic trend process

Comparing these three figures gives a good idea of the differences between the properties of a stationary, a stochastic trend and a deterministic trend process. In Figure 8.3, a white noise process visibly has no trending behaviour, and it frequently crosses its mean value of zero. The random walk (thick line) and random walk with drift (faint line) processes of Figure 8.4 exhibit 'long swings' away from their mean value, which they cross very rarely. A comparison of the two lines in this graph reveals that the positive drift leads to a series that is more likely to rise over time than to fall; obviously, the effect of the drift on the series becomes greater and greater the further the two processes are tracked. Finally, the deterministic trend process of Figure 8.5 clearly does not have a constant mean, and exhibits completely random fluctuations about its upward trend. If the trend were removed from the series, a plot similar to the white noise process of Figure 8.3 would result. In this author's opinion, more time series in finance and economics look like Figure 8.4 than either Figure 8.3 or 8.5. Consequently, as stated above, the stochastic trend model will be the focus of the remainder of this chapter.

Finally, Figure 8.6 plots the value of an autoregressive process of order 1 with different values of the autoregressive coefficient as given by equation (8.4). Values of $\phi = 0$ (i.e., a white noise process), $\phi = 0.8$ (i.e., a stationary AR(1)) and $\phi = 1$ (i.e., a random walk) are plotted over time.

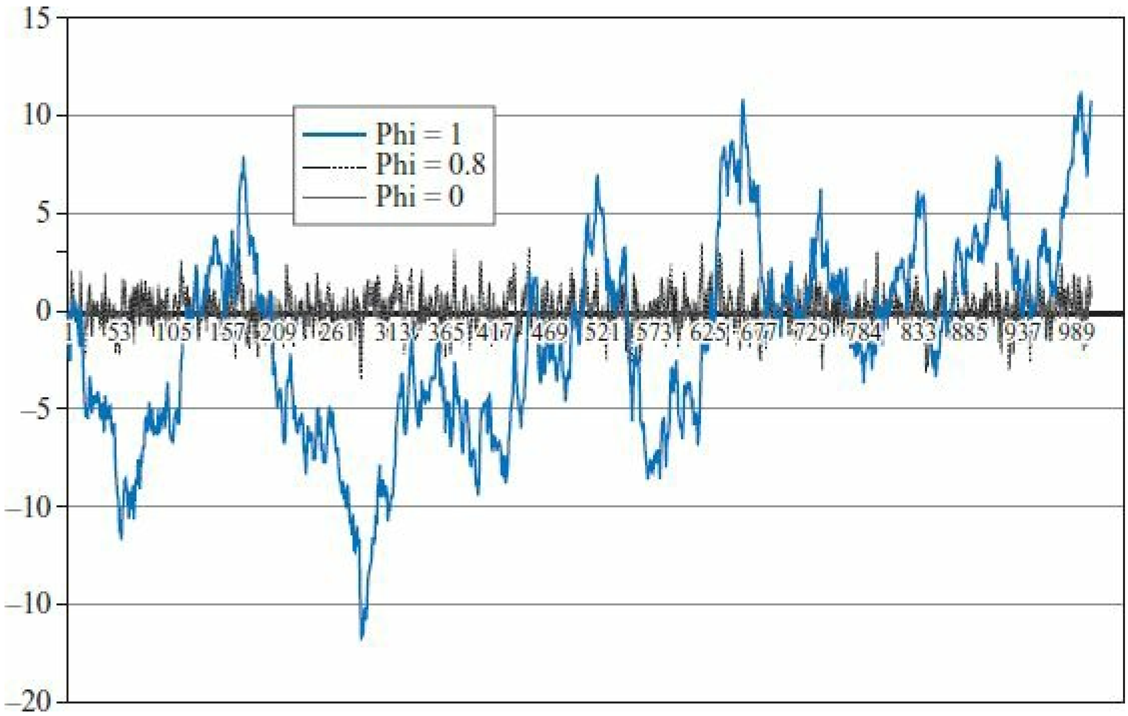


Figure 8.6 Autoregressive processes with differing values of ϕ (0, 0.8, 1)

8.1.3 Some More Definitions and Terminology

If a non-stationary series, y_t must be differenced d times before it becomes stationary, then it is said to be integrated of order d. This would be written $y_t \sim I(d)$. So if $y_t \sim I(d)$ then $\Delta^d y_t \sim I(0)$. This latter piece of terminology states that applying the difference operator, Δ, d times, leads to an I(0) process, i.e., a process with no unit roots. In fact, applying the difference operator more than d times to an I(d) process will still result in a stationary series (but with an MA error structure). An I(0) series is a stationary series, while an I(1) series contains one unit root. For example, consider the random walk

$$y_t = y_{t-1} + u_t \tag{8.24}$$

An I(2) series contains two unit roots and so would require differencing twice to induce stationarity. I(1) and I(2) series can wander a long way from their mean value and cross this mean value rarely, while I(0) series should cross the mean frequently. The majority of financial and economic time series contain a single unit root, although some are stationary and some have been argued to possibly contain two unit roots (series such as nominal consumer prices and nominal wages). The efficient markets hypothesis together with rational expectations suggest that asset prices (or the natural logarithms of asset prices) should follow a random walk or a random walk with drift, so that their differences are unpredictable (or only predictable to their long-term average value).

To see what types of data generating process could lead to an I(2) series, consider the equation

$$y_t = 2y_{t-1} - y_{t-2} + u_t \tag{8.25}$$

taking all of the terms in y over to the LHS, and then applying the lag operator notation

$$y_t - 2y_{t-1} + y_{t-2} = u_t \tag{8.26}$$

$$(1 - 2L + L^2)y_t = u_t \tag{8.27}$$

$$(1 - L)(1 - L)y_t = u_t \tag{8.28}$$

It should be evident now that this process for y_t contains two unit roots, and would require differencing twice to induce stationarity.

What would happen if y_t in equation (8.25) were differenced only once? Taking first differences of equation (8.25), i.e., subtracting y_{t-1} from both sides

$$y_t - y_{t-1} = y_{t-1} - y_{t-2} + u_t \tag{8.29}$$

$$y_t - y_{t-1} = (y_t - y_{t-1})_{-1} + u_t \tag{8.30}$$

$$\Delta y_t = \Delta y_{t-1} + u_t \tag{8.31}$$

$$(1 - L)\Delta y_t = u_t \tag{8.32}$$

First differencing would therefore have removed one of the unit roots, but there is still a unit root remaining in the new variable, Δy_t.

8.1.4 Testing for a Unit Root

One immediately obvious (but inappropriate) method that readers may think of to test for a unit root would be to examine the autocorrelation function of the series of interest. However, although shocks to a unit root process will remain in the system indefinitely, the acf for a unit root process (a random walk) will often be seen to decay away very slowly to zero. Thus, such a process may be mistaken for a highly persistent but stationary process. Hence it is not possible to use the acf or pacf to determine whether a series is characterised by a unit root or not.

Furthermore, even if the true data generating process for y_t contains a unit root, the results of the tests for a given sample could lead one to believe that the process is stationary. Therefore, what is required is some kind of formal hypothesis testing procedure that answers the question, 'given the sample of data to hand, is it plausible that the true data generating process for y contains one or more unit roots?'

The early and pioneering work on testing for a unit root in time series was done by Dickey and Fuller (Fuller, 1976; Dickey and Fuller, 1979). The basic objective of the test is to examine the null hypothesis that $\phi = 1$ in

$$y_t = \phi y_{t-1} + u_t \tag{8.33}$$

against the one-sided alternative $\phi < 1$. Thus the hypotheses of interest are H_0: series contains a unit root versus H_1: series is stationary.

In practice, the following regression is employed, rather than equation (8.33), for ease of computation and interpretation

$$\Delta y_t = \psi y_{t-1} + u_t \tag{8.34}$$

so that a test of $\phi = 1$ is equivalent to a test of $\psi = 0$ (since $\phi - 1 = \psi$).

Dickey–Fuller (DF) tests are also known as τ-tests, and can be conducted allowing for an intercept, or an intercept and deterministic trend, or neither, in the test regression. The model for the unit root test in each case is

$$y_t = \phi y_{t-1} + \mu + \lambda t + u_t \tag{8.35}$$

The tests can also be written, by subtracting y_{t-1} from each side of the equation, as

$$\Delta y_t = \psi y_{t-1} + \mu + \lambda t + u_t \tag{8.36}$$

In another paper, Dickey and Fuller (1981) provide a set of additional test statistics and their critical values for joint tests of the significance of the lagged y, and the constant and trend terms. These are not examined further here. The test statistics for the original DF tests are defined as

$$test\ statistic = \frac{\hat{\psi}}{\hat{SE}(\hat{\psi})} \qquad \mathbf{(8.37)}$$

The test statistics do not follow the usual *t*-distribution under the null hypothesis, since the null is one of non-stationarity, but rather they follow a non-standard distribution. Critical values are derived from simulations experiments in, for example, Fuller (1976); see also Chapter 13 in this book. Relevant examples of the distribution are shown in Table 8.1. A full set of DF critical values is given in the Appendix of Statistical Tables at the end of this book (Appendix 2). A discussion and example of how such critical values (CV) are derived using simulations methods are presented in Chapter 13.

Table 8.1 Critical values for DF tests (Fuller, 1976, p. 373)

Significance level	10%	5%	1%
CV for constant but no trend	−2.57	−2.86	−3.43
CV for constant and trend	−3.12	−3.41	−3.96

Comparing these with the standard normal critical values, it can be seen that the DF critical values are much bigger in absolute terms (i.e., more negative). Thus more evidence against the null hypothesis is required in the context of unit root tests than under standard *t*-tests. This arises partly from the inherent instability of the unit root process, the fatter distribution of the *t*-ratios in the context of non-stationary data (see Figure 8.2), and the resulting uncertainty in inference. The null hypothesis of a unit root is rejected in favour of the stationary alternative in each case if the test statistic is more negative than the critical value.

The tests above are valid only if u_t is white noise. In particular, u_t is assumed not to be autocorrelated, but would be so if there was autocorrelation in the dependent variable of the regression (Δy_t) which has not been modelled. If this is the case, the test would be 'oversized', meaning that the true size of the test (the proportion of times a correct null hypothesis is incorrectly rejected) would be higher than the nominal size used (e.g., 5%). The solution is to 'augment' the test using p lags of the dependent variable. The alternative model in case (i) (equation (8.34)) is now written

$$\Delta y_t = \psi y_{t-1} + \sum_{i=1}^{p} \alpha_i \Delta y_{t-i} + u_t \qquad \textbf{(8.38)}$$

The lags of Δy_t now 'soak up' any dynamic structure present in the dependent variable, to ensure that u_t is not autocorrelated. The test is known as an augmented Dickey–Fuller (ADF) test and is still conducted on ψ, and the same critical values from the DF tables are used as before.

A problem now arises in determining the optimal number of lags of the dependent variable. Although several ways of choosing p have been proposed, they are all somewhat arbitrary, and are thus not presented here. Instead, the following two simple rules of thumb are suggested. First, the *frequency of the data* can be used to decide. So, for example, if the data are monthly, use twelve lags, if the data are quarterly, use four lags, and so on. Clearly, there would not be an obvious choice for the number of lags to use in a regression containing higher frequency financial data (e.g., hourly or daily)! Second, an *information criterion* can be used to decide. So choose the number of lags that minimises the value of an information criterion, as outlined in Chapter 7.

It is quite important to attempt to use an optimal number of lags of the dependent variable in the test regression, and to examine the sensitivity of the outcome of the test to the lag length chosen. In most cases, hopefully the conclusion will not be qualitatively altered by small changes in p, but sometimes it will. Including too few lags will not remove all of the autocorrelation, thus biasing the results, while using too many will increase the coefficient standard errors. The latter effect arises since an increase in the number of parameters to estimate uses up degrees of freedom. Therefore, everything else being equal, the absolute values of the test statistics will be reduced. This will result in a reduction in the power of the test, implying that for a stationary process the null hypothesis of a unit root will be rejected less frequently than would otherwise have been the case.

8.1.5 Testing for Higher Orders of Integration

Consider the simple regression

$$\Delta y_t = \psi y_{t-1} + u_t \qquad \textbf{(8.39)}$$

H_0: $\psi = 0$ is tested against H_1: $\psi < 0$.

If H_0 is rejected, it would simply be concluded that y_t does not contain a unit root. But what should be the conclusion if H_0 is not rejected? The series contains a unit root, but is that it? No! What if $y_t \sim I(2)$? The null hypothesis would still not have been rejected. It is now necessary to perform a test of

$$H_0 : y_t \sim I(2) \text{ vs. } H_1 : y_t \sim I(1)$$

$\Delta^2 y_t (= \Delta y_t - y_{t-1})$ would now be regressed on Δy_{t-1} (plus lags of $\Delta^2 y_t$ to augment the test if necessary). Thus, testing H_0: $\Delta y_t \sim I(1)$ is equivalent to H_0: $y_t \sim I(2)$. So in this case, if H_0 is not rejected (very unlikely in practice), it would be concluded that y_t is at least I(2). If H_0 is rejected, it would be concluded that y_t contains a single unit root. The tests should continue for a further unit root until H_0 is rejected.

Dickey and Pantula (1987) have argued that an ordering of the tests as described above (i.e., testing for I(1), then I(2), and so on) is, strictly speaking, invalid. The theoretically correct approach would be to start by assuming some highest plausible order of integration (e.g., I(2)), and to test I(2) against I(1). If I(2) is rejected, then test I(1) against I(0). In practice, however, to the author’s knowledge, no financial time series contain more than a single unit root, so that this matter is of less concern in finance.

8.1.6 Phillips–Perron (PP) Tests

Phillips and Perron have developed a more comprehensive theory of unit root non-stationarity. The tests are similar to ADF tests, but they incorporate an automatic correction to the DF procedure to allow for autocorrelated residuals. The tests often give the same conclusions as, and suffer from most of the same important limitations as, the ADF tests.

8.1.7 Criticisms of Dickey–Fuller- and Phillips–Perron-Type Tests

The most important criticism that has been levelled at unit root tests is that their power is low if the process is stationary but with a root close to the non-stationary boundary. So, for example, consider an AR(1) data generating process with coefficient 0.95. If the true data generating process is

$$y_t = 0.95y_{t-1} + u_t \tag{8.40}$$

the null hypothesis of a unit root should be rejected. It has been thus argued that the tests are poor at deciding, for example, whether $\phi = 1$ or $\phi = 0.95$, especially with small sample sizes. The source of this problem is that, under the classical hypothesis-testing framework, the null hypothesis is never accepted, it is simply stated that it is either rejected or not rejected. This means that a failure to reject the null hypothesis could occur either because the null was correct, or because there is insufficient information in the sample to enable rejection. One way to get around this problem is to use a stationarity test as well as a unit root test, as described in Box 8.1.

BOX 8.1 Stationarity tests

Stationarity tests have stationarity under the null hypothesis, thus reversing the null and alternatives under the Dickey–Fuller approach. Thus, under stationarity tests, the data will appear stationary by default if there is little information in the sample. One such stationarity test is the KPSS test (Kwaitkowski *et al.*, 1992). The computation of the test statistic is not discussed here but the test is available within standard econometrics software such as EViews. The results of these tests can be compared with the ADF/PP procedure to see if the same conclusion is obtained. The null and alternative hypotheses under each testing approach are as follows:

ADF/PP	*KPSS*
$H_0: y_t \sim I(1)$	$H_0: y_t \sim I(0)$
$H_1: y_t \sim I(0)$	$H_1: y_t \sim I(1)$

There are four possible outcomes

(1)	Reject H_0	and	Do not reject H_0
(2)	Do not reject H_0	and	Reject H_0
(3)	Reject H_0	and	Reject H_0
(4)	Do not reject H_0	and	Do not reject H_0

For the conclusions to be robust, the results should fall under outcomes (1) or (2), which would be the case when both tests concluded that the series is stationary or non-stationary, respectively. Outcomes (3) or (4) imply conflicting results. The joint use of stationarity and unit root tests is known as *confirmatory data analysis*.

8.2 Tests for Unit Roots in the Presence of Structural Breaks

8.2.1 Motivation

The standard Dickey-Fuller-type unit root tests presented above do not perform well if there are one or more structural breaks in the series under investigation, either in the intercept or the slope of the regression. More specifically, the tests have low power in such circumstances and they fail to reject the unit root null hypothesis when it is incorrect as the slope parameter in the regression of y_t on y_{t-1} is biased towards unity by an unparameterised structural break. In general, the larger the break and the smaller the sample, the lower the power of the test. As Leybourne, Mills and Newbold (1998) have shown, unit root tests are also oversized in the presence of structural breaks, so they reject the null hypothesis too frequently when it is correct.[1]

Perron's (1989) work is important since he was able to demonstrate that if we allow for structural breaks in the testing framework, a whole raft of macroeconomic series that Nelson and Plosser (1982) had identified as non-stationary may turn out to be stationary. He argues that most economic time series are best characterised by *broken trend stationary processes*, where the data generating process is a deterministic trend but with a structural break around 1929 that permanently changed the levels (i.e., the intercepts) of the series.

8.2.2 The Perron (1989) Procedure

Recall from above that the flexible framework for unit root testing involves a regression of the form

$$\Delta y_t = \psi y_{t-1} + \mu + \lambda t + \sum_{i=1}^{p} \alpha_i \Delta y_{t-i} + u_t \tag{8.41}$$

where μ is an intercept and λt captures the time trend, one or both of which could be excluded from the regression if they were thought to be unnecessary.

Perron (1989) proposes three test equations differing dependent on the type of break that was thought to be present. The first he terms a 'crash' model that allows a break in the level (i.e., the intercept) of the series; the second is a ‘changing growth’ model that allows for a break in the growth rate (i.e., the slope) of the series; the final model allows for both types of break to occur at the same time, changing both the intercept and the slope of the trend. If we define the break point in the data as T_b, and D_t is a dummy variable defined as

$$D_t = \begin{cases} 0 & \text{if } t < T_b \\ 1 & \text{if } t \geq T_b \end{cases}$$

the general equation for the third type of test (i.e., the most general) is

$$\Delta y_t = \psi y_{t-1} + \mu + \alpha_1 D_t + \alpha_2 (t - T_b) D_t + \lambda t + \sum_{i=1}^{p} \alpha_i \Delta y_{t-i} + u_t \qquad \textbf{(8.42)}$$

For the crash only model, set $\alpha_2 = 0$, while for the changing growth only model, set $\alpha_1 = 0$. In all three cases, there is a unit root with a structural break at T_b under the null hypothesis and a series that is a stationary process with a break under the alternative.

While Perron (1989) commences a new literature on testing for unit roots in the presence of structural breaks, an important limitation of this approach is that it assumes that the break date is known in advance and the test is constructed using this information. It is possible, and perhaps even likely, however, that the date will not be known and must be determined from the data. More seriously, Christiano (1992) has argued that the critical values employed with the test will presume the break date to be chosen exogenously, and yet most researchers will select a break point based on an examination of the data and thus the asymptotic theory assumed will no longer hold.

As a result, Banerjee, Lumsdaine and Stock (1992) and Zivot and Andrews (1992) introduce an approach to testing for unit roots in the presence of structural change that allows the break date to be selected endogenously. Their methods are based on recursive, rolling and sequential tests. For the recursive and rolling tests, Banerjee *et al*. propose four specifications. First, the standard Dickey–Fuller test on the whole sample, which they term $\hat{t}_{DF}$; second, the ADF test is conducted repeatedly on the sub-samples and the minimal DF statistic, $\hat{t}_{DF}^{min}$, is obtained; third, the maximal DF statistic is obtained from the sub-samples, $\hat{t}_{DF}^{max}$; finally, the difference between the maximal and minimal statistics, $\hat{t}_{DF}^{diff} = \hat{t}_{DF}^{max} - \hat{t}_{DF}^{min}$, is taken. For the sequential test, the whole sample is used each time with the following regression being run

$$\Delta y_t = \psi y_{t-1} + \mu + \alpha \tau_t(t_{used}) + \lambda t + \sum_{i=1}^{p} \alpha_i \Delta y_{t-i} + u_t \quad \textbf{(8.43)}$$

where $t_{used} = T_b/T$. The test is run repeatedly for different values of T_b over as much of the data as possible (a 'trimmed sample') that excludes the first few and the last few observations (since it is not possible to reliably detect breaks there). Clearly it is $\tau_t(t_{used})$ that allows for the break, which can either be in the level (where $\tau_t(t_{used}) = 1$ if $t > t_{used}$ and 0 otherwise); or the break can be in the deterministic trend (where $\tau_t(t_{used}) = t - t_{used}$ if $t > t_{used}$ and 0 otherwise). For each specification, a different set of critical values is required, and these can be found in Banerjee *et al.* (1992).

Perron (1997) proposes an extension of the Perron (1989) technique but using a sequential procedure that estimates the test statistic allowing for a break at any point during the sample to be determined by the data. This technique is very similar to that of Zivot and Andrews, except that his is more flexible, and therefore arguably preferable, since it allows for a break under both the null and alternative hypotheses, whereas according to Zivot and Andrews' model it can only arise under the alternative.

A further extension would be to allow for more than one structural break in the series – for example, Lumsdaine and Papell (1997) enhance the Zivot and Andrews (1992) approach to allow for two structural breaks. It is also possible to allow for structural breaks in the cointegrating relationship between series (see Section 8.4 below for a thorough discussion of cointegration) using an extension of the first step in the Engle-Granger approach – see Gregory and Hansen (1996).

8.2.3 An Example: Testing for Unit Roots in EuroSterling Interest Rates

Section 8.11 discusses the expectations hypothesis of the term structure of interest rates based on cointegration between the long and short rates. Clearly, key to this analysis is the question as to whether the interest rates themselves are I(1) or I(0) processes. Perhaps surprisingly, there is not a consensus in the empirical literature on whether this is the case. Brooks and Rew (2002) examine whether EuroSterling interest rates are best viewed as unit root process or not, allowing for the possibility of structural breaks in the series.[2] They argue that failure to account for structural breaks that may be present in the data (caused, for example, by changes in monetary policy or the removal of exchange rate controls) may lead to incorrect inferences regarding the validity or otherwise of the expectations hypothesis. Their sample covers the period 1 January 1981 to 1 September 1997 to total 4,348 data points.

Brooks and Rew (2002) use the standard Dickey–Fuller test, the recursive and sequential tests of Banerjee *et al.* (1992), and their results are presented in Table 8.2. They also employ the rolling test, the Perron (1997) approach and several other techniques that are not shown here due to space limitations.

Table 8.2 Recursive unit root tests for interest rates allowing for structural breaks

Maturity	t_{DF}	Recursive statistics			Sequential statistics	
		$\hat{t}^{max}_{DF}$	$\hat{t}^{min}_{DF}$	$\hat{t}^{diff}_{DF}$	$\bar{t}^{min}_{DF,trend}$	$\bar{t}^{min}_{DF,mean}$
Short rate	−2.44	−1.33	−3.29	1.96	−2.99	−4.79
7-days	−1.95	−1.33	−3.19	1.86	−2.44	−5.65
1-month	−1.82	−1.07	−2.90	1.83	−2.32	−4.78
3-months	−1.80	−1.02	−2.75	1.73	−2.28	−4.02
6-months	−1.86	−1.00	−2.85	1.85	−2.28	−4.10
1-year	−1.97	−0.74	−2.88	2.14	−2.35	−4.55
Critical values	−3.13	−1.66	−3.88	3.21	−4.11	−4.58

Notes: Source: Brooks and Garrett (2002), taken from Tables 1, 4 and 5. $\bar{t}^{min}_{DF,trend}$ denotes the sequential test statistic allowing for a break in the trend, while $\bar{t}^{min}_{DF,mean}$ is the test statistic allowing for a break in the level. The final row presents the 10% level critical values for each type of test obtained from Banerjee *et al.* (1992, p. 278, Table 2).

The findings for the recursive tests are the same as those for the standard DF test, and show that the unit root null should not be rejected at the 10% level for any of the maturities examined. For the sequential tests, the results are slightly more mixed with the break in trend model still showing no signs of rejecting the null hypothesis, while it is rejected for the short, seven-day and the one-month rates when a structural break is allowed for in the mean.

Brooks and Rew's overall conclusion is that the weight of evidence across all the tests they examine indicates that short term interest rates are best viewed as unit root processes that have a structural break in their level around the time of 'Black Wednesday' (16 September 1992) when the UK dropped out of the European Exchange Rate Mechanism (ERM). The longer term-rates, on the other hand, are I(1) processes with no breaks.

8.2.4 Seasonal Unit Roots

As we will discuss in detail in Chapter 10, many time series exhibit seasonal patterns. One approach to capturing such characteristics would be to use deterministic dummy variables at the frequency of the data (e.g., monthly dummy variables if the data are monthly). However, if the seasonal characteristics of the data are themselves changing over time so that their mean is not constant, then the use of dummy variables will be inadequate. Instead, we can entertain the possibility that a series may contain seasonal unit roots, so that it requires seasonal differencing to induce stationarity. We would use the notation $I(d, D)$ to denote a series that is integrated of order d, D and requires differencing d times and seasonal differencing D times to obtain a stationary process. Osborn (1990) develops a test for seasonal unit roots based on a natural extension of the Dickey–Fuller approach. Groups of series with seasonal unit roots may also be seasonally cointegrated. However, Osborn also shows that only a small proportion of macroeconomic series exhibit seasonal unit roots; the majority have seasonal patterns that can better be characterised using dummy variables, which may explain why the concept of seasonal unit roots has not been widely adopted.[3]

8.3 Cointegration

In most cases, if two variables that are I(1) are linearly combined, then the combination will also be I(1). More generally, if a set of variables $X_{i,t}$ with differing orders of integration are combined, the combination will have an order of integration equal to the largest. If $X_{i,t} \sim I(d_i)$ for $i = 1, 2, 3, \ldots, k$ so that there are k variables each integrated of order d_i, and letting

$$z_t = \sum_{i=1}^{k} \alpha_i X_{i,t} \tag{8.44}$$

Then $z_t \sim I(\max d_i)$. z_t in this context is simply a linear combination of the k variables X_i. Rearranging equation (8.44)

$$X_{1,t} = \sum_{i=2}^{k} \beta_i X_{i,t} + z'_t \tag{8.45}$$

where $\beta_i = -\frac{\alpha_i}{\alpha_1}, z'_t = \frac{z_t}{\alpha_1}, i = 2, \ldots, k$. All that has been done is to take one of the variables, $X_{1,t}$, and to rearrange equation (8.44) to make it the subject. It could also be said that the equation has been normalised on $X_{1,t}$. But viewed another way, equation (8.45) is just a regression equation where z'_t is a disturbance term. These disturbances would have some very undesirable properties: in general, z'_t will not be stationary and is autocorrelated if all of the X_i are I(1).

As a further illustration, consider the following regression model containing variables y_t, x_{2t}, x_{3t} which are all I(1)

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + u_t \tag{8.46}$$

For the estimated model, the SRF would be written

$$y_t = \hat{\beta}_1 + \hat{\beta}_2 x_{2t} + \hat{\beta}_3 x_{3t} + \hat{u}_t \tag{8.47}$$

Taking everything except the residuals to the LHS

$$y_t - \hat{\beta}_1 - \hat{\beta}_2 x_{2t} - \hat{\beta}_3 x_{3t} = \hat{u}_t \tag{8.48}$$

Again, the residuals when expressed in this way can be considered a linear combination of the variables. Typically, this linear combination of I(1) variables will itself be I(1), but it would obviously be desirable to obtain residuals that are I(0). Under what circumstances will this be the case? The answer is that a linear combination of I(1) variables will be I(0), in other words stationary, if the variables are *cointegrated*.

8.3.1 Definition of Cointegration (Engle and Granger, 1987)

Let w_t be a $k \times 1$ vector of variables, then the components of w_t are integrated of order (d, b) if

(1) All components of w_t are I(d)

(2) There is at least one vector of coefficients α such that

$$\alpha' w_t \sim \text{I}(d - b)$$

In practice, many financial variables contain one unit root, and are thus I(1), so that the remainder of this chapter will restrict analysis to the case where $d = b = 1$. In this context, a set of variables is defined as cointegrated if a linear combination of them is stationary. Many time series are non-stationary but ‘move together’ over time – that is, there exist some influences on the series (for example, market forces), which imply that the two series are bound by some relationship in the long run. A cointegrating relationship may also be seen as a long-term or equilibrium phenomenon, since it is possible that cointegrating variables may deviate from their relationship in the short run, but their association would return in the long run.

8.3.2 Examples of Possible Cointegrating Relationships in Finance

Financial theory should suggest where two or more variables would be expected to hold some long-run relationship with one another. There are many examples in finance of areas where cointegration might be expected to hold, including

- Spot and futures prices for a given commodity or asset
- Ratio of relative prices and an exchange rate
- Equity prices and dividends.

In all three cases, market forces arising from no-arbitrage conditions suggest that there should be an equilibrium relationship between the series concerned. The easiest way to understand this notion is perhaps to consider what would be the effect if the series were not cointegrated. If there were no cointegration, there would be no long-run relationship binding the series together, so that the series could wander apart without bound. Such an effect would arise since all linear combinations of the series would be non-stationary, and hence would not have a constant mean that would be returned to frequently.

Spot and futures prices may be expected to be cointegrated since they are obviously prices for the same asset at different points in time, and hence will be affected in very similar ways by given pieces of information. The long-run relationship between spot and futures prices would be given by the cost of carry.

Purchasing power parity (PPP) theory states that a given representative basket of goods and services should cost the same wherever it is bought when converted into a common currency. Further discussion of PPP occurs in Section 8.9, but for now suffice it to say that PPP implies that the ratio of relative prices in two countries and the exchange rate between them should be cointegrated. If they did not cointegrate, assuming zero transactions costs, it would be profitable to buy goods in one country, sell them in another, and convert the money obtained back to the currency of the original country.

Finally, if it is assumed that some stock in a particular company is held to perpetuity (i.e., for ever), then the only return that would accrue to that investor would be in the form of an infinite stream of future dividend payments. Hence the discounted dividend model argues that the appropriate price to pay for a share today is the present value of all future dividends. Hence, it may be argued that one would not expect current prices to 'move out of line' with future anticipated dividends in the long run, thus implying that share prices and dividends should be cointegrated.

An interesting question to ask is whether a potentially cointegrating regression should be estimated using the levels of the variables or the logarithms of the levels of the variables. Financial theory may provide an answer as to the more appropriate functional form, but fortunately even if not, Hendry and Juselius (2000) note that if a set of series is cointegrated in levels, they will also be cointegrated in log levels.

8.4 Equilibrium Correction or Error Correction Models

When the concept of non-stationarity was first considered in the 1970s, a usual response was to independently take the first differences of each of the I(1) variables and then to use these first differences in any subsequent modelling process. In the context of univariate modelling (e.g., the construction of ARMA models), this is entirely the correct approach. However, when the relationship between variables is important, such a procedure is inadvisable. While this approach is statistically valid, it does have the problem that pure first difference models have no long-run solution. For example, consider two series, y_t and x_t, that are both I(1). The model that one may consider estimating is

$$\Delta y_t = \beta \Delta x_t + u_t \tag{8.49}$$

One definition of the long run that is employed in econometrics implies that the variables have converged upon some long-term values and are no longer changing, thus $y_t = y_{t-1} = y$; $x_t = x_{t-1} = x$. Hence all the difference terms will be zero in equation (8.49), i.e., $\Delta y_t = 0$; $\Delta x_t = 0$, and thus everything in the equation cancels. Model equation (8.49) has no long-run solution and it therefore has nothing to say about whether x and y have an equilibrium relationship (see Chapter 5).

Fortunately, there is a class of models that can overcome this problem by using combinations of first differenced and lagged levels of cointegrated variables. For example, consider the following equation

$$\Delta y_t = \beta_1 \Delta x_t + \beta_2 (y_{t-1} - \gamma x_{t-1}) + u_t \tag{8.50}$$

This model is known as an *error correction model* or an *equilibrium correction model*, and $y_{t-1} - \gamma x_{t-1}$ is known as the *error correction term*. Provided that y_t and x_t are cointegrated with cointegrating coefficient γ, then $(y_{t-1} - \gamma x_{t-1})$ will be I(0) even though the constituents are I(1). It is thus valid to use OLS and standard procedures for statistical inference on equation (8.50). It is of course possible to have an intercept in either the cointegrating term (e.g., $y_{t-1} - \alpha - \gamma x_{t-1}$) or in the model for Δy_t (e.g., $\Delta y_t = \beta_0 + \beta_1 \Delta x_t + \beta_2(y_{t-1} - \gamma x_{t-1}) + u_t$) or both. Whether a constant is included or not could be determined on the basis of financial theory, considering the arguments on the importance of a constant discussed in Chapter 5.

The error correction model is sometimes termed an equilibrium correction model, and the two terms will be used synonymously for the purposes of this book. Error correction models are interpreted as follows. y is purported to change between $t - 1$ and t as a result of changes in the values of the explanatory variable(s), x, between $t - 1$ and t, and also in part to correct for any disequilibrium that existed during the previous period. Note that the error correction term $(y_{t-1} - \gamma x_{t-1})$ appears in equation (8.50) with a lag. It would be implausible for the term to appear without any lag (i.e., as $y_t - \gamma x_t$), for this would imply that y changes between $t - 1$ and t in response to a disequilibrium at time t. γ defines the long-run relationship between x and y, while β_1 describes the short-run relationship between changes in x and changes in y. Broadly, β_2 describes the speed of adjustment back to equilibrium, and its strict definition is that it measures the proportion of last period's equilibrium error that is corrected for.

Of course, an error correction model can be estimated for more than two variables. For example, if there were three variables, x_t, w_t, y_t, that were cointegrated, a possible error correction model would be

$$\Delta y_t = \beta_1 \Delta x_t + \beta_2 \Delta w_t + \beta_3 (y_{t-1} - \gamma_1 x_{t-1} - \gamma_2 w_{t-1}) + \varepsilon_t \quad \textbf{(8.51)}$$

The *Granger representation theorem* states that if there exists a dynamic linear model with stationary disturbances and the data are I(1), then the variables must be cointegrated of order (1,1).

8.5 Testing for Cointegration in Regression: A Residuals-Based Approach

The model for the equilibrium correction term can be generalised further to include k variables (y and the $k - 1$ xs)

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \cdots + \beta_k x_{kt} + u_t \quad \textbf{(8.52)}$$

u_t should be I(0) if the variables y_t, x_{2t}, …x_{kt} are cointegrated, but u_t will still be non-stationary if they are not.

Thus it is necessary to test the residuals of equation (8.52) to see whether they are non-stationary or stationary. The DF or ADF test can be used on $\hat{u}_t$, using a regression of the form

$$\Delta \hat{u}_t = \psi \hat{u}_{t-1} + v_t \quad \textbf{(8.53)}$$

with v_t an iid error term.

However, since this is a test on residuals of a model, $\hat{u}_t$, then the critical values are changed compared to a DF or an ADF test on a series of raw data. Engle and Granger (1987) have tabulated a new set of critical values for this application and hence the test is known as the Engle–Granger (*EG*) test. The reason that modified critical values are required is that the test is now operating on the residuals of an estimated model rather than on raw data. The residuals have been constructed from a particular set of coefficient estimates, and the sampling estimation error in those coefficients will change the distribution of the test statistic. Engle and Yoo (1987) tabulate a new set of critical values that are larger in absolute value (i.e., more negative) than the DF critical values, also given at the end of this book. The critical values also become more negative as the number of variables in the potentially cointegrating regression increases.

It is also possible to use the Durbin–Watson (*DW*) test statistic or the Phillips–Perron (*PP*) approach to test for non-stationarity of $\hat{u}_t$. If the *DW* test is applied to the residuals of the potentially cointegrating regression, it is known as the Cointegrating Regression Durbin Watson (*CRDW*). Under the null hypothesis of a unit root in the errors, *CRDW* ≈ 0, so the null of a unit root is rejected if the *CRDW* statistic is larger than the relevant critical value (which is approximately 0.5).

What are the null and alternative hypotheses for any unit root test applied to the residuals of a potentially cointegrating regression?

$$\begin{aligned} H_0 &: u_t \sim I(1) \\ H_1 &: u_t \sim I(0). \end{aligned}$$

Thus, under the null hypothesis there is a unit root in the potentially cointegrating regression residuals, while under the alternative, the residuals are stationary. Under the null hypothesis, therefore, a stationary linear combination of the non-stationary variables has not been found. Hence, if this null hypothesis is not rejected, there is no cointegration. The appropriate strategy for econometric modelling in this case would be to employ specifications in first differences only. Such models would have no long-run equilibrium solution, but this would not matter since no cointegration implies that there is no long-run relationship anyway.

On the other hand, if the null of a unit root in the potentially cointegrating regression's residuals is rejected, it would be concluded that a stationary linear combination of the non-stationary variables had been found. Therefore, the variables would be classed as cointegrated. The appropriate strategy for econometric modelling in this case would be to form and estimate an error correction model, using a method described in the following section.

8.6 Methods of Parameter Estimation in Cointegrated Systems

What should be the modelling strategy if the data at hand are thought to be non-stationary and possibly cointegrated? There are (at least) three methods that could be used: Engle–Granger, Engle–Yoo and Johansen. The first and third of these will be considered in some detail below.

8.6.1 The Engle–Granger 2-Step Method

This is a single equation technique, which is conducted as follows:

Step 1

Make sure that all the individual variables are I(1). Then estimate the cointegrating regression using OLS. Note that it is not possible to perform any inferences on the coefficient estimates in this regression – all that can be done is to estimate the parameter values. Save the residuals of the cointegrating regression, $\hat{u}_t$. Test these residuals to ensure that they are I(0). If they are I(0), proceed to Step 2; if they are I(1), estimate a model containing only first differences.

Step 2

Use the step 1 residuals as one variable in the error correction model, e.g.,

$$\Delta y_t = \beta_1 \Delta x_t + \beta_2(\hat{u}_{t-1}) + v_t \qquad \textbf{(8.54)}$$

where $\hat{u}_{t-1} = y_{t-1} - \hat{\tau} x_{t-1}$. The stationary, linear combination of non-stationary variables is also known as the *cointegrating vector*. In this case, the cointegrating vector would be $[1 - \hat{\tau}]$. Additionally, any linear transformation of the cointegrating vector will also be a cointegrating vector. So, for example, $-10y_{t-1} + 10\hat{\tau} x_{t-1}$ will also be stationary. In equation (8.48) above, the cointegrating vector would be $[1 \; -\hat{\beta}_1 - \hat{\beta}_2 - \hat{\beta}_3]$. It is now valid to perform inferences in the second-stage regression, i.e., concerning the parameters β_1 and β_2 (provided that there are no other forms of misspecification, of course), since all variables in this regression are stationary.

The Engle–Granger 2-step method suffers from a number of problems

(1) The usual finite sample problem of a *lack of power in unit root and cointegration tests* discussed above.

(2) There could be a *simultaneous equations bias* if the causality between *y* and *x* runs in both directions, but this single equation approach requires the researcher to normalise on one variable (i.e., to specify one variable as the dependent variable and the others as independent variables). The researcher is forced to treat *y* and *x* asymmetrically, even though there may have been no theoretical reason for doing so. A further issue is the following. Suppose that the following specification had been estimated as a potential cointegrating regression

$$y_t = \alpha_1 + \beta_1 x_t + u_{1t} \qquad \textbf{(8.55)}$$

What if instead the following equation was estimated?

$$x_t = \alpha_2 + \beta_2 y_t + u_{2t} \tag{8.56}$$

If it is found that $u_{1t} \sim I(0)$, does this imply automatically that $u_{2t} \sim I(0)$? The answer in theory is 'yes', but in practice different conclusions may be reached in finite samples. Also, if there is an error in the model specification at stage 1, this will be carried through to the cointegration test at stage 2, as a consequence of the sequential nature of the computation of the cointegration test statistic.

(3) It is not possible to perform any *hypothesis tests* about the actual cointegrating relationship estimated at stage 1.

(4) There may be more than one cointegrating relationship – see Box 8.2.

BOX 8.2 Multiple cointegrating relationships

In the case where there are only two variables in an equation, y_t, and x_t, say, there can be at most only one linear combination of y_t, and x_t that is stationary – i.e., at most one cointegrating relationship.

However, suppose that there are k variables in a system (ignoring any constant term), denoted $y_t, x_{2t}, \ldots x_{kt}$. In this case, there may be up to r linearly independent cointegrating relationships (where $r \leq k - 1$). This potentially presents a problem for the OLS regression approach described above, which is capable of finding at most one cointegrating relationship no matter how many variables there are in the system. And if there are multiple cointegrating relationships, how can one know if there are others, or whether the 'best' or strongest cointegrating relationship has been found?

An OLS regression will find the minimum variance stationary linear combination of the variables, but there may be other linear combinations of the variables that have more intuitive appeal. The answer to this problem is to use a systems approach to cointegration, which will allow determination of all r cointegrating relationships. One such approach is Johansen's method – see Section 8.9.

Problems (1) and (2) are small sample problems that should disappear asymptotically. Problem (3) is addressed by another method due to Engle and Yoo. There is also another alternative technique, which overcomes problems (2) and (3) by adopting a different approach based on estimation of a VAR system – see Section 8.8.

8.6.2 The Engle and Yoo 3-Step Method

The Engle and Yoo (1987) 3-step procedure takes its first two steps from Engle–Granger (EG). Engle and Yoo then add a third step giving updated estimates of the cointegrating vector and its standard errors. The Engle and Yoo (EY) third step is algebraically technical and additionally, EY suffers from all of the remaining problems of the EG approach. There is arguably a far superior procedure available to remedy the lack of testability of hypotheses concerning the cointegrating relationship – namely, the Johansen (1988) procedure. For these reasons, the Engle–Yoo procedure is rarely employed in empirical applications and is not considered further here.

There now follows an application of the Engle–Granger procedure in the context of spot and futures markets.

8.7 Lead–Lag and Long-Term Relationships Between Spot and Futures Markets

8.7.1 Background

If the markets are frictionless and functioning efficiently, changes in the (log of the) spot price of a financial asset and its corresponding changes in the (log of the) futures price would be expected to be perfectly contemporaneously correlated and not to be cross-autocorrelated. Mathematically, these notions would be represented as

$$\text{corr}(\Delta\ln(f_t), \Delta\ln(s_t)) \approx 1 \qquad \textbf{(a)}$$

$$\text{corr}(\Delta\ln(f_t), \Delta\ln(s_{t-k})) \approx 0 \quad \forall\, k > 0 \qquad \textbf{(b)}$$

$$\text{corr}(\Delta\ln(f_{t-j}), \Delta\ln(s_t)) \approx 0 \quad \forall\, j > 0 \qquad \textbf{(c)}$$

In other words, changes in spot prices and changes in futures prices are expected to occur at the same time (condition (a)). The current change in the futures price is also expected not to be related to previous changes in the spot price (condition (b)), and the current change in the spot price is expected not to be related to previous changes in the futures price (condition (c)). The changes in the log of the spot and futures prices are also of course known as the spot and futures returns.

For the case when the underlying asset is a stock index, the equilibrium relationship between the spot and futures prices is known as the *cost of carry model*, given by

$$F_t^* = S_t e^{(r-d)(T-t)} \quad \textbf{(8.57)}$$

where F_t^* is the fair futures price, S_t is the spot price, r is a continuously compounded risk-free rate of interest, d is the continuously compounded yield in terms of dividends derived from the stock index until the futures contract matures, and $(T - t)$ is the time to maturity of the futures contract. Taking logarithms of both sides of (8.57) gives

$$f_t^* = s_t + (r - d)(T - t) \quad \textbf{(8.58)}$$

where f_t^* is the log of the fair futures price and s_t is the log of the spot price. Equation (8.58) suggests that the long-term relationship between the logs of the spot and futures prices should be one to one. Thus the basis, defined as the difference between the futures and spot prices (and if necessary adjusted for the cost of carry) should be stationary, for if it could wander without bound, arbitrage opportunities would arise, which would be assumed to be quickly acted upon by traders such that the relationship between spot and futures prices will be brought back to equilibrium.

The notion that there should not be any lead–lag relationships between the spot and futures prices and that there should be a long-term one to one relationship between the logs of spot and futures prices can be tested using simple linear regressions and cointegration analysis. This book will now examine the results of two related papers – Tse (1995), who employs daily data on the Nikkei Stock Average (NSA) and its futures contract, and Brooks, Brooks, Rew, and Ritson (2001), who examine high-frequency data from the FTSE 100 stock index and index futures contract.

The data employed by Tse (1995) consists of 1,055 daily observations on NSA stock index and stock index futures values from December 1988 to April 1993. The data employed by Brooks *et al.* comprises 13,035 ten-minutely observations for all trading days in the period June 1996–May 1997, provided by FTSE International. In order to form a statistically adequate model, the variables should first be checked as to whether they can be considered stationary. The results of applying a DF test to the logs of the spot and futures prices of the ten-minutely FTSE data are shown in Table 8.3.

Table 8.3 DF tests on log-prices and returns for high frequency FTSE data

	Futures	Spot
Dickey–Fuller statistics for log-price data	−0.1329	−0.7335
Dickey–Fuller statistics for returns data	−84.9968	−114.1803

As one might anticipate, both studies conclude that the two log–price series contain a unit root, while the returns are stationary. Of course, it may be necessary to augment the tests by adding lags of the dependent variable to allow for autocorrelation in the errors (i.e., an ADF test). Results for such tests are not presented, since the conclusions are not altered. A statistically valid model would therefore be one in the returns. However, a formulation containing only first differences has no long-run equilibrium solution. Additionally, theory suggests that the two series should have a long-run relationship. The solution is therefore to see whether there exists a cointegrating relationship between f_t and s_t which would mean that it is valid to include levels terms along with returns in this framework. This is tested by examining whether the residuals, $\hat{z}_t$, of a regression of the form

$$s_t = \gamma_0 + \gamma_1 f_t + z_t \quad \textbf{(8.59)}$$

are stationary, using a DF test, where z_t is the error term. The coefficient values for the estimated equation (8.59) and the DF test statistic are given in Table 8.4.

Table 8.4 Estimated potentially cointegrating equation and test for cointegration for high frequency FTSE data

Coefficient	Estimated value
$\hat{\gamma}_0$	0.1345

$\hat{\gamma}_1$	0.9834
DF test on residuals	**Test statistic**
$\hat{z}_t$	−14.7303

Source: Brooks, Rew, and Ritson (2001).

Clearly, the residuals from the cointegrating regression can be considered stationary. Note also that the estimated slope coefficient in the cointegrating regression takes on a value close to unity, as predicted from the theory. It is not possible to formally test whether the true population coefficient could be one, however, since there is no way in this framework to test hypotheses about the cointegrating relationship.

The final stage in building an error correction model using the Engle–Granger two-step approach is to use a lag of the first-stage residuals, $\hat{z}_t$, as the equilibrium correction term in the general equation. The overall model is

$$\Delta \ln s_t = \beta_0 + \delta \hat{z}_{t-1} + \beta_1 \Delta \ln s_{t-1} + \alpha_1 \Delta \ln f_{t-1} + v_t \quad \textbf{(8.60)}$$

where v_t is an error term. The coefficient estimates for this model are presented in Table 8.5.

Table 8.5 Estimated error correction model for high frequency FTSE data

Coefficient	Estimated value	*t*-ratio
$\hat{\beta}_0$	9.6713E−06	1.6083
$\hat{\delta}$	−0.8388	−5.1298
$\hat{\beta}_1$	0.1799	19.2886
$\hat{\alpha}_1$	0.1312	20.4946

Source: Brooks, Rew, and Ritson (2001).

Consider first the signs and significances of the coefficients (these can now be interpreted validly since all variables used in this model are stationary). $\hat{\alpha}_1$ is positive and highly significant, indicating that the futures market does indeed lead the spot market, since lagged changes in futures prices lead to a positive change in the subsequent spot price. $\hat{\beta}_1$ is positive and highly significant, indicating on average a positive autocorrelation in spot returns. $\hat{\delta}$, the coefficient on the error correction term, is negative and significant, indicating that if the difference between the logs of the spot and futures prices is positive in one period, the spot price will fall during the next period to restore equilibrium, and vice versa.

8.7.2 Forecasting Spot Returns

Both Brooks, Rew, and Ritson (2001) and Tse (1995) show that it is possible to use an error correction formulation to model changes in the log of a stock index. An obvious related question to ask is whether such a model can be used to forecast the future value of the spot series for a holdout sample of data not used previously for model estimation. Both sets of researchers employ forecasts from three other models for comparison with the forecasts of the error correction model. These are an error correction model with an additional term that allows for the cost of carry, an ARMA model (with lag length chosen using an information criterion) and an unrestricted VAR model (with lag length chosen using a multivariate information criterion).

The results are evaluated by comparing their root-mean squared errors, mean absolute errors and percentage of correct direction predictions. The forecasting results from the Brooks, Rew and Ritson paper are given in Table 8.6.

Table 8.6 Comparison of out-of-sample forecasting accuracy

	ECM	ECM-COC	ARIMA	VAR
RMSE	0.0004382	0.0004350	0.0004531	0.0004510
MAE	0.4259	0.4255	0.4382	0.4378
% Correct direction	67.69%	68.75%	64.36%	66.80%

Source: Brooks, Rew, and Ritson (2001).

It can be seen from Table 8.6 that the error correction models have both the lowest mean squared and mean absolute errors, and the highest proportion of correct direction predictions. There is, however, little to choose between the models, and all four have over 60% of the signs of the next returns predicted correctly.

It is clear that on statistical grounds the out-of-sample forecasting performances of the error correction models are better than those of their competitors, but this does not necessarily mean that such forecasts have any practical use. Many studies have questioned the usefulness of statistical measures of forecast accuracy as indicators of the profitability of using these forecasts in a practical trading setting (see, for example, Leitch and Tanner, 1991). Brooks, Rew, and Ritson (2001) investigate this proposition directly by developing a set of trading rules based on the forecasts of the error correction model with the cost of carry term, the best statistical forecasting model. The trading period is an out-of-sample data series not used in model estimation, running from 1 May–30 May 1997. The error correction model with cost of carry (ECM-COC) model yields ten-minutely one-step-ahead forecasts. The trading strategy involves analysing the forecast for the spot return, and incorporating the decision dictated by the trading rules described below. It is assumed that the original investment is £1,000, and if the holding in the stock index is zero, the investment earns the risk-free rate. Five trading strategies are employed, and their profitabilities are compared with that obtained by passively buying and holding the index. There are of course an infinite number of strategies that could be adopted for a given set of spot return forecasts, but Brooks, Rew and Ritson use the following

- *Liquid trading strategy* This trading strategy involves making a round-trip trade (i.e., a purchase and sale of the FTSE 100 stocks) every ten minutes that the return is predicted to be positive by the model. If the return is predicted to be negative by the model, no trade is executed and the investment earns the risk-free rate.
- *Buy-and-hold while forecast positive strategy* This strategy allows the trader to continue holding the index if the return at the next predicted investment period is positive, rather than making a round-trip transaction for each period.
- *Filter strategy: better predicted return than average* This strategy involves purchasing the index only if the predicted returns are greater than the average positive return (there is no trade for negative returns therefore the average is only taken of the positive returns).
- *Filter strategy: better predicted return than first decile* This strategy is similar to the previous one, but rather than utilising the average as previously, only the returns predicted to be in the top 10% of all returns are traded on.
- *Filter strategy: high arbitrary cutoff* An arbitrary filter of 0.0075% is imposed, which will result in trades only for returns that are predicted to be extremely large for a ten-minute interval.

The results from employing each of the strategies using the forecasts for the spot returns obtained from the ECM-COC model are presented in Table 8.7.

Table 8.7 Trading profitability of the error correction model with cost of carry

Trading strategy	Terminal wealth (£)	Return(%) annualised	Terminal wealth (£) with slippage	Return(%) annualised with slippage	Number of trades
Passive investment	1040.92	4.09 {49.08}	1040.92	4.09 {49.08}	1
Liquid trading	1156.21	15.62 {187.44}	1056.38	5.64 {67.68}	583
Buy-and-hold while forecast positive	1156.21	15.62 {187.44}	1055.77	5.58 {66.96}	383
Filter I	1144.51	14.45 {173.40}	1123.57	12.36 {148.32}	135
Filter II	1100.01	10.00 {120.00}	1046.17	4.62 {55.44}	65
Filter III	1019.82	1.98 {23.76}	1003.23	0.32 {3.84}	8

Source: Brooks, Rew, and Ritson (2001).

The test month of May 1997 was a particularly bullish one, with a pure buy-and-hold-the-index strategy netting a return of 4%, or almost 50% on an annualised basis. Ideally, the forecasting exercise would be conducted over a much longer period than one month, and preferably over different market conditions. However, this was simply impossible due to the lack of availability of very high frequency data over a long time period. Clearly, the forecasts have some market timing ability in the sense that they seem to ensure trades that, on average, would have invested in the index when it rose, but be out of the market when it fell. The most profitable trading strategies in gross terms are those that trade on the basis of every positive spot return forecast, and all rules except the strictest filter make more money than a passive investment. The strict filter appears not to work well since it is out of the index for too long during a period when the market is rising strongly.

However, the picture of immense profitability painted thus far is somewhat misleading for two reasons: slippage time and transactions costs. First, it is unreasonable to assume that trades can be executed in the market the minute they are requested, since it may take some time to find counterparties for all the trades required to 'buy the index'. (Note, of course, that in practice, a similar returns profile to the index can be achieved with a very much smaller number of stocks.) Brooks, Rew and Ritson therefore allow for ten minutes of 'slippage time', which assumes that it takes ten minutes from when the trade order is placed to when it is executed. Second, it is unrealistic to consider gross profitability, since transactions costs in the spot market are non-negligible and the strategies examined suggested a lot of trades. Sutcliffe (1997, p. 47) suggests that total round-trip transactions costs for FTSE stocks are of the order of 1.7% of the investment.

The effect of slippage time is to make the forecasts less useful than they would otherwise have been. For example, if the spot price is forecast to rise, and it does, it may have already risen and then stopped rising by the time that the order is executed, so that the forecasts lose their market timing ability. Terminal wealth appears to fall substantially when slippage time is allowed for, with the monthly return falling by between 1.5% and 10%, depending on the trading rule.

Finally, if transactions costs are allowed for, none of the trading rules can outperform the passive investment strategy, and all in fact make substantial losses.

8.7.3 Conclusions

If the markets are frictionless and functioning efficiently, changes in the spot price of a financial asset and its corresponding futures price would be expected to be perfectly contemporaneously correlated and not to be cross-autocorrelated. Many academic studies, however, have documented that the futures market systematically 'leads' the spot market, reflecting news more quickly as a result of the fact that the stock index is not a single entity. The latter implies that

- Some components of the index are infrequently traded, implying that the observed index value contains 'stale' component prices
- It is more expensive to transact in the spot market and hence the spot market reacts more slowly to news
- Stock market indices are recalculated only every minute so that new information takes longer to be reflected in the index.

Clearly, such spot market impediments cannot explain the inter-daily lead–lag relationships documented by Tse (1995). In any case, however, since it appears impossible to profit from these relationships, their existence is entirely consistent with the absence of arbitrage opportunities and is in accordance with modern definitions of the efficient markets hypothesis.

8.8 Testing for and Estimating Cointegration in Systems Using the Johansen Technique based on VARs

Suppose that a set of g variables ($g \geq 2$) are under consideration that are I(1) and which are thought may be cointegrated. A VAR with k lags containing these variables could be set up:

$$\underset{g\times 1}{y_t} = \underset{g\times g}{\beta_1}\underset{g\times 1}{y_{t-1}} + \underset{g\times g}{\beta_2}\underset{g\times 1}{y_{t-2}} + \cdots + \underset{g\times g}{\beta_k}\underset{g\times 1}{y_{t-k}} + \underset{g\times 1}{u_t} \qquad \textbf{(8.61)}$$

In order to use the Johansen test, the VAR (8.61) above needs to be turned into a vector error correction model (VECM) of the form

$$\Delta y_t = \Pi y_{t-k} + \Gamma_1 \Delta y_{t-1} + \Gamma_2 \Delta y_{t-2} + \cdots + \Gamma_{k-1} \Delta y_{t-(k-1)} + u_t \qquad \textbf{(8.62)}$$

where $\Pi = (\sum_{i=1}^{k} \beta_i) - I_g$ and $\Gamma_i = (\sum_{j=1}^{i} \beta_j) - I_g$

This VAR contains g variables in first differenced form on the LHS, and $k - 1$ lags of the dependent variables (differences) on the RHS, each with a Γ coefficient matrix attached to it. In fact, the Johansen test can be affected by the lag length employed in the VECM, and so it is useful to attempt to select the lag length optimally, as outlined in Chapter 7. The Johansen test centres around an examination of the Π matrix. Π can be interpreted as a long-run coefficient matrix, since in equilibrium, all the Δy_{t-i} will be zero, and setting the error terms, u_t, to their expected value of zero will leave $\Pi y_{t-k} = 0$. Notice the comparability between this set of equations and the testing equation for an ADF test, which has a first differenced term as the dependent variable, together with a lagged levels term and lagged differences on the RHS.

The test for cointegration between the ys is calculated by looking at the rank of the Π matrix via its eigenvalues.[2] The rank of a matrix is equal to the number of its characteristic roots (eigenvalues) that are different from zero (see Section 1.7.5 for some algebra and examples). The eigenvalues, denoted λ_i are put in descending order $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_g$. If the λs are roots, in this context they must be less than one in absolute value and positive, and λ_1 will be the largest (i.e., the closest to one), while λ_g will be the smallest (i.e., the closest to zero). If the variables are not cointegrated, the rank of will not be significantly different from zero, so $\lambda_i \approx 0 \ \forall \ i$. The test statistics actually incorporate $\ln(1 - \lambda_i)$, rather than the λ_i themselves, but still, when $\lambda_i = 0$, $\ln(1 - \lambda_i) = 0$.

Suppose now that rank $(\Pi) = 1$, then $\ln(1 - \lambda_1)$ will be negative and $\ln(1 - \lambda_i) = 0 \, \forall \, i > 1$. If the eigenvalue i is non-zero, then $\ln(1 - \lambda_i) < 0 \, \forall \, i \geq 1$. That is, for to have a rank of 1, the largest eigenvalue must be significantly non-zero, while others will not be significantly different from zero.

There are two test statistics for cointegration under the Johansen approach, which are formulated as

$$\lambda_{trace}(r) = -T \sum_{i=r+1}^{g} \ln(1 - \hat{\lambda}_i) \tag{8.63}$$

and

$$\lambda_{max}(r, r+1) = -T \ln(1 - \hat{\lambda}_{r+1}) \tag{8.64}$$

where r is the number of cointegrating vectors under the null hypothesis and $\hat{\lambda}_i$ is the estimated value for the ith ordered eigenvalue from the Π matrix. Intuitively, the larger is $\hat{\lambda}_i$, the more large and negative will be $\ln(1 - \hat{\lambda}_i)$ and hence the larger will be the test statistic. Each eigenvalue will have associated with it a different cointegrating vector, which will be an eigenvector. A significantly non-zero eigenvalue indicates a significant cointegrating vector.

λ_{trace} is a joint test where the null is that the number of cointegrating vectors is less than or equal to r against an unspecified or general alternative that there are more than r. It starts with p eigenvalues, and then successively the largest is removed. $\lambda_{trace} = 0$ when all the $\lambda_i = 0$, for $i = 1, \ldots, g$.

λ_{max} conducts separate tests on each eigenvalue, and has as its null hypothesis that the number of cointegrating vectors is r against an alternative of $r + 1$.

Johansen and Juselius (1990) provide critical values for the two statistics. The distribution of the test statistics is non-standard, and the critical values depend on the value of $g - r$, the number of non-stationary components and whether constants are included in each of the equations. Intercepts can be included either in the cointegrating vectors themselves or as additional terms in the VAR. The latter is equivalent to including a trend in the data generating processes for the levels of the series. Osterwald-Lenum (1992) provides a more complete set of critical values for the Johansen test, some of which are also given in the Appendix of Statistical Tables (Appendix 2) at the end of this book.

If the test statistic is greater than the critical value from Johansen's tables, reject the null hypothesis that there are r cointegrating vectors in favour of the alternative that there are $r + 1$ (for λ_{max}) or more than r (for λ_{trace}). The testing is conducted in a sequence and under the null, $r = 0, 1, \ldots, g - 1$ so that the hypotheses for λ_{trace} are

$$\begin{array}{lll} H_0: r = 0 & \text{versus} & H_1: 0 < r \leq g \\ H_0: r = 1 & \text{versus} & H_1: 1 < r \leq g \\ H_0: r = 2 & \text{versus} & H_1: 2 < r \leq g \\ \vdots & \vdots & \vdots \\ H_0: r = g - 1 & \text{versus} & H_1: r = g \end{array}$$

The first test involves a null hypothesis of no cointegrating vectors (corresponding to Π having zero rank). If this null is not rejected, it would be concluded that there are no cointegrating vectors and the testing would be completed. However, if $H_0 : r = 0$ is rejected, the null that there is one cointegrating vector (i.e., $H_0 : r = 1$) would be tested and so on. Thus the value of r is continually increased until the null is no longer rejected.

But how does this correspond to a test of the rank of the Π matrix? r is the rank of Π. Π cannot be of full rank (g) since this would correspond to the original y_t being stationary. If Π has zero rank, then by analogy to the univariate case, Δy_t depends only on Δy_{t-j} and not on y_{t-1}, so that there is no long-run relationship between the elements of y_{t-1}. Hence there is no cointegration. For $1 \leq \text{rank}(\Pi) < g$, there are r cointegrating vectors. Π is then defined as the product of two matrices, α and β', of dimension $(g \times r)$ and $(r \times g)$, respectively, i.e.,

$$\Pi = \alpha\beta' \tag{8.65}$$

The matrix β gives the cointegrating vectors, while α gives the amount of each cointegrating vector entering each equation of the VECM, also known as the ‘adjustment parameters’.

For example, suppose that $g = 4$, so that the system contains four variables. The elements of the Π matrix would be written

$$\Pi = \begin{pmatrix} \pi_{11} & \pi_{12} & \pi_{13} & \pi_{14} \\ \pi_{21} & \pi_{22} & \pi_{23} & \pi_{24} \\ \pi_{31} & \pi_{32} & \pi_{33} & \pi_{34} \\ \pi_{41} & \pi_{42} & \pi_{43} & \pi_{44} \end{pmatrix} \tag{8.66}$$

If $r = 1$, so that there is one cointegrating vector, then α and β will be (4×1)

$$\Pi = \alpha\beta' = \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \\ \alpha_{14} \end{pmatrix} \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \end{pmatrix} \tag{8.67}$$

If $r = 2$, so that there are two cointegrating vectors, then α and β will be (4×2)

$$\Pi = \alpha\beta' = \begin{pmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \\ \alpha_{13} & \alpha_{23} \\ \alpha_{14} & \alpha_{24} \end{pmatrix} \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \end{pmatrix} \tag{8.68}$$

and so on for $r = 3, \ldots$

Suppose now that $g = 4$, and $r = 1$, as in equation (8.67), so that there are four variables in the system, y_1, y_2, y_3, and y_4, that exhibit one cointegrating vector. Then Πy_{t-k} will be given by

$$\Pi = \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \\ \alpha_{14} \end{pmatrix} \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}_{t-k} \tag{8.69}$$

Equation (8.69) can also be written

$$\Pi = \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \\ \alpha_{14} \end{pmatrix} (\beta_{11}y_1 + \beta_{12}y_2 + \beta_{13}y_3 + \beta_{14}y_4)_{t-k} \tag{8.70}$$

Given equation (8.70), it is possible to write out the separate equations for each variable Δy_t. It is also common to 'normalise' on a particular variable, so that the coefficient on that variable in the cointegrating vector is one. For example, normalising on y_1 would make the cointegrating term in the equation for Δy_1

$$\alpha_{11}\left(y_1 + \frac{\beta_{12}}{\beta_{11}}y_2 + \frac{\beta_{13}}{\beta_{11}}y_3 + \frac{\beta_{14}}{\beta_{11}}y_4\right)_{t-k}, \text{etc.}$$

Finally, it must be noted that the above description is not exactly how the Johansen procedure works, but is an intuitive approximation to it.

8.8.1 Tests for Cointegration with Mixed Orders of Integration

Suppose that we have a set of variables which we believe are related to one another and where there may potentially be a long-term relationship between some of them but where the individual variables are of different orders of integration. In the context of the Engle-Granger single equation approach, the test for cointegration will still be applicable, but the order of integration of the residuals in the potentially cointegrating regression will be the highest of the individual variables if they are not cointegrated and $I(0)$ if they are cointegrated. In practice we will again only be considering variables that are either $I(1)$ or $I(0)$, so suppose we have a set of three variables which are individually $I(1)$, $I(1)$, and $I(0)$. If the variables are cointegrated then the residuals will be $I(0)$ since these residuals will be a stationary linear combination of the two $I(1)$ variables and the variable which was already stationary ($I(0)$), whereas if they are not cointegrated then the residuals will be $I(1)$. Thus the $I(0)$ variable effectively acts like a constant from the perspective of non-stationarity.

Within the Johansen framework, if the number of variables in the system is N, then the cointegrating rank is equal to the sum of the number of linearly independent cointegrating vectors and the number of $I(0)$ variables in the system.

8.8.2 Hypothesis Testing using Johansen

Engle–Granger did not permit the testing of hypotheses on the cointegrating relationships themselves, but the Johansen setup does permit the testing of hypotheses about the equilibrium relationships between the variables. Johansen allows a researcher to test a hypothesis about one or more coefficients in the cointegrating relationship by viewing the hypothesis as a restriction on the Π matrix. If there exist r cointegrating vectors, only these linear combinations or linear transformations of them, or combinations of the cointegrating vectors, will be stationary. In fact, the matrix of cointegrating vectors β can be multiplied by any non-singular conformable matrix to obtain a new set of cointegrating vectors.

A set of required long-run coefficient values or relationships between the coefficients does not necessarily imply that the cointegrating vectors have to be restricted. This is because any combination of cointegrating vectors is also a cointegrating vector. So it may be possible to combine the cointegrating vectors thus far obtained to provide a new one or, in general, a new set, having the required properties. The simpler and fewer are the required properties, the more likely that this recombination process (called *renormalisation*) will automatically yield cointegrating vectors with the required properties. However, as the restrictions become more numerous or involve more of the coefficients of the vectors, it will eventually become impossible to satisfy all of them by renormalisation. After this point, all other linear combinations of the variables will be non-stationary. If the restriction does not affect the model much, i.e., if the restriction is not binding, then the eigenvectors should not change much following imposition of the restriction. A test statistic to test this hypothesis is given by

$$\text{test statistic} = -T \sum_{i=1}^{r} [\ln(1-\lambda_i) - \ln(1-\lambda_i^*)] \sim \chi^2(m) \qquad \textbf{(8.71)}$$

where λ_i^* are the characteristic roots of the restricted model, λ_i are the characteristic roots of the unrestricted model, r is the number of non-zero characteristic roots in the unrestricted model and m is the number of over-identifying restrictions.

Restrictions are actually imposed by substituting them into the relevant α or β matrices as appropriate, so that tests can be conducted on either the cointegrating vectors or their loadings in each equation in the system (or both). For example, considering equations (8.66)–(8.68) above, it may be that theory suggests that the coefficients on the loadings of the cointegrating vector(s) in each equation should take on certain values, in which case it would be relevant to test restrictions on the elements of α (e.g. $\alpha_{11} = 1$, $\alpha_{23} = -1$, etc.). Equally, it may be of interest to examine whether only a sub-set of the variables in y_t is actually required to obtain a stationary linear combination. In that case, it would be appropriate to test restrictions of elements of β. For example, to test the hypothesis that y_4 is not necessary to form a long-run relationship, set $\beta_{14} = 0$, $\beta_{24} = 0$, etc.

For an excellent detailed treatment of cointegration in the context of both single equation and multiple equation models, see Harris (1995). Several applications of tests for cointegration and modelling cointegrated systems in finance will now be given.

8.9 Purchasing Power Parity

Purchasing power parity (PPP) states that the equilibrium or long-run exchange rate between two countries is equal to the ratio of their relative price levels. Purchasing power parity implies that the real exchange rate, Q_t, is stationary. The real exchange rate can be defined as

$$Q_t = \frac{E_t P_t^*}{P_t} \tag{8.72}$$

where E_t is the nominal exchange rate in domestic currency per unit of foreign currency, P_t is the domestic price level and P_t^* is the foreign price level. Taking logarithms of equation (8.72) and rearranging, another way of stating the PPP relation is obtained

$$e_t - p_t + p_t^* = q_t \tag{8.73}$$

where the lower case letters in equation (8.73) denote logarithmic transforms of the corresponding upper case letters used in equation (8.72). A necessary and sufficient condition for PPP to hold is that the variables on the LHS of equation (8.73) – that is the log of the exchange rate between countries A and B, and the logs of the price levels in countries A and B be cointegrated with cointegrating vector [1 – 1 1].

A test of this form is conducted by Chen (1995) using monthly data from Belgium, France, Germany, Italy and the Netherlands over the period April 1973 to December 1990. Pair-wise evaluations of the existence or otherwise of cointegration are examined for all combinations of these countries (ten country pairs). Since there are three variables in the system (the log exchange rate and the two log nominal price series) in each case, and that the variables in their log-levels forms are nonstationary, there can be at most two linearly independent cointegrating relationships for each country pair. The results of applying Johansen's trace test are presented in Chen's Table 1, adapted and presented here as Table 8.8.

Table 8.8 Cointegration tests of PPP with European data

Tests for cointegration between	$r = 0$	$r \leq 1$	$r \leq 2$	α_1	α_2
FRF–DEM	34.63*	17.10	6.26	1.33	−2.50
FRF–ITL	52.69*	15.81	5.43	2.65	−2.52
FRF–NLG	68.10*	16.37	6.42	0.58	−0.80
FRF–BEF	52.54*	26.09*	3.63	0.78	−1.15
DEM–ITL	42.59*	20.76*	4.79	5.80	−2.25
DEM–NLG	50.25*	17.79	3.28	0.12	−0.25
DEM–BEF	69.13*	27.13*	4.52	0.87	−0.52
ITL–NLG	37.51*	14.22	5.05	0.55	−0.71
ITL–BEF	69.24*	32.16*	7.15	0.73	−1.28
NLG–BEF	64.52*	21.97*	3.88	1.69	−2.17
Critical values	31.52	17.95	8.18	–	–

Notes: FRF – French franc; DEM – German mark; NLG – Dutch guilder; ITL – Italian lira; BEF – Belgian franc.

Source: Chen (1995). Reprinted with the permission of Taylor and Francis Ltd (www.tandf.co.uk).

As can be seen from the results, the null hypothesis of no cointegrating vectors is rejected for all country pairs, and the null of one or fewer cointegrating vectors is rejected for France–Belgium, Germany–Italy, Germany–Belgium, Italy–Belgium, Netherlands–Belgium. In no cases is the null of two or less cointegrating vectors rejected. It is therefore concluded that the PPP hypothesis is upheld and that there are either one or two cointegrating relationships between the series depending on the country pair. Estimates of α_1 and α_2 are given in the last two columns of Table 8.8. PPP suggests that the estimated values of these coefficients should be 1 and −1, respectively. In most cases, the coefficient estimates are a long way from these expected values. Of course, it would be possible to impose this restriction and to test it in the Johansen framework as discussed above, but Chen does not conduct this analysis.

8.10 Cointegration Between International Bond Markets

Often, investors will hold bonds from more than one national market in the expectation of achieving a reduction in risk via the resulting diversification. If international bond markets are very strongly correlated in the long run, diversification will be less effective than if the bond markets operated independently of one another. An important indication of the degree to which long-run diversification is available to international bond market investors is given by determining whether the markets are cointegrated. This book will now study two examples from the academic literature that consider this issue: Clare, Maras and Thomas (1995), and Mills and Mills (1991).

8.10.1 Cointegration Between International Bond Markets: A Univariate Approach

Clare, Maras and Thomas (1995) use the Dickey–Fuller and Engle–Granger single-equation method to test for cointegration using a pair-wise analysis of four countries' bond market indices: US, UK, Germany and Japan. Monthly Salomon Brothers' total return government bond index data from January 1978 to April 1990 are employed. An application of the Dickey–Fuller test to the log of the indices reveals the following results (adapted from their Table 1), given in Table 8.9.

Table 8.9 DF tests for international bond indices

Panel A: test on log-index for country	**DF Statistic**
Germany	−0.395
Japan	−0.799

UK	−0.884
US	0.174
Panel B: test on log-returns for country	
Germany	−10.37
Japan	−10.11
UK	−10.56
US	−10.64

Source: Clare, Maras and Thomas (1995). Reprinted with the permission of Blackwell Publishers.

Neither the critical values, nor a statement of whether a constant or trend are included in the test regressions, are offered in the paper. Nevertheless, the results are clear. Recall that the null hypothesis of a unit root is rejected if the test statistic is smaller (more negative) than the critical value. For samples of the size given here, the 5% critical value would be somewhere between −1.95 and −3.50. It is thus demonstrated quite conclusively that the logarithms of the indices are non-stationary, while taking the first difference of the logs (that is, constructing the returns) induces stationarity.

Given that all logs of the indices in all four cases are shown to be I(1), the next stage in the analysis is to test for cointegration by forming a potentially cointegrating regression and testing its residuals for non-stationarity. Clare, Maras and Thomas use regressions of the form

$$B_i = \alpha_0 + \alpha_1 B_j + u \quad \textbf{(8.74)}$$

with time subscripts suppressed and where B_i and B_j represent the log-bond indices for any two countries i and j. The results are presented in their Tables 3 and 4, which are combined into Table 8.10 here. They offer findings from applying seven different tests, while we present the results for only the Cointegrating Regression Durbin Watson (CRDW), Dickey–Fuller and Augmented Dickey–Fuller tests (although the lag lengths for the latter are not given in their paper).

Table 8.10 Cointegration tests for pairs of international bond indices

Test	UK–Germany	UK–Japan	UK–US	Germany–Japan	Germany–US	Japa US
CRDW	0.189	0.197	0.097	0.230	0.169	0.13[illegible]
DF	2.970	2.770	2.020	3.180	2.160	2.16[illegible]
ADF	3.160	2.900	1.800	3.360	1.640	1.89[illegible]

Source: Clare, Maras and Thomas (1995). Reprinted with the permission of Blackwell Publishers.

In this case, the null hypothesis of a unit root in the residuals from regression (8.74) cannot be rejected. The conclusion is therefore that there is no cointegration between any pair of bond indices in this sample.

8.10.2 Cointegration Between International Bond Markets: A Multivariate Approach

Mills and Mills (1991) also consider the issue of cointegration or non-cointegration between the same four international bond markets. However, unlike Clare *et al.* (1995), who use bond price indices, Mills and Mills employ daily closing observations on the redemption yields. The latter's sample period runs from 1 April 1986 to 29 December 1989, giving 960 observations. They employ a Dickey–Fuller-type regression procedure to test the individual series for non-stationarity and conclude that all four yields series are I(1).

The Johansen systems procedure is then used to test for cointegration between the series. Unlike Clare *et al.*, Mills and Mills consider all four indices together rather than investigating them in a pair-wise fashion. Therefore, since there are four variables in the system (the redemption yield for each country), i.e., $g = 4$, there can be at most three linearly independent cointegrating vectors, i.e., $r \leq 3$. The trace statistic is employed, and it takes the form

$$\lambda_{trace}(r) = -T \sum_{i=r+1}^{g} \ln(1 - \hat{\lambda}_i) \qquad \textbf{(8.75)}$$

where λ_i are the ordered eigenvalues. The results are presented in their Table 2, which is modified slightly here, and presented in Table 8.11.

Table 8.11 Johansen tests for cointegration between international bond yields

r (number of cointegrating vectors under the null hypothesis)	Test statistic	Critical values	
		10%	5%
0	22.06	35.6	38.6
1	10.58	21.2	23.8
2	2.52	10.3	12.0
3	0.12	2.9	4.2

Source: Mills and Mills (1991). Reprinted with the permission of Blackwell Publishers.

Looking at the first row under the heading, it can be seen that the test statistic is smaller than the critical value, so the null hypothesis that $r = 0$ cannot be rejected, even at the 10% level. It is thus not necessary to look at the remaining rows of the table. Hence, reassuringly, the conclusion from this analysis is the same as that of Clare *et al.* – i.e., that there are no cointegrating vectors.

Given that there are no linear combinations of the yields that are stationary, and therefore that there is no error correction representation, Mills and Mills then continue to estimate a VAR for the first differences of the yields. The VAR is of the form

$$\Delta X_t = \sum_{i=1}^{k} \Gamma_i \Delta X_{t-i} + v_t \quad \textbf{(8.76)}$$

where

$$X_t = \begin{bmatrix} X(US)_t \\ X(UK)_t \\ X(WG)_t \\ X(JAP)_t \end{bmatrix}, \Gamma_i = \begin{bmatrix} \Gamma_{11i} & \Gamma_{12i} & \Gamma_{13i} & \Gamma_{14i} \\ \Gamma_{21i} & \Gamma_{22i} & \Gamma_{23i} & \Gamma_{24i} \\ \Gamma_{31i} & \Gamma_{32i} & \Gamma_{33i} & \Gamma_{34i} \\ \Gamma_{41i} & \Gamma_{42i} & \Gamma_{43i} & \Gamma_{44i} \end{bmatrix}, v_t = \begin{bmatrix} v_{1t} \\ v_{2t} \\ v_{3t} \\ v_{4t} \end{bmatrix}$$

They set k, the number of lags of each change in the yield in each regression, to 8, arguing that likelihood ratio tests rejected the possibility of smaller numbers of lags. Unfortunately, and as one may anticipate for a regression of daily yield changes, the R^2 values for the VAR equations are low, ranging from 0.04 for the US to 0.17 for Germany. Variance decompositions and impulse responses are calculated for the estimated VAR. Two orderings of the variables are employed: one based on a previous study and one based on the chronology of the opening (and closing) of the financial markets considered: Japan → Germany → UK → US. Only results for the latter, adapted from Tables 4 and 5 of Mills and Mills (1991), are presented here. The variance decompositions and impulse responses for the VARs are given in Tables 8.12 and 8.13, respectively.

Table 8.12 Variance decompositions for VAR of international bond yields

Explaining movements in	Days ahead	Explained by movements in			
		US	UK	Germany	Japan
US	1	95.6	2.4	1.7	0.3
	5	94.2	2.8	2.3	0.7
	10	92.9	3.1	2.9	1.1
	20	92.8	3.2	2.9	1.1
UK	1	0.0	98.3	0.0	1.7
	5	1.7	96.2	0.2	1.9
	10	2.2	94.6	0.9	2.3
	20	2.2	94.6	0.9	2.3
Germany	1	0.0	3.4	94.6	2.0
	5	6.6	6.6	84.8	3.0
	10	8.3	6.5	82.9	3.6
	20	8.4	6.5	82.7	3.7
Japan	1	0.0	0.0	1.4	100.0
	5	1.3	1.4	1.1	96.2
	10	1.5	2.1	1.8	94.6
	20	1.6	2.2	1.9	94.2

Source: Mills and Mills (1991). Reprinted with the permission of Blackwell Publishers.

Table 8.13 Impulse responses for VAR of international bond yields

Days after shock	Response of US to innovations in			
	US	UK	Germany	Japan
0	0.98	0.00	0.00	0.00
1	0.06	0.01	−0.10	0.05
2	−0.02	0.02	−0.14	0.07
3	0.09	−0.04	0.09	0.08
4	−0.02	−0.03	0.02	0.09
10	−0.03	−0.01	−0.02	−0.01
20	0.00	0.00	−0.10	−0.01

Days after shock	Response of UK to innovations in			
	US	UK	Germany	Japan
0	0.19	0.97	0.00	0.00
1	0.16	0.07	0.01	−0.06
2	−0.01	−0.01	−0.05	0.09
3	0.06	0.04	0.06	0.05
4	0.05	−0.01	0.02	0.07
10	0.01	0.01	−0.04	−0.01
20	0.00	0.00	−0.01	0.00

Days after shock	Response of Germany to innovations in			
	US	UK	Germany	Japan
0	0.07	0.06	0.95	0.00
1	0.13	0.05	0.11	0.02
2	0.04	0.03	0.00	0.00
3	0.02	0.00	0.00	0.01
4	0.01	0.00	0.00	0.09
10	0.01	0.01	−0.01	0.02
20	0.00	0.00	0.00	0.00
0	0.03	0.05	0.12	0.97
1	0.06	0.02	0.07	0.04
2	0.02	0.02	0.00	0.21
3	0.01	0.02	0.06	0.07
4	0.02	0.03	0.07	0.06
10	0.01	0.01	0.01	0.04
20	0.00	0.00	0.00	0.01

Source: Mills and Mills (1991). Reprinted with the permission of Blackwell Publishers.

As one may expect from the low R^2 of the VAR equations, and the lack of cointegration, the bond markets seem very independent of one another. The variance decompositions, which show the proportion of the movements in the dependent variables that are due to their 'own' shocks, versus shocks to the other variables, seem to suggest that the US, UK and Japanese markets are to a certain extent exogenous in this system. That is, little of the movement of the US, UK or Japanese series can be explained by movements other than their own bond yields. In the German case, however, after twenty days, only 83% of movements in the German yield are explained by German shocks. The German yield seems particularly influenced by US (8.4% after twenty days) and UK (6.5% after twenty days) shocks. It also seems that Japanese shocks have the least influence on the bond yields of other markets.

A similar pattern emerges from the impulse response functions, which show the effect of a unit shock applied separately to the error of each equation of the VAR. The markets appear relatively independent of one another, and also informationally efficient in the sense that shocks work through the system very quickly. There is never a response of more than 10% to shocks in any series three days after they have happened; in most cases, the shocks have worked through the system in two days. Such a result implies that the possibility of making excess returns by trading in one market on the basis of 'old news' from another appears very unlikely.

8.10.3 Cointegration in International Bond Markets: Conclusions

A single set of conclusions can be drawn from both of these papers. Both approaches have suggested that international bond markets are not cointegrated. This implies that investors can gain substantial diversification benefits. This is in contrast to results reported for other markets, such as foreign exchange (Baillie and Bollerslev, 1989), commodities (Baillie, 1989) and equities (Taylor and Tonks, 1989). Clare, Maras and Thomas (1995) suggest that the lack of long-term integration between the markets may be due to 'institutional idiosyncrasies', such as heterogeneous maturity and taxation structures, and differing investment cultures, issuance patterns and macroeconomic policies between countries, which imply that the markets operate largely independently of one another.

8.11 Testing the Expectations Hypothesis of the Term Structure of Interest Rates

The following notation replicates that employed by Campbell and Shiller (1991) in their seminal paper. The single, linear expectations theory of the term structure used to represent the expectations hypothesis (hereafter EH), defines a relationship between an n-period interest rate or yield, denoted $R_t^{(n)}$, and an m-period interest rate, denoted $R_t^{(m)}$, where $n > m$. Hence $R_t^{(n)}$ is the interest rate or yield on a longer-term instrument relative to a shorter-term interest rate or yield, $R_t^{(m)}$. More precisely, the EH states that the expected return from investing in an n-period rate will equal the expected return from investing in m-period rates up to $n - m$ periods in the future plus a constant risk-premium, c, which can be expressed as

$$R_t^{(n)} = \frac{1}{q}\sum_{i=0}^{q-1} E_t R_{t+mi}^{(m)} + c \tag{8.77}$$

where $q = n/m$. Consequently, the longer-term interest rate, $R_t^{(n)}$, can be expressed as a weighted-average of current and expected shorter-term interest rates, $R_t^{(m)}$, plus a constant risk premium, c. If equation (8.77) is considered, it can be seen that by subtracting $R_t^{(m)}$ from both sides of the relationship we have

$$R_t^{(n)} - R_t^{(m)} = \frac{1}{q}\sum_{i=0}^{q-1}\sum_{j=1}^{j=i} E_t[\Delta^{(m)} R_{t+jm}^{(m)}] + c \tag{8.78}$$

Examination of equation (8.78) generates some interesting restrictions. If the interest rates under analysis, say $R_t^{(n)}$ and $R_t^{(m)}$, are I(1) series, then, by definition, $\Delta R_t^{(n)}$ and $\Delta R_t^{(m)}$ will be stationary series. There is a general acceptance that interest rates, Treasury bill yields, etc. are well described as I(1) processes and this can be seen in Campbell and Shiller (1988) and Stock and Watson (1988). Further, since c is a constant then it is by definition a stationary series. Consequently, if the EH is to hold, given that c and $\Delta R_t^{(m)}$ are I(0) implying that the RHS of equation (8.78) is stationary, then $R_t^{(n)} - R_t^{(m)}$ must by definition be stationary, otherwise we will have an inconsistency in the order of integration between the RHS and LHS of the relationship. $R_t^{(n)} - R_t^{(m)}$ is commonly known as the *spread* between the n-period and m-period rates, denoted $S_t^{(n,m)}$, which in turn gives an indication of the slope of the term structure. Consequently, it follows that if the EH is to hold, then the spread will be found to be stationary and therefore $R_t^{(n)}$ and $R_t^{(m)}$ will cointegrate with a cointegrating vector (1, −1) for $[R_t^{(n)}, R_t^{(m)}]$. Therefore, the integrated process driving each of the two rates is common to both and hence it can be said that the rates have a common stochastic trend. As a result, since the EH predicts that each interest rate series will cointegrate with the one-period interest rate, it must be true that the stochastic process driving all the rates is the same as that driving the one-period rate, i.e., any combination of rates formed to create a spread should be found to cointegrate with a cointegrating vector (1, −1).

Many examinations of the expectations hypothesis of the term structure have been conducted in the literature, and still no overall consensus appears to have emerged concerning its validity. One such study that tested the expectations hypothesis using a standard data set due to McCulloch (1987) was conducted by Shea (1992). The data comprises a zero coupon term structure for various maturities from one month to twenty-five years, covering the period January 1952–February 1987. Various techniques are employed in Shea's paper, while only his application of the Johansen technique is discussed here. A vector X_t containing the interest rate at each of the maturities is constructed

$$X_t = [R_t \; R_t^{(2)} \; \ldots \; R_t^{(n)}]' \tag{8.79}$$

where R_t denotes the spot interest rate. It is argued that each of the elements of this vector is non-stationary, and hence the Johansen approach is used to model the system of interest rates and to test for cointegration between the rates. Both the λ_{max} and λ_{trace} statistics are employed, corresponding to the use of the maximum eigenvalue and the cumulated eigenvalues, respectively. Shea tests for cointegration between various combinations of the interest rates, measured as returns to maturity. A selection of Shea's results is presented in Table 8.14.

Table 8.14 Tests of the expectations hypothesis using the US zero coupon yield curve with monthly data

Sample period	Interest rates included	Lag length of VAR	Hypothesis is	λ_{max}	λ_{trace}
1952M1–	$X_t = [R_t \; R_t^{(6)}]'$	2	$r = 0$	47.54***	49.82*

1978*M*12					
			$r \leq 1$	2.28	2.28
1952*M*1–1987*M*2	$X_t = [R_t \, R_t^{(120)}]'$	2	$r = 0$	40.66***	43.73*
			$r \leq 1$	3.07	3.07
1952*M*1–1987*M*2	$X_t = [R_t \, R_t^{(60)} \, R_t^{(120)}]'$	2	$r = 0$	40.13***	42.63*
			$r \leq 1$	2.50	2.50
1973*M*5–1987*M*2	$X_t = [R_t \, R_t^{(60)} \, R_t^{(120)}$				
	$R_t^{(180)} \, R_t^{(240)}]'$	7	$r = 0$	34.78***	75.50*
			$r \leq 1$	23.31*	40.72
			$r \leq 2$	11.94	17.41
			$r \leq 3$	3.80	5.47
			$r \leq 4$	1.66	1.66

Notes: *, ** and *** denote significance at the 20%, 10% and 5% levels, respectively; r is the number of cointegrating vectors under the null hypothesis.

Source: Shea (1992). Reprinted with the permission of American Statistical Association. All rights reserved.

The results below, together with the other results presented by Shea, seem to suggest that the interest rates at different maturities are typically cointegrated, usually with one cointegrating vector. As one may have expected, the cointegration becomes weaker in the cases where the analysis involves rates a long way apart on the maturity spectrum. However, cointegration between the rates is a necessary but not sufficient condition for the expectations hypothesis of the term structure to be vindicated by the data. Validity of the expectations hypothesis also requires that any combination of rates formed to create a spread should be found to cointegrate with a cointegrating vector (1, −1). When comparable restrictions are placed on the β estimates associated with the cointegrating vectors, they are typically rejected, suggesting only limited support for the expectations hypothesis.

A Note on Long-memory Models

It is widely believed that (the logs of) asset prices contain a unit root. However, asset return series evidently do not possess a further unit root, although this does not imply that the returns are independent. In particular, it is possible (and indeed, it has been found to be the case with some financial and economic data) that observations from a given series taken some distance apart, show signs of dependence. Such series are argued to possess *long memory*. One way to represent this phenomenon is using a ‘fractionally integrated’ model. In simple terms, a series is integrated of a given order d if it becomes stationary on differencing a minimum of d times. In the fractionally integrated framework, d is allowed to take on non-integer values. This framework has been applied to the estimation of ARMA models (see, for example, Mills, 2008). Under fractionally integrated models, the corresponding autocorrelation function (ACF) will decline hyperbolically, rather than exponentially to zero. Thus, the ACF for a fractionally integrated model dies away considerably more slowly than that of an ARMA model with $d = 0$. The notion of long memory has also been applied to GARCH models (discussed in Chapter 9), where volatility has been found to exhibit longrange dependence. A new class of models known as fractionally integrated GARCH (FIGARCH) have been proposed to allow for this phenomenon (see Ding, Granger, and Engle, 1993 or Bollerslev and Mikkelsen, 1996).

KEY CONCEPTS

The key terms to be able to define and explain from this chapter are

- non-stationary
- unit root
- augmented Dickey–Fuller test
- error correction model
- Johansen technique
- eigenvalues
- explosive process
- spurious regression
- cointegration
- Engle–Granger 2-step approach
- vector error correction model

SELF-STUDY QUESTIONS

1. (a) What kinds of variables are likely to be non-stationary? How can such variables be made stationary?

(b) Why is it in general important to test for non-stationarity in time series data before attempting to build an empirical model?

(c) Define the following terms and describe the processes that they represent

(i) Weak stationarity

(ii) Strict stationarity

(iii) Deterministic trend

(iv) Stochastic trend.

2. A researcher wants to test the order of integration of some time-series data. He decides to use the DF test. He estimates a regression of the form

$$\Delta y_t = \mu + \psi y_{t-1} + u_t$$

and obtains the estimate $\hat{\psi} = -0.02$ with standard error = 0.31.

(a) What are the null and alternative hypotheses for this test?

(b) Given the data, and a critical value of −2.88, perform the test.

(c) What is the conclusion from this test and what should be the next step?

(d) Why is it not valid to compare the estimated test statistic with the corresponding critical value from a *t*-distribution, even though the test statistic takes the form of the usual *t*-ratio?

3. Using the same regression as for Question 2, but on a different set of data, the researcher now obtains the estimate $\hat{\psi} = -0.52$ with standard error = 0.16.

(a) Perform the test.

(b) What is the conclusion, and what should be the next step?

(c) Another researcher suggests that there may be a problem with this methodology since it assumes that the disturbances (u_t) are white noise. Suggest a possible source of difficulty and how the researcher might in practice get around it.

4. (a) Consider a series of values for the spot and futures prices of a given commodity. In the context of these series, explain the concept of cointegration. Discuss how a researcher might test for cointegration between the variables using the Engle–Granger approach. Explain also the steps involved in the formulation of an error correction model.

(b) Give a further example from finance where cointegration between a set of variables may be expected. Explain, by reference to the implication of non-cointegration, why cointegration between the series might be expected.

5. (a) Briefly outline Johansen's methodology for testing for cointegration between a set of variables in the context of a VAR.

(b) A researcher uses the Johansen procedure and obtains the following test statistics (and critical values)

r	λ_{max}	5% critical value
0	38.962	33.178
1	29.148	27.169
2	16.304	20.278
3	8.861	14.036
4	1.994	3.962

Determine the number of cointegrating vectors.

(c) 'If two series are cointegrated, it is not possible to make inferences regarding the cointegrating relationship using the Engle–Granger technique since the residuals from the cointegrating regression are likely to be autocorrelated.' How does Johansen circumvent this problem to test hypotheses about the cointegrating relationship?

(d) Give one or more examples from the academic finance literature of where the Johansen systems technique has been employed. What were the main results and conclusions of this research?

(e) Compare the Johansen maximal eigenvalue test with the test based on the trace statistic. State clearly the null and alternative hypotheses in each case.

6. (a) Suppose that a researcher has a set of three variables, y_t (t = 1, …, T), i.e., y_t denotes a p-variate, or $p \times 1$ vector, that she wishes to test for the existence of cointegrating relationships using the Johansen procedure.

   What is the implication of finding that the rank of the appropriate matrix takes on a value of

   (i) 0 (ii) 1 (iii) 2 (iv) 3?

   (b) The researcher obtains results for the Johansen test using the variables outlined in part (a) of the question as follows

r	λ_{max}	5% critical value
0	38.65	30.26
1	26.91	23.84
2	10.67	17.72
3	8.55	10.71

   Determine the number of cointegrating vectors, explaining your answer.

7. Compare and contrast the Engle–Granger and Johansen methodologies for testing for cointegration and modelling cointegrated systems. Which, in your view, represents the superior approach and why?

8. (a) What issues arise when testing for a unit root if there is a structural break in the series under investigation?

   (b) What are the limitations of the Perron (1989) approach for dealing with structural breaks in testing for a unit root?

[1] This material is fairly specialised and thus is not well covered by most of the standard textbooks. But for any readers wishing to see more detail, there is a useful and accessible chapter by Perron in Rao (1994). There is also a chapter on structural change in Maddala and Kim (1999).

[2] EuroSterling interest rates are those at which money is loaned/borrowed in British pounds but outside of the UK.

[3] For further reading on this topic, the book by Harris (1995) provides an extremely clear introduction to unit roots and cointegration, including a section on seasonal unit roots.

[2] Strictly, the eigenvalues used in the test statistics are taken from rank-restricted product moment matrices and not of Π itself.

9

Modelling Volatility and Correlation

LEARNING OUTCOMES

In this chapter, you will learn how to

- Discuss the features of data that motivate the use of GARCH models
- Explain how conditional volatility models are estimated
- Test for 'ARCH-effects' in time-series data
- Produce forecasts from GARCH models
- Contrast various models from the GARCH family
- Discuss the three hypothesis testing procedures available under maximum likelihood estimation
- Construct multivariate conditional volatility models and compare between alternative specifications

9.1 Motivations: An Excursion into Non-Linearity Land

All of the models that have been discussed in Chapters 3–8 of this book have been linear in nature – that is, the model is linear in the parameters, so that there is one parameter multiplied by each variable in the model. For example, a structural model could be something like

$$y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + u \quad \textbf{(9.1)}$$

or more compactly $y = X\beta + u$. It was additionally assumed that $u_t \sim N(0, \sigma^2)$.

The linear paradigm as described above is a useful one. The properties of linear estimators are very well researched and very well understood. Many models that appear, *prima facie,* to be non-linear, can be made linear by taking logarithms or some other suitable transformation. However, it is likely that many relationships in finance are intrinsically non-linear. As Campbell, Lo and MacKinlay (1997) state, the payoffs to options are non-linear in some of the input variables, and investors' willingness to trade off returns and risks are also non-linear. These observations provide clear motivations for consideration of non-linear models in a variety of circumstances in order to capture better the relevant features of the data.

Linear structural (and time series) models such as equation 9.1 are also unable to explain a number of important features common to much financial data, including

- *Leptokurtosis* – that is, the tendency for financial asset returns to have distributions that exhibit fat tails and excess peakedness at the mean.
- *Volatility clustering or volatility pooling* – the tendency for volatility in financial markets to appear in bunches. Thus large returns (of either sign) are expected to follow large returns, and small returns (of either sign) to follow small returns. A plausible explanation for this phenomenon, which seems to be an almost universal feature of asset return series in finance, is that the information arrivals which drive price changes themselves occur in bunches rather than being evenly spaced over time.
- *Leverage effects* – the tendency for volatility to rise more following a large price fall than following a price rise of the same magnitude.

Campbell *et al.* (1997) broadly define a non-linear data generating process as one where the current value of the series is related non-linearly to current and previous values of the error term

$$y_t = f(u_t, u_{t-1}, u_{t-2}, \ldots) \tag{9.2}$$

where u_t is an iid error term and f is a non-linear function. According to Campbell *et al.*, a more workable and slightly more specific definition of a non-linear model is given by the equation

$$y_t = g(u_{t-1}, u_{t-2}, \ldots) + u_t\, \sigma^2(u_{t-1}, u_{t-2}, \ldots) \tag{9.3}$$

where g is a function of past error terms only, and σ^2 can be interpreted as a variance term, since it is multiplied by the current value of the error. Campbell *et al.* usefully characterise models with non-linear $g(\bullet)$ as being non-linear in mean, while those with non-linear $\sigma(\bullet)^2$ are characterised as being non-linear in variance.

Models can be linear in mean and variance (e.g., the CLRM, ARMA models) or linear in mean, but non-linear in variance (e.g., GARCH models). Models could also be classified as non-linear in mean but linear in variance (e.g., bicorrelations models, a simple example of which is of the following form (see Brooks and Heravi, 1999))

$$y_t = \alpha_0 + \alpha_1 y_{t-1} y_{t-2} + u_t \tag{9.4}$$

Finally, models can be non-linear in both mean and variance (e.g., the hybrid threshold model with GARCH errors employed by Brooks, 2001).

9.1.1 Types of Non-Linear Models

There are an infinite number of different types of non-linear model. However, only a small number of non-linear models have been found to be useful for modelling financial data. The most popular non-linear financial models are the ARCH or GARCH models used for modelling and forecasting volatility, and switching models, which allow the behaviour of a series to follow different processes at different points in time. Models for volatility and correlation will be discussed in this chapter, with switching models being covered in Chapter 10.

9.1.2 Testing for Non-Linearity

How can it be determined whether a non-linear model may potentially be appropriate for the data? The answer to this question should come at least in part from financial theory: a non-linear model should be used where financial theory suggests that the relationship between variables should be such as to require a non-linear model. But the linear versus non-linear choice may also be made partly on statistical grounds – deciding whether a linear specification is sufficient to describe all of the most important features of the data at hand.

So what tools are available to detect non-linear behaviour in financial time series? Unfortunately, 'traditional' tools of time-series analysis (such as estimates of the autocorrelation or partial autocorrelation function, or 'spectral analysis', which involves looking at the data in the frequency domain) are likely to be of little use. Such tools may find no evidence of linear structure in the data, but this would not necessarily imply that the same observations are independent of one another.

However, there are a number of tests for non-linear patterns in time series that are available to the researcher. These tests can broadly be split into two types: general tests and specific tests. General tests, also sometimes called 'portmanteau' tests, are usually designed to detect many departures from randomness in data. The implication is that such tests will detect a variety of non-linear structures in data, although these tests are unlikely to tell the researcher which type of non-linearity is present! Perhaps the simplest general test for non-linearity is Ramsey's RESET test discussed in Chapter 4, although there are many other popular tests available. One of the most widely used tests is known as the BDS test (see Brock, Hsieh and LeBaron, 1991) named after the three authors who first developed it. BDS is a pure hypothesis test. That is, it has as its null hypothesis that the data are pure noise (completely random), and it has been argued to have power to detect a variety of departures from randomness – linear or non-linear stochastic processes, deterministic chaos, etc. (see Brock *et al.*, 1991). The BDS test follows a standard normal distribution under the null hypothesis. The details of this test, and others, are technical and beyond the scope of this book, although computer code for BDS estimation is now widely available free of charge on the internet.

As well as applying the BDS test to raw data in an attempt to 'see if there is anything there', another suggested use of the test is as a model diagnostic. The idea is that a proposed model (e.g., a linear model, GARCH, or some other non-linear model) is estimated, and the test applied to the (standardised) residuals in order to 'see what is left'. If the proposed model is adequate, the standardised residuals should be white noise, while if the postulated model is insufficient to capture all of the relevant features of the data, the BDS test statistic for the standardised residuals will be statistically significant. This is an excellent idea in theory, but has difficulties in practice. First, if the postulated model is a non-linear one (such as GARCH), the asymptotic distribution of the test statistic will be altered, so that it will no longer follow a normal distribution. This requires new critical values to be constructed via simulation for every type of non-linear model whose residuals are to be tested. More seriously, if a non-linear model is fitted to the data, any remaining structure is typically garbled, resulting in the test either being unable to detect additional structure present in the data (see Brooks and Henry, 2000) or selecting as adequate a model which is not even in the correct class for that data generating process (see Brooks and Heravi, 1999).

Other popular tests for non-linear structure in time-series data include the bispectrum test due to Hinich (1982), the bicorrelation test (see Hsieh, 1993; Hinich, 1996; or Brooks and Heravi, 1999 for its multivariate generalisation).

Most applications of the above tests conclude that there is non-linear dependence in financial asset returns series, but that the dependence is best characterised by a GARCH-type process (see Hinich and Patterson, 1985; Baillie and Bollerslev, 1989; Brooks, 1996; and the references therein for applications of non-linearity tests to financial data).

Specific tests, on the other hand, are usually designed to have power to find specific types of non-linear structure. Specific tests are unlikely to detect other forms of nonlinearities in the data, but their results will by definition offer a class of models that should be relevant for the data at hand. Examples of specific tests will be offered later in this and subsequent chapters.

9.1.3 Chaos in Financial Markets

Econometricians have searched long and hard for chaos in financial, macroeconomic and microeconomic data, with very limited success to date. *Chaos theory* is a notion taken from the physical sciences that suggests that there could be a deterministic, non-linear set of equations underlying the behaviour of financial series or markets. Such behaviour will appear completely random to the standard statistical tests developed for application to linear models. The motivation behind this endeavour is clear: a positive sighting of chaos implies that while, by definition, long-term forecasting would be futile, short-term forecastability and controllability are possible, at least in theory, since there is some deterministic structure underlying the data. Varying definitions of what actually constitutes chaos can be found in the literature, but a robust definition is that a system is chaotic if it exhibits sensitive dependence on initial conditions (SDIC). The concept of SDIC embodies the fundamental characteristic of chaotic systems that if an infinitesimal change is made to the initial conditions (the initial state of the system), then the corresponding change iterated through the system for some arbitrary length of time will grow exponentially. Although several statistics are commonly used to test for the presence of chaos, only one is arguably a true test for chaos, namely estimation of the largest Lyapunov exponent.

The largest Lyapunov exponent measures the rate at which information is lost from a system. A positive largest Lyapunov exponent implies sensitive dependence, and therefore that evidence of chaos has been obtained. This has important implications for the predictability of the underlying system, since the fact that all initial conditions are in practice estimated with some error (owing either to measurement error or exogenous noise), will imply that long-term forecasting of the system is impossible as all useful information is likely to be lost in just a few time steps.

Chaos theory was hyped and embraced by both the academic literature and in financial markets worldwide in the 1980s. However, almost without exception, applications of chaos theory to financial markets have been unsuccessful. Consequently, although the ideas generate continued debate owing to the interesting mathematical properties and the possibility of finding a prediction holy grail, academic and practitioner interest in chaotic models for financial markets has arguably almost disappeared. The primary reason for the failure of the chaos theory approach appears to be the fact that financial markets are extremely complex, involving a very large number of different participants, each with different objectives and different sets of information – and, above all, each of whom are human with human emotions and irrationalities. The consequence of this is that financial and economic data are usually far noisier and 'more random' than data from other disciplines, making the specification of a deterministic model very much harder and possibly even futile.

9.1.4 Neural Network Models

Artificial neural networks (ANNs) are a class of models whose structure is broadly motivated by the way that *the brain performs computation*. ANNs have been widely employed in finance for tackling time-series and classification problems. Recent applications have included forecasting financial asset returns, volatility, bankruptcy and takeover prediction. Applications are contained in the books by Trippi and Turban (1993), Van Eyden (1996) and Refenes (1995). A technical collection of papers on the econometric aspects of neural networks is given by White (1992), while an excellent general introduction and a description of the issues surrounding neural network model estimation and analysis is contained in Franses and van Dijk (2000).

Neural networks have virtually no theoretical motivation in finance (they are often termed a 'black box' technology), but owe their popularity to their ability to fit any functional relationship in the data to an arbitrary degree of accuracy. The most common class of ANN models in finance are known as *feedforward network models*. These have a set of inputs (akin to regressors) linked to one or more outputs (akin to the regressand) via one or more ‘hidden’ or intermediate layers. The size and number of hidden layers can be modified to give a closer or less close fit to the data sample, while a feedforward network with no hidden layers is simply a standard linear regression model.

Neural network models are likely to work best in situations where financial theory has virtually nothing to say about the likely functional form for the relationship between a set of variables. However, their popularity has arguably waned over the past five years or so as a consequence of several perceived problems with their employment. First, the coefficient estimates from neural networks do not have any real theoretical interpretation. Second, virtually no diagnostic or specification tests are available for estimated models to determine whether the model under consideration is adequate. Third, ANN models can provide excellent fits in-sample to a given set of ‘training’ data, but typically provide poor out-of-sample forecast accuracy. The latter result usually arises from the tendency of neural networks to fit closely to sample-specific data features and ‘noise’, and therefore their inability to generalise. Various methods of resolving this problem exist, including ‘pruning’ (removing some parts of the network) or the use of information criteria to guide the network size. Finally, the non-linear estimation of neural network models can be cumbersome and computationally time-intensive, particularly, for example, if the model must be estimated rolling through a sample to produce a series of one-step-ahead forecasts.

9.2 Models for Volatility

Modelling and forecasting stock market volatility has been the subject of vast empirical and theoretical investigation over the past decade or so by academics and practitioners alike. There are a number of motivations for this line of inquiry. Arguably, volatility is one of the most important concepts in the whole of finance. Volatility, as measured by the standard deviation or variance of returns, is often used as a crude measure of the total risk of financial assets. Many value-at-risk models for measuring market risk require the estimation or forecast of a volatility parameter. The volatility of stock market prices also enters directly into the Black–Scholes formula for deriving the prices of traded options.

The next few sections will discuss various models that are appropriate to capture the stylised features of volatility, discussed below, that have been observed in the literature.

9.3 Historical Volatility

The simplest model for volatility is the historical estimate. Historical volatility simply involves calculating the variance (or standard deviation) of returns in the usual way over some historical period, and this then becomes the volatility forecast for all future periods. The historical average variance (or standard deviation) was traditionally used as the volatility input to options pricing models, although there is a growing body of evidence suggesting that the use of volatility predicted from more sophisticated time-series models will lead to more accurate option valuations (see, for example, Akgiray, 1989; or Chu and Freund, 1996). Historical volatility is still useful as a benchmark for comparing the forecasting ability of more complex time models.

9.4 Implied Volatility Models

All pricing models for financial options require a volatility estimate or forecast as an input. Given the price of a traded option obtained from transactions data, it is possible to determine the volatility forecast over the lifetime of the option implied by the option's valuation. For example, if the standard Black–Scholes model is used, the option price, the time to maturity, a risk-free rate of interest, the strike price and the current value of the underlying asset, are all either specified in the details of the options contracts or are available from market data. Therefore, given all of these quantities, it is possible to use a numerical procedure, such as the method of bisections or Newton–Raphson to derive the volatility implied by the option (see Watsham and Parramore, 2004). This implied volatility is the market's forecast of the volatility of underlying asset returns over the lifetime of the option.

9.5 Exponentially Weighted Moving Average Models

The exponentially weighted moving average (EWMA) is essentially a simple extension of the historical average volatility measure, which allows more recent observations to have a stronger impact on the forecast of volatility than older data points. Under an EWMA specification, the latest observation carries the largest weight, and weights associated with previous observations decline exponentially over time. This approach has two advantages over the simple historical model. First, volatility is in practice likely to be affected more by recent events, which carry more weight, than events further in the past. Second, the effect on volatility of a single given observation declines at an exponential rate as weights attached to recent events fall. On the other hand, the simple historical approach could lead to an abrupt change in volatility once the shock falls out of the measurement sample. And if the shock is still included in a relatively long measurement sample period, then an abnormally large observation will imply that the forecast will remain at an artificially high level even if the market is subsequently tranquil. The exponentially weighted moving average model can be expressed in several ways, e.g.,

$$\sigma_t^2 = (1-\lambda)\sum_{j=0}^{\infty} \lambda^j (r_{t-j} - \bar{r})^2 \tag{9.5}$$

where σ_t^2 is the estimate of the variance for period t, which also becomes the forecast of future volatility for all periods, $\bar{r}$ is the average return estimated over the observations and λ is the 'decay factor', which determines how much weight is given to recent versus older observations. The decay factor could be estimated, but in many studies is set at 0.94 as recommended by RiskMetrics, producers of popular risk measurement software. Note also that RiskMetrics and many academic papers assume that the average return, $\bar{r}$, is zero. For data that is of daily frequency or higher, this is not an unreasonable assumption, and is likely to lead to negligible loss of accuracy since it will typically be very small. Obviously, in practice, an infinite number of observations will not be available on the series, so that the sum in equation (9.5) must be truncated at some fixed lag. As with exponential smoothing models, the forecast from an EWMA model for all prediction horizons is the most recent weighted average estimate.

It is worth noting two important limitations of EWMA models. First, while there are several methods that could be used to compute the EWMA, the crucial element in each case is to remember that when the infinite sum in equation (9.5) is replaced with a finite sum of observable data, the weights from the given expression will now sum to less than one. In the case of small samples, this could make a large difference to the computed EWMA and thus a correction may be necessary. Second, most time series models, such as GARCH (see below), will have forecasts that tend towards the unconditional variance of the series as the prediction horizon increases.

This is a good property for a volatility forecasting model to have, since it is well known that volatility series are 'mean-reverting'. This implies that if they are currently at a high level relative to their historic average, they will have a tendency to fall back towards their average level, while if they are at a low level relative to their historic average, they will have a tendency to rise back towards the average. This feature is accounted for in GARCH volatility forecasting models, but not by EWMAs.

9.6 Autoregressive Volatility Models

Autoregressive volatility models are a relatively simple example from the class of stochastic volatility specifications. The idea is that a time series of observations on some volatility proxy are obtained. The standard Box–Jenkins-type procedures for estimating autoregressive (or ARMA) models can then be applied to this series. If the quantity of interest in the study is a daily volatility estimate, two natural proxies have been employed in the literature: squared daily returns, or daily range estimators. Producing a series of daily squared returns trivially involves taking a column of observed returns and squaring each observation. The squared return at each point in time, *t*, then becomes the daily volatility estimate for day *t*. A range estimator typically involves calculating the log of the ratio of the highest observed price to the lowest observed price for trading day *t*, which then becomes the volatility estimate for day *t*

$$\sigma_t^2 = \ln\left(\frac{high_t}{low_t}\right) \tag{9.6}$$

Given either the squared daily return or the range estimator, a standard autoregressive model is estimated, with the coefficients β_i estimated using OLS (or maximum likelihood – see below). The forecasts are also produced in the usual fashion discussed in Chapter 6 in the context of ARMA models

$$\sigma_t^2 = \beta_0 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 + \varepsilon_t \tag{9.7}$$

9.7 Autoregressive Conditionally Heteroscedastic (ARCH) Models

One particular non-linear model in widespread usage in finance is known as an ‘ARCH’ model (ARCH stands for ‘autoregressive conditionally heteroscedastic’). To see why this class of models is useful, recall that a typical structural model could be expressed by an equation of the form given in equation (9.1) on p. 384 with $u_t \sim N(0, \sigma^2)$. The assumption of the CLRM that the variance of the errors is constant is known as *homoscedasticity* (i.e., it is assumed that $\text{var}(u_t) = \sigma^2$). If the variance of the errors is not constant, this would be known as *heteroscedasticity*. As was explained in Chapter 5, if the errors are heteroscedastic, but assumed homoscedastic, an implication would be that standard error estimates could be wrong. It is unlikely in the context of financial time series that the variance of the errors will be constant over time, and hence it makes sense to consider a model that does not assume that the variance is constant, and which describes how the variance of the errors evolves.

Another important feature of many series of financial asset returns that provides a motivation for the ARCH class of models, is known as ‘volatility clustering’ or ‘volatility pooling’. Volatility clustering describes the tendency of large changes in asset prices (of either sign) to follow large changes and small changes (of either sign) to follow small changes. In other words, the current level of volatility tends to be positively correlated with its level during the immediately preceding periods. This phenomenon is demonstrated in Figure 9.1, which plots daily S&P500 returns for August 2003–July 2018.

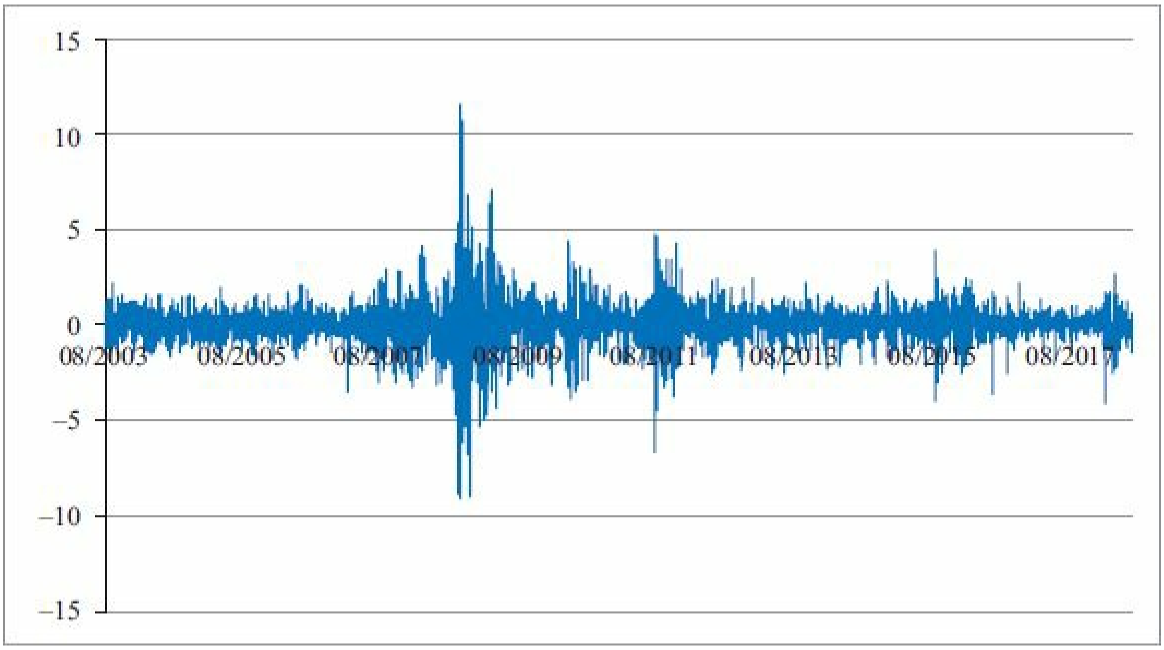


Figure 9.1 Daily S&P returns for August 2003–July 2018

The important point to note from Figure 9.1 is that *volatility occurs in bursts*. There appears to have been a prolonged period of relative tranquillity in the market during the 2003 to 2008 period until the financial crisis began, evidenced by only relatively small positive and negative returns until that point. On the other hand, during mid-2008 to mid-2009, there was far more volatility, when many large positive and large negative returns were observed during a short space of time. Abusing the terminology slightly, it could be stated that 'volatility is autocorrelated'.

How could this phenomenon, which is common to many series of financial asset returns, be parameterised (modelled)? One approach is to use an ARCH model. To understand how the model works, a definition of the conditional variance of a random variable, u_t, is required. The distinction between the conditional and unconditional variances of a random variable is exactly the same as that of the conditional and unconditional mean. The conditional variance of u_t may be denoted σ_t^2, which is written as

$$\sigma_t^2 = \text{var}(u_t \mid u_{t-1}, u_{t-2}, \ldots) = \text{E}[(u_t - \text{E}(u_t))^2 \mid u_{t-1}, u_{t-2}, \ldots] \quad \textbf{(9.8)}$$

It is usually assumed that $\text{E}(u_t) = 0$, so

$$\sigma_t^2 = \text{var}(u_t \mid u_{t-1}, u_{t-2}, \ldots) = \text{E}\left[u_t^2 \mid u_{t-1}, u_{t-2}, \ldots\right] \quad \textbf{(9.9)}$$

Equation (9.9) states that the conditional variance of a zero mean normally distributed random variable u_t is equal to the conditional expected value of the square of u_t. Under the ARCH model, the 'autocorrelation in volatility' is modelled by allowing the conditional variance of the error term, σ_t^2, to depend on the immediately previous value of the squared error

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 \quad \textbf{(9.10)}$$

The above model is known as an ARCH(1), since the conditional variance depends on only one lagged squared error. Notice that equation (9.10) is only a partial model, since nothing has been said yet about the conditional mean. Under ARCH, the conditional mean equation (which describes how the dependent variable, y_t, varies over time) could take almost any form that the researcher wishes. One example of a full model would be

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 x_{4t} + u_t \qquad u_t \sim \mathrm{N}(0, \sigma_t^2) \tag{9.11}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 \tag{9.12}$$

The model given by equations (9.11) and (9.12) could easily be extended to the general case where the error variance depends on q lags of squared errors, which would be known as an ARCH(q) model:

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_q u_{t-q}^2 \tag{9.13}$$

Instead of calling the conditional variance σ_t^2, in the literature it is often called h_t, so that the model would be written

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 x_{4t} + u_t \qquad u_t \sim \mathrm{N}(0, h_t) \tag{9.14}$$

$$h_t = \alpha_0 + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_q u_{t-q}^2 \tag{9.15}$$

The remainder of this chapter will use σ_t^2 to denote the conditional variance at time t, except for computer instructions where h_t will be used since it is easier not to use Greek letters.

9.7.1 Another Way of Expressing ARCH Models

For illustration, consider an ARCH(1). The model can be expressed in two ways that look different but are in fact identical. The first is as given in equations (9.11) and (9.12) above. The second way would be as follows

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 x_{4t} + u_t \tag{9.16}$$

$$u_t = v_t \sigma_t \qquad v_t \sim \mathrm{N}(0, 1) \tag{9.17}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 \tag{9.18}$$

The form of the model given in equations (9.11) and (9.12) is more commonly presented, although specifying the model as in equations (9.16)–(9.18) is required in order to use a GARCH process in a simulation study (see Chapter 13). To show that the two methods for expressing the model are equivalent, consider that in equation (9.17), v_t is normally distributed with zero mean and unit variance, so that u_t will also be normally distributed with zero mean and variance σ_t^2.

9.7.2 Non-Negativity Constraints

Since h_t is a conditional variance, its value must always be strictly positive; a negative variance at any point in time would be meaningless. The variables on the RHS of the conditional variance equation are all squares of lagged errors, and so by definition will not be negative. In order to ensure that these always result in positive conditional variance estimates, all of the coefficients in the conditional variance are usually required to be non-negative. If one or more of the coefficients were to take on a negative value, then for a sufficiently large lagged squared innovation term attached to that coefficient, the fitted value from the model for the conditional variance could be negative. This would clearly be nonsensical. So, for example, in the case of equation (9.18), the non-negativity condition would be $\alpha_0 \geq 0$ and $\alpha_1 \geq 0$. More generally, for an ARCH(q) model, all coefficients would be required to be non-negative: $\alpha_i \geq 0 \; \forall \; i = 0, 1, 2, \ldots, q$. In fact, this is a sufficient but not necessary condition for non-negativity of the conditional variance (i.e., it is a slightly stronger condition than is actually necessary).

9.7.3 Testing for 'ARCH Effects'

A test for determining whether 'ARCH effects' are present in the residuals of an estimated model may be conducted using the steps outlined in Box 9.1.

BOX 9.1 Testing for 'ARCH effects'

(1) Run any postulated linear regression of the form given in the equation above, e.g.,

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 x_{4t} + u_t \tag{9.19}$$

saving the residuals, $\hat{u}_t$.

(2) Square the residuals, and regress them on q own lags to test for ARCH of order q, i.e., run the regression

$$\hat{u}_t^2 = \gamma_0 + \gamma_1 \hat{u}_{t-1}^2 + \gamma_2 \hat{u}_{t-2}^2 + \cdots + \gamma_q \hat{u}_{t-q}^2 + v_t \tag{9.20}$$

where v_t is an error term.

Obtain R^2 from this regression.

(3) The test statistic is defined as TR^2 (the number of observations multiplied by the coefficient of multiple correlation) from the last regression, and is distributed as a $\chi^2(q)$

(4) The null and alternative hypotheses are

$$H_0 : \gamma_1 = 0 \text{ and } \gamma_2 = 0 \text{ and } \gamma_3 = 0 \text{ and} \ldots \text{and } \gamma_q = 0$$
$$H_1 : \gamma_1 \neq 0 \text{ or } \gamma_2 \neq 0 \text{ or } \gamma_3 \neq 0 \text{ or} \ldots \text{or } \gamma_q \neq 0$$

Thus, the test is one of a joint null hypothesis that all q lags of the squared residuals have coefficient values that are not significantly different from zero. If the value of the test statistic is greater than the critical value from the χ^2 distribution, then reject the null hypothesis. The test can also be thought of as a test for autocorrelation in the squared residuals. As well as testing the residuals of an estimated model, the ARCH test is frequently applied to raw returns data.

9.7.4 Limitations of ARCH(q) Models

ARCH provided a framework for the analysis and development of time series models of volatility. However, ARCH models themselves have rarely been used in the last decade or more, since they bring with them a number of difficulties

- How should *the value of q*, the number of lags of the squared residual in the model, be decided? One approach to this problem would be the use of a likelihood ratio test, discussed later in this chapter, although there is no clearly best approach.
- The value of q, the number of lags of the squared error that are required to capture all of the dependence in the conditional variance, might be *very large*. This would result in a large conditional variance model that was not parsimonious. Engle (1982) circumvented this problem by specifying an arbitrary linearly declining lag length on an ARCH(4)

$$\sigma_t^2 = \gamma_0 + \gamma_1\left(0.4\hat{u}_{t-1}^2 + 0.3\hat{u}_{t-2}^2 + 0.2\hat{u}_{t-3}^2 + 0.1\hat{u}_{t-4}^2\right) \qquad \textbf{(9.21)}$$

  such that only two parameters are required in the conditional variance equation (γ_0 and γ_1), rather than the five which would be required for an unrestricted ARCH(4).
- *Non-negativity constraints might be violated*. Everything else equal,

the more parameters there are in the conditional variance equation, the more likely it is that one or more of them will have negative estimated values.

A natural extension of an ARCH(q) model which overcomes some of these problems is a GARCH model. In contrast with ARCH, GARCH models are extremely widely employed in practice.

9.8 Generalised ARCH (GARCH) Models

The GARCH model was developed independently by Bollerslev (1986) and Taylor (1986). The GARCH model allows the conditional variance to be dependent upon previous own lags, so that the conditional variance equation in the simplest case is now

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta\sigma_{t-1}^2 \quad \textbf{(9.22)}$$

This is a GARCH(1,1) model. σ_t^2 is known as the *conditional variance* since it is a one-period ahead estimate for the variance calculated based on any past information thought relevant. Using the GARCH model it is possible to interpret the current fitted variance, h_t, as a weighted function of a long-term average value (dependent on α_0), information about volatility during the previous period $(\alpha_1 u_{t-1}^2)$ and the fitted variance from the model during the previous period $(\beta\sigma_{t-1}^2)$. Note that the GARCH model can be expressed in a form that shows that it is effectively an ARMA model for the conditional variance. To see this, consider that the squared return at time t relative to the conditional variance is given by

$$\varepsilon_t = u_t^2 - \sigma_t^2 \quad \textbf{(9.23)}$$

or

$$\sigma_t^2 = u_t^2 - \varepsilon_t \quad \textbf{(9.24)}$$

Using the latter expression to substitute in for the conditional variance in equation (9.22)

$$u_t^2 - \varepsilon_t = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta\left(u_{t-1}^2 - \varepsilon_{t-1}\right) \quad \textbf{(9.25)}$$

Rearranging

$$u_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta u_{t-1}^2 - \beta\varepsilon_{t-1} + \varepsilon_t \quad \textbf{(9.26)}$$

so that

$$u_t^2 = \alpha_0 + (\alpha_1 + \beta)u_{t-1}^2 - \beta\varepsilon_{t-1} + \varepsilon_t \quad \textbf{(9.27)}$$

This final expression is an ARMA(1,1) process for the squared errors.

Why is GARCH a better and therefore a far more widely used model than ARCH? The answer is that the former is more parsimonious, and avoids overfitting. Consequently, the model is less likely to breach non-negativity constraints. In order to illustrate why the model is parsimonious, first take the conditional variance equation in the GARCH(1,1) case, subtract 1 from each of the time subscripts of the conditional variance equation in equation (9.22), so that the following expression would be obtained

$$\sigma_{t-1}^2 = \alpha_0 + \alpha_1 u_{t-2}^2 + \beta\sigma_{t-2}^2 \quad \textbf{(9.28)}$$

and subtracting 1 from each of the time subscripts again

$$\sigma_{t-2}^2 = \alpha_0 + \alpha_1 u_{t-3}^2 + \beta\sigma_{t-3}^2 \quad \textbf{(9.29)}$$

Substituting into equation (9.22) for σ_{t-1}^2

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta\left(\alpha_0 + \alpha_1 u_{t-2}^2 + \beta\sigma_{t-2}^2\right) \quad \textbf{(9.30)}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \alpha_0\beta + \alpha_1\beta u_{t-2}^2 + \beta^2\sigma_{t-2}^2 \quad \textbf{(9.31)}$$

Now substituting into equation (9.31) for σ_{t-2}^2

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \alpha_0\beta + \alpha_1\beta u_{t-2}^2 + \beta^2\left(\alpha_0 + \alpha_1 u_{t-3}^2 + \beta\sigma_{t-3}^2\right) \quad \textbf{(9.32)}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \alpha_0\beta + \alpha_1\beta u_{t-2}^2 + \alpha_0\beta^2 + \alpha_1\beta^2 u_{t-3}^2 + \beta^3\sigma_{t-3}^2 \quad \textbf{(9.33)}$$

$$\sigma_t^2 = \alpha_0(1 + \beta + \beta^2) + \alpha_1 u_{t-1}^2(1 + \beta L + \beta^2 L^2) + \beta^3\sigma_{t-3}^2 \quad \textbf{(9.34)}$$

An infinite number of successive substitutions of this kind would yield

$$\sigma_t^2 = \alpha_0(1 + \beta + \beta^2 + \cdots) + \alpha_1 u_{t-1}^2(1 + \beta L + \beta^2 L^2 + \cdots) + \beta^\infty\sigma_0^2 \quad \textbf{(9.35)}$$

The first expression on the RHS of equation (9.35) is simply a constant, and as the number of observations tends to infinity, β^∞ will tend to zero. Hence, the GARCH(1,1) model can be written as

$$\sigma_t^2 = \gamma_0 + \alpha_1 u_{t-1}^2 (1 + \beta L + \beta^2 L^2 + \cdots) \quad \textbf{(9.36)}$$
$$= \gamma_0 + \gamma_1 u_{t-1}^2 + \gamma_2 u_{t-2}^2 + \cdots , \quad \textbf{(9.37)}$$

which is a restricted infinite order ARCH model. Thus the GARCH(1,1) model, containing only three parameters in the conditional variance equation, is a very parsimonious model, that allows an infinite number of past squared errors to influence the current conditional variance.

The GARCH(1,1) model can be extended to a GARCH(*p*,*q*) formulation, where the current conditional variance is parameterised to depend upon *q* lags of the squared error and *p* lags of the conditional variance

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_q u_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-2}^2 + \cdots + \beta_p \sigma_{t-p}^2 \quad \textbf{(9.38)}$$
$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i u_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 \quad \textbf{(9.39)}$$

But in general a GARCH(1,1) model will be sufficient to capture the volatility clustering in the data, and rarely is any higher order model estimated or even entertained in the academic finance literature.

9.8.1 The Unconditional Variance Under a GARCH Specification

The conditional variance is changing, but the unconditional variance of u_t is constant and given by

$$\text{var}(u_t) = \frac{\alpha_0}{1 - (\alpha_1 + \beta)} \quad \textbf{(9.40)}$$

so long as $\alpha_1 + \beta < 1$. For $\alpha_1 + \beta \geq 1$, the unconditional variance of u_t is not defined, and this would be termed 'non-stationarity in variance'. $\alpha_1 + \beta = 1$ would be known as a 'unit root in variance', also termed 'Integrated GARCH' or IGARCH. Non-stationarity in variance does not have a strong theoretical motivation for its existence, as would be the case for non-stationarity in the mean (e.g., of a price series). Furthermore, a GARCH model whose coefficients imply non-stationarity in variance would have some highly undesirable properties. One illustration of these relates to the forecasts of variance made from such models. For stationary GARCH models, conditional variance forecasts converge upon the long-term average value of the variance as the prediction horizon increases (see below). For IGARCH processes, this convergence will not happen, while for $\alpha_1 + \beta > 1$, the conditional variance forecast will tend to infinity as the forecast horizon increases.

9.9 Estimation of ARCH/GARCH Models

Since the model is no longer of the usual linear form, OLS cannot be used for GARCH model estimation. There are a variety of reasons for this, but the simplest and most fundamental is that OLS minimises the *RSS*. The *RSS* depends only on the parameters in the conditional mean equation, and not the conditional variance, and hence RSS minimisation is no longer an appropriate objective.

In order to estimate models from the GARCH family, another technique known as *maximum likelihood* is employed. Essentially, the method works by finding the most likely values of the parameters given the actual data. More specifically, a log-likelihood function (*LLF*) is formed and the values of the parameters that maximise it are sought. Maximum likelihood estimation can be employed to find parameter values for both linear and non-linear models. The steps involved in actually estimating an ARCH or GARCH model are shown in Box 9.2. The following section will elaborate on points (2) and (3) presented in the box, explaining how the *LLF* is derived.

BOX 9.2 Estimating an ARCH or GARCH model

(1) Specify the appropriate equations for the mean and the variance – e.g. an AR(1)-GARCH(1,1) model

$$y_t = \mu + \phi y_{t-1} + u_t, u_t \sim \mathrm{N}(0, \sigma_t^2) \tag{9.41}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta \sigma_{t-1}^2 \tag{9.42}$$

(2) Specify the log-likelihood function (*LLF*) to maximise under a normality assumption for the disturbances

$$L = -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\ln\left(\sigma_t^2\right) - \frac{1}{2}\sum_{t=1}^{T}(y_t - \mu - \phi y_{t-1})^2/\sigma_t^2 \tag{9.43}$$

(3) The computer will maximise the function and generate parameter values that maximise the *LLF* and will construct their standard errors.

9.9.1 Parameter Estimation Using Maximum Likelihood

As stated above, under maximum likelihood estimation, a set of parameter values are chosen that are most likely to have produced the observed data. This is done by first forming a *likelihood function*, denoted *LF*. *LF* will be a multiplicative function of the actual data, which will consequently be difficult to maximise with respect to the parameters. Therefore, its logarithm is taken in order to turn *LF* into an additive function of the sample data, i.e., the *LLF*. A derivation of the maximum likelihood (ML) estimator in the context of the simple bivariate regression model with homoscedasticity is given in the appendix to this chapter. Essentially, deriving the ML estimators involves differentiating the *LLF* with respect to the parameters. But how does this help in estimating heteroscedastic models? How can the method outlined in Appendix 9.1 to this chapter for homoscedastic models be modified for application to GARCH model estimation?

In the context of conditional heteroscedasticity models, the model is $y_t = \mu + \phi y_{t-1} + u_t,\ u_t \sim \mathrm{N}(0, \sigma_t^2)$, so that the variance of the errors has been modified from being assumed constant, σ^2, to being time-varying, σ_t^2, with the equation for the conditional variance as previously. The *LLF* relevant for a GARCH model can be constructed in the same way as for the homoscedastic case by replacing

$$\frac{T}{2}\ln\sigma^2$$

with the equivalent for time-varying variance

$$\frac{1}{2}\sum_{t=1}^{T}\ln\sigma_t^2$$

and replacing σ^2 in the denominator of the last part of the expression with σ_t^2 (see Appendix 9.1 to this chapter). Derivation of this result from first principles is beyond the scope of this text, but the log-likelihood function for the above model with time-varying conditional variance and normally distributed errors is given by equation (9.43) in Box 9.2.

Intuitively, maximising the *LLF* involves jointly minimising

$$\sum_{t=1}^{T} \ln \sigma_t^2$$

and

$$\sum_{t=1}^{T} \frac{(y_t - \mu - \phi y_{t-1})^2}{\sigma_t^2}$$

(since these terms appear preceded with a negative sign in the *LLF*, and

$$-\frac{T}{2} \ln(2\pi)$$

is just a constant with respect to the parameters). Minimising these terms jointly also implies minimising the error variance, as described in Chapter 4. Unfortunately, maximising the *LLF* for a model with time-varying variances is trickier than in the homoscedastic case. Analytical derivatives of the *LLF* in equation (9.43) with respect to the parameters have been developed, but only in the context of the simplest examples of GARCH specifications. Moreover, the resulting formulae are complex, so a numerical procedure is often used instead to maximise the log-likelihood function.

Essentially, all methods work by ‘searching’ over the parameter-space until the values of the parameters that maximise the log-likelihood function are found. Most software packages employ an iterative technique for maximising the *LLF*. This means that, given a set of initial guesses for the parameter estimates, these parameter values are updated at each iteration until the program determines that an optimum has been reached. If the *LLF* has only one maximum with respect to the parameter values, any optimisation method should be able to find it – although some methods will take longer than others. A detailed presentation of the various methods available is beyond the scope of this book. However, as is often the case with non-linear models such as GARCH, the *LLF* can have many local maxima, so that different algorithms could find different local maxima of the *LLF*. Hence readers should be warned that different optimisation procedures could lead to different coefficient estimates and especially different estimates of the standard errors (see Brooks, 2001 or 2003 for details). In such instances, a good set of initial parameter guesses is essential.

Local optima or multimodalities in the likelihood surface present potentially serious drawbacks with the maximum likelihood approach to estimating the parameters of a GARCH model, as shown in Figure 9.2.

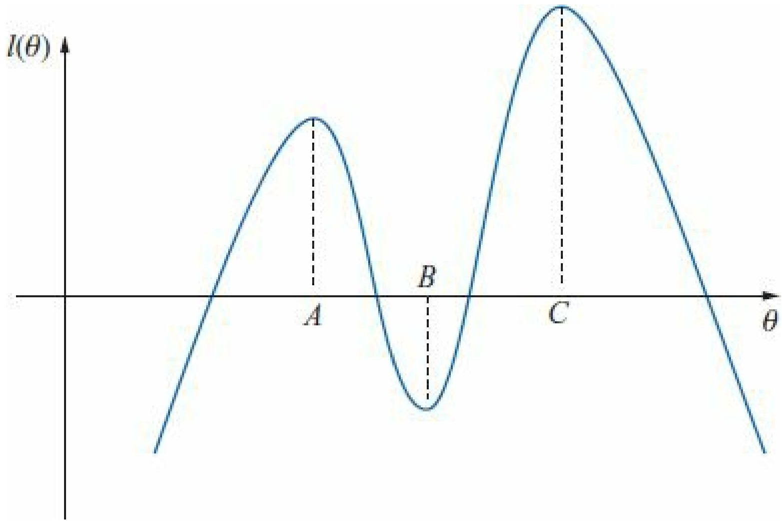


Figure 9.2 The problem of local optima in maximum likelihood estimation

Suppose that the model contains only one parameter, θ, so that the log-likelihood function is to be maximised with respect to this one parameter. In Figure 9.2, the value of the *LLF* for each value of θ is denoted $l(\theta)$. Clearly, $l(\theta)$ reaches a global maximum when $\theta = C$, and a local maximum when $\theta = A$. This demonstrates the importance of good initial guesses for the parameters. Any initial guesses to the left of *B* are likely to lead to the selection of *A* rather than *C*. The situation is likely to be even worse in practice, since the log-likelihood function will be maximised with respect to several parameters, rather than one, and there could be many local optima. Another possibility that would make optimisation difficult is when the *LLF* is flat around the maximum. So, for example, if the peak corresponding to *C* in Figure 9.2, were flat rather than sharp, a range of values for θ could lead to very similar values for the *LLF*, making it difficult to choose between them.

So, to explain again in more detail, the optimisation is done in the way shown in Box 9.3. The optimisation methods employed by many software packages such as EViews are based on the determination of the first and second derivatives of the log-likelihood function with respect to the parameter values at each iteration, known as the gradient and Hessian (the matrix of second derivatives of the *LLF* w.r.t the parameters), respectively. An algorithm for optimisation due to Berndt *et al.* (1974), known as BHHH, is available in EViews. BHHH employs only first derivatives (calculated numerically rather than analytically) and approximations to the second derivatives are calculated. Not calculating the actual Hessian at each iteration at each time step increases computational speed, but the approximation may be poor when the *LLF* is a long way from its maximum value, requiring more iterations to reach the optimum. The Marquardt algorithm, available in EViews, is a modification of BHHH (both of which are variants on the Gauss–Newton method) that incorporates a 'correction', the effect of which is to push the coefficient estimates more quickly to their optimal values. All of these optimisation methods are described in detail in Press *et al.* (1992).

BOX 9.3 Using maximum likelihood estimation in practice

(1) *Set up* the *LLF*.

(2) Use regression to get *initial estimates* for the mean parameters.

(3) Choose some initial guesses for the *conditional variance parameters*. In most software packages, the default initial values for the conditional variance parameters would be zero. This is unfortunate since zero parameter values often yield a local maximum of the likelihood function. So if possible, set plausible initial values away from zero.

(4) Specify a *convergence criterion* – either by criterion or by value. When 'by criterion' is selected, the package will continue to search for 'better' parameter values that give a higher value of the *LLF* until the change in the value of the *LLF* between iterations is less than the specified convergence criterion. Choosing 'by value' will lead to the software searching until the change in the coefficient estimates are small enough. For example, the default convergence criterion for EViews is 0.001, which means that convergence is achieved and the program will stop searching if the biggest percentage change in any of the coefficient estimates for the most recent iteration is smaller than 0.1%.

9.9.2 Non-Normality and Maximum Likelihood

Recall that the conditional normality assumption for u_t is essential in specifying the likelihood function. It is possible to test for non-normality using the following representation

$$u_t = v_t \sigma_t, v_t \sim \mathrm{N}(0, 1) \tag{9.44}$$

$$\sigma_t = \sqrt{\alpha_0 + \alpha_1 u_{t-1}^2 + \beta\sigma_{t-1}^2} \tag{9.45}$$

Note that one would not expect u_t to be normally distributed – it is a $N(0, \sigma_t^2)$ disturbance term from the regression model, which will imply it is likely to have fat tails. A plausible method to test for normality would be to construct the statistic

$$v_t = \frac{u_t}{\sigma_t} \tag{9.46}$$

which would be the model disturbance at each point in time t divided by the conditional standard deviation at that point in time. Thus, it is the v_t that are assumed to be normally distributed, not u_t. The sample counterpart would be

$$\hat{v}_t = \frac{\hat{u}_t}{\hat{\sigma}_t} \tag{9.47}$$

which is known as a standardised residual. Whether the $\hat{v}_t$ are normal can be examined using any standard normality test, such as the Bera–Jarque. Typically, $\hat{v}_t$ are still found to be leptokurtic, although less so than the $\hat{u}_t$. The upshot is that the GARCH model is able to capture some, although not all, of the leptokurtosis in the unconditional distribution of asset returns.

Is it a problem if $\hat{v}_t$ are not normally distributed? Well, the answer is 'not really'. Even if the conditional normality assumption does not hold, the parameter estimates will still be consistent if the equations for the mean and variance are correctly specified. However, in the context of non-normality, the usual standard error estimates will be inappropriate, and a different variance–covariance matrix estimator that is robust to non-normality, due to Bollerslev and Wooldridge (1992), should be used. This procedure (i.e., maximum likelihood with Bollerslev–Wooldridge standard errors) is known as *quasi-maximum likelihood*, or QML.

9.10 Extensions to the Basic GARCH Model

Since the GARCH model was developed, a huge number of extensions and variants have been proposed. A couple of the most important examples will be highlighted here. Interested readers who wish to investigate further are directed to a comprehensive survey by Bollerslev, Chou and Kroner (1992).

Many of the extensions to the GARCH model have been suggested as a consequence of perceived problems with standard GARCH(p, q) models. First, the non-negativity conditions may be violated by the estimated model. The only way to avoid this for sure would be to place artificial constraints on the model coefficients in order to force them to be non-negative. Second, GARCH models cannot account for leverage effects (explained below), although they can account for volatility clustering and leptokurtosis in a series. Finally, the model does not allow for any direct feedback between the conditional variance and the conditional mean.

Some of the most widely used and influential modifications to the model will now be examined. These may remove some of the restrictions or limitations of the basic model.

9.11 Asymmetric GARCH Models

One of the primary restrictions of GARCH models is that they enforce a symmetric response of volatility to positive and negative shocks. This arises since the conditional variance in equations such as equation (9.39) is a function of the magnitudes of the lagged residuals and not their signs (in other words, by squaring the lagged error in equation (9.39), the sign is lost). However, it has been argued that a negative shock to financial time series is likely to cause volatility to rise by more than a positive shock of the same magnitude. In the case of equity returns, such asymmetries are typically attributed to *leverage effects*, whereby a fall in the value of a firm's stock causes the firm's debt to equity ratio to rise. This leads shareholders, who bear the residual risk of the firm, to perceive their future cashflow stream as being relatively more risky.

An alternative view is provided by the 'volatility-feedback' hypothesis. Assuming constant dividends, if expected returns increase when stock price volatility increases, then stock prices should fall when volatility rises. Although asymmetries in returns series other than equities cannot be attributed to changing leverage, there is equally no reason to suppose that such asymmetries only exist in equity returns.

Two popular asymmetric formulations are explained below: the GJR model, named after the authors Glosten, Jagannathan and Runkle (1993), and the exponential GARCH (EGARCH) model proposed by Nelson (1991).

9.12 The GJR model

The GJR model is a simple extension of GARCH with an additional term added to account for possible asymmetries. The conditional variance is now given by

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta\sigma_{t-1}^2 + \gamma u_{t-1}^2 I_{t-1} \tag{9.48}$$

where $I_{t-1} = 1$ if $u_{t-1} < 0$
$= 0$ otherwise

For a leverage effect, we would see $\gamma > 0$. Notice now that the condition for nonnegativity will be $\alpha_0 > 0$, $\alpha_1 > 0$, $\beta \geq 0$, and $\alpha_1 + \gamma \geq 0$. That is, the model is still admissible, even if $\gamma < 0$, provided that $\alpha_1 + \gamma \geq 0$.

EXAMPLE 9.1

To offer an illustration of the GJR approach, using monthly S&P500 returns from December 1979 until June 1998, the following results would be obtained, with *t*-ratios in parentheses

$$\underset{(3.198)}{y_t = 0.172} \tag{9.49}$$

$$\sigma_t^2 = \underset{(16.372)}{1.243} + \underset{(0.437)}{0.015}u_{t-1}^2 + \underset{(14.999)}{0.498}\sigma_{t-1}^2 + \underset{(5.772)}{0.604}u_{t-1}^2 I_{t-1} \tag{9.50}$$

Note that the asymmetry term, γ, has the correct sign and is significant. To see how volatility rises more after a large negative shock than a large positive one, suppose that $\sigma_{t-1}^2 = 0.823$, and consider $\hat{u}_{t-1} = \pm 0.5$. If $\hat{u}_{t-1} = 0.5$, this implies that $\sigma_t^2 = 1.65$. However, a shock of the same magnitude but of opposite sign, $\hat{u}_{t-1} = -0.5$, implies that the fitted conditional variance for time t will be $\sigma_t^2 = 1.80$.

9.13 The EGARCH Model

The exponential GARCH model was proposed by Nelson (1991). There are various ways to express the conditional variance equation, but one possible specification is given by

$$\ln(\sigma_t^2) = \omega + \beta \ln(\sigma_{t-1}^2) + \gamma \frac{u_{t-1}}{\sqrt{\sigma_{t-1}^2}} + \alpha \left[\frac{|u_{t-1}|}{\sqrt{\sigma_{t-1}^2}} - \sqrt{\frac{2}{\pi}} \right] \tag{9.51}$$

The model has several advantages over the pure GARCH specification. First, since the $\ln(\sigma_t^2)$ is modelled, then even if the parameters are negative, σ_t^2 will be positive. There is thus no need to artificially impose non-negativity constraints on the model parameters. Second, asymmetries are allowed for under the EGARCH formulation, since if the relationship between volatility and returns is negative, γ, will be negative.

Note that in the original formulation, Nelson assumed a generalised error distribution (GED) structure for the errors. GED is a very broad family of distributions that can be used for many types of series. However, owing to its computational ease and intuitive interpretation, almost all applications of EGARCH employ conditionally normal errors as discussed above rather than using GED.

9.14 Tests for Asymmetries in Volatility

Engle and Ng (1993) have proposed a set of tests for asymmetry in volatility, known as sign and size bias tests. The Engle and Ng tests should thus be used to determine whether an asymmetric model is required for a given series, or whether the symmetric GARCH model can be deemed adequate. In practice, the Engle–Ng tests are usually applied to the residuals of a GARCH fit to the returns data. Define S_{t-1}^- as an indicator dummy that takes the value 1 if $\hat{u}_{t-1} < 0$ and zero otherwise. The test for sign bias is based on the significance or otherwise of ϕ_1 in

$$\hat{u}_t^2 = \phi_0 + \phi_1 S_{t-1}^- + \upsilon_t \tag{9.52}$$

where υ_t is an iid error term. If positive and negative shocks to $\hat{u}_{t-1}$ impact differently upon the conditional variance, then ϕ_1 will be statistically significant.

It could also be the case that the magnitude or size of the shock will affect whether the response of volatility to shocks is symmetric or not. In this case, a negative size bias test would be conducted, based on a regression where S_{t-1}^- is now used as a slope dummy variable. Negative size bias is argued to be present if ϕ_1 is statistically significant in the regression

$$\hat{u}_t^2 = \phi_0 + \phi_1 S_{t-1}^- u_{t-1} + \upsilon_t \quad \textbf{(9.53)}$$

Finally, defining $S_{t-1}^+ = 1 - S_{t-1}^-$, so that S_{t-1}^+ picks out the observations with positive innovations, Engle and Ng propose a joint test for sign and size bias based on the regression

$$\hat{u}_t^2 = \phi_0 + \phi_1 S_{t-1}^- + \phi_2 S_{t-1}^- u_{t-1} + \phi_3 S_{t-1}^+ u_{t-1} + \upsilon_t \quad \textbf{(9.54)}$$

Significance of ϕ_1 indicates the presence of sign bias, where positive and negative shocks have differing impacts upon future volatility, compared with the symmetric response required by the standard GARCH formulation. On the other hand, the significance of ϕ_2 or ϕ_3 would suggest the presence of size bias, where not only the sign but the magnitude of the shock is important. A joint test statistic is formulated in the standard fashion by calculating TR^2 from regression equation (9.54), which will asymptotically follow a χ^2 distribution with three degrees of freedom under the null hypothesis of no asymmetric effects.

9.14.1 News Impact Curves

A pictorial representation of the degree of asymmetry of volatility to positive and negative shocks is given by the news impact curve introduced by Pagan and Schwert (1990). The news impact curve plots the next-period volatility (σ_t^2) that would arise from various positive and negative values of u_{t-1}, given an estimated model. The curves are drawn by using the estimated conditional variance equation for the model under consideration, with its given coefficient estimates, and with the lagged conditional variance set to the unconditional variance. Then, successive values of u_{t-1} are used in the equation to determine what the corresponding values of σ_t^2 derived from the model would be. For example, suppose that model estimates are constructed for GARCH and GJR models for S&P500 data. Values of u_{t-1} in the range (−1, +1) are substituted into the equations in each case to investigate the impact on the conditional variance during the next period. The resulting news impact curves for the GARCH and GJR models are given in Figure 9.3.

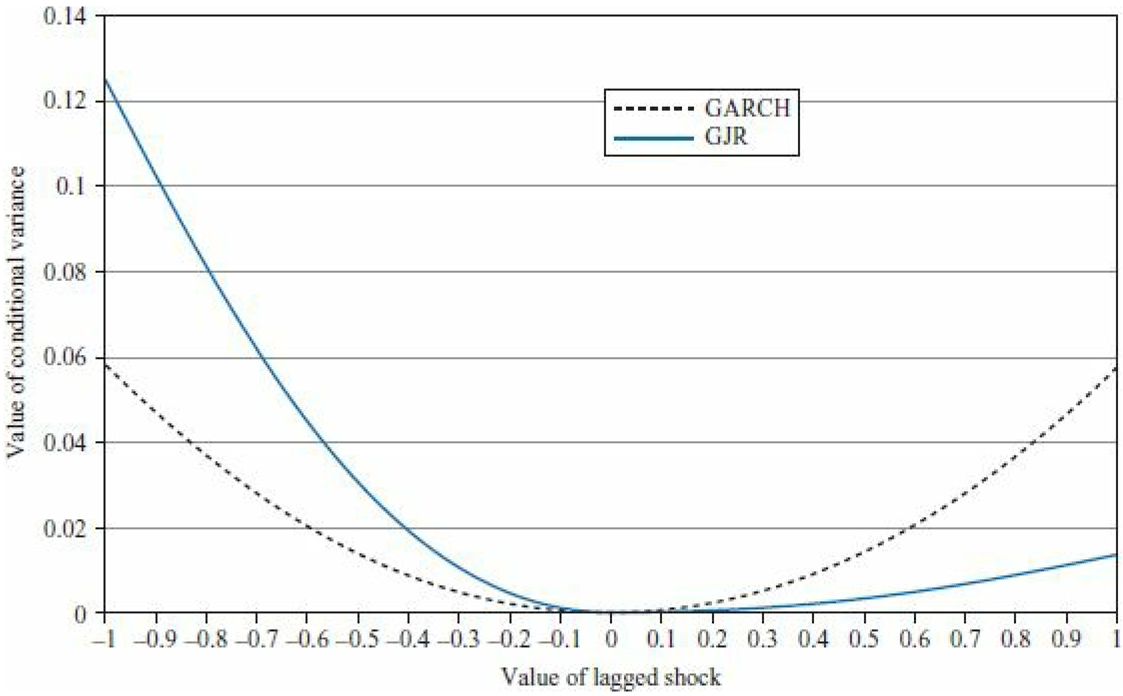


Figure 9.3 News impact curves for S&P500 returns using coefficients implied from GARCH and GJR model estimates

As can be seen from Figure 9.3, the GARCH news impact curve (the light blue line) is of course symmetrical about zero, so that a shock of given magnitude will have the same impact on the future conditional variance whatever its sign. On the other hand, the GJR news impact curve (the dark blue line) is asymmetric, with negative shocks having more impact on future volatility than positive shocks of the same magnitude. It can also be seen that a negative shock of given magnitude will have a bigger impact under GJR than would be implied by a GARCH model, while a positive shock of given magnitude will have more impact under GARCH than GJR. The latter result arises as a result of the reduction in the value of α_1, the coefficient on the lagged squared error, when the asymmetry term is included in the model.

9.15 GARCH-in-Mean

Most models used in finance suppose that investors should be rewarded for taking additional risk by obtaining a higher return. One way to operationalise this concept is to let the return of a security be partly determined by its risk. Engle, Lilien and Robins (1987) suggested an ARCH-M specification, where the conditional variance of asset returns enters into the conditional mean equation. Since GARCH models are now considerably more popular than ARCH, it is more common to estimate a GARCH-M model. An example of a GARCH-M model is given by the specification

$$y_t = \mu + \delta\sigma_{t-1} + u_t, u_t \sim \mathrm{N}(0, \sigma_t^2) \tag{9.55}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta\sigma_{t-1}^2 \tag{9.56}$$

If δ is positive and statistically significant, then increased risk, given by an increase in the conditional variance, leads to a rise in the mean return. Thus δ can be interpreted as a risk premium. In some empirical applications, the conditional variance term, σ_{t-1}^2, appears directly in the conditional mean equation, rather than in square root form, σ_{t-1}. Also, in some applications the term is contemporaneous, σ_t^2, rather than lagged.

9.16 Uses of GARCH-Type Models Including Volatility Forecasting

Essentially GARCH models are useful because they can be used to model the volatility of a series over time. It is possible to combine together more than one of the time series models that have been considered so far in this book, to obtain more complex 'hybrid' models. Such models can account for a number of important features of financial series at the same time – e.g., an ARMA–EGARCH(1,1)-M model; the potential complexity of the model is limited only by the imagination!

GARCH-type models can be used to forecast volatility. GARCH is a model to describe movements in the conditional variance of an error term, u_t, which may not appear particularly useful. But it is possible to show that

$$\mathrm{var}(y_t \mid y_{t-1}, y_{t-2}, \ldots) = \mathrm{var}(u_t \mid u_{t-1}, u_{t-2}, \ldots) \tag{9.57}$$

So the conditional variance of y, given its previous values, is the same as the conditional variance of u, given its previous values. Hence, modelling σ_t^2 will give models and forecasts for the variance of y_t as well. Thus, if the dependent variable in a regression, y_t is an asset return series, forecasts of σ_t^2 will be forecasts of the future variance of y_t. So one primary usage of GARCH-type models is in forecasting volatility. This can be useful in, for example, the pricing of financial options where volatility is an input to the pricing model. For example, the value of a 'plain vanilla' call option is a function of the current value of the underlying, the strike price, the time to maturity, the risk-free interest rate and volatility. The required volatility, to obtain an appropriate options price, is really the volatility of the underlying asset expected over the lifetime of the option. As stated previously, it is possible to use a simple historical average measure as the forecast of future volatility, but another method that seems more appropriate would be to use a time series model such as GARCH to compute the volatility forecasts. The forecasting ability of various models is considered in a paper by Day and Lewis (1992), discussed in detail below.

Producing forecasts from models of the GARCH class is relatively simple, and the algebra involved is very similar to that required to obtain forecasts from ARMA models. An illustration is given by Example 9.2.

EXAMPLE 9.2

Consider the following GARCH(1,1) model

$$y_t = \mu + u_t,\ u_t \sim \mathrm{N}(0, \sigma_t^2) \qquad \textbf{(9.58)}$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta\sigma_{t-1}^2 \qquad \textbf{(9.59)}$$

Suppose that the researcher had estimated the above GARCH model for a series of returns on a stock index and obtained the following parameter estimates: $\hat{\mu} = 0.0023$, $\hat{\alpha}_0 = 0.0172$, $\hat{\beta} = 0.7811$, $\hat{\alpha}_1 = 0.1251$. If the researcher has data available up to and including time T, write down a set of equations in σ_t^2 and u_t^2 and their lagged values,which could be employed to produce one-, two-, and three-step-ahead forecasts for the conditional variance of y_t.

What is needed is to generate forecasts of $\sigma_{T+1}{}^2|\Omega_T$, $\sigma_{T+2}{}^2|\Omega_T$, …, $\sigma_{T+s}{}^2|\Omega_T$ where Ω_T denotes all information available up to and including observation T. For time T, the conditional variance equation is given by equation (9.59). Adding one to each of the time subscripts of this equation, and then two, and then three would yield equations (9.60)–(9.62)

$$\sigma_{T+1}{}^2 = \alpha_0 + \alpha_1 u_T^2 + \beta\sigma_T^2 \qquad \textbf{(9.60)}$$
$$\sigma_{T+2}{}^2 = \alpha_0 + \alpha_1 u_{T+1}^2 + \beta\sigma_{T+1}^2 \qquad \textbf{(9.61)}$$
$$\sigma_{T+3}{}^2 = \alpha_0 + \alpha_1 u_{T+2}^2 + \beta\sigma_{T+2}^2 \qquad \textbf{(9.62)}$$

Let $\sigma_{1,T}^{f^2}$ be the one-step-ahead forecast for σ^2 made at time T. This is easy to calculate since, at time T, the values of all the terms on the RHS are known. $\sigma^{f^2}_{1,T}$ would be obtained by taking the conditional expectation of equation **(9.60)**.

Given $\sigma^{f^2}_{1,T}$, how is $\sigma^{f^2}_{2,T}$, the two-step-ahead forecast for σ^2 made at time T, calculated?

$$\sigma^{f^2}_{1,T} = \alpha_0 + \alpha_1 u_T^2 + \beta\sigma_T^2 \tag{9.63}$$

From equation (9.61), it is possible to write

$$\sigma^{f^2}_{2,T} = \alpha_0 + \alpha_1 \mathrm{E}(u_{T+1}^2 \mid \Omega_T) + \beta\sigma^{f^2}_{1,T} \tag{9.64}$$

where $\mathrm{E}(u_{T+1}^2 \mid \Omega_T)$ is the expectation, made at time T, of u_{T+1}^2, squared disturbance term. It is necessary to find $\mathrm{E}(u_{T+1}^2 \mid \Omega_T)$, using the expression for the variance of a random variable u_t. The model assumes that the series u_t has zero mean, so that the variance can be written

$$\mathrm{var}\,(u_t) = \mathrm{E}[(u_t - \mathrm{E}(u_t))^2] = \mathrm{E}\left(u_t^2\right). \tag{9.65}$$

The conditional variance of u_t is σ_t^2, so

$$\sigma_t^2 \mid \Omega_t = \mathrm{E}(u_t^2) \tag{9.66}$$

Turning this argument around, and applying it to the problem at hand

$$\mathrm{E}(u_{T+1}^2 \mid \Omega_t)^2 = \sigma_{T+1} \tag{9.67}$$

but σ_{T+1}^2 is not known at time T, so it is replaced with the forecast for it, $\sigma^{f^2}_{1,T}$, so that equation (9.64) becomes

$$\sigma^{f^2}_{2,T} = \alpha_0 + \alpha_1\sigma^{f^2}_{1,T} + \beta\sigma^{f^2}_{1,T} \tag{9.68}$$

$$\sigma^{f^2}_{2,T} = \alpha_0 + (\alpha_1 + \beta)\sigma^{f^2}_{1,T} \tag{9.69}$$

What about the three-step-ahead forecast?

By similar arguments,

$$\sigma^{f^2}_{3,T} = \mathrm{E}_T\left(\alpha_0 + \alpha_1 u_{T+2}^2 + \beta\sigma_{T+2}^2\right) \tag{9.70}$$

$$\sigma^{f^2}_{3,T} = \alpha_0 + (\alpha_1 + \beta)\sigma^{f^2}_{2,T} \tag{9.71}$$

$$\sigma^{f^2}_{3,T} = \alpha_0 + (\alpha_1 + \beta)\left[\alpha_0 + (\alpha_1 + \beta)\sigma^{f^2}_{1,T}\right] \tag{9.72}$$

$$\sigma^{f^2}_{3,T} = \alpha_0 + \alpha_0(\alpha_1 + \beta) + (\alpha_1 + \beta)^2 \sigma^{f^2}_{1,T} \quad \textbf{(9.73)}$$

Any *s*-step-ahead forecasts would be produced by

$$\sigma^{f^2}_{s,T} = \alpha_0 \sum_{i=1}^{s-1} (\alpha_1 + \beta)^{i-1} + (\alpha_1 + \beta)^{s-1} \sigma^{f^2}_{1,T} \quad \textbf{(9.74)}$$

for any value of $s \geq 2$.

It is worth noting at this point that variances, and therefore variance forecasts, are additive over time. This is a very useful property. Suppose, for example, that using daily foreign exchange returns, one-, two-, three-, four-, and five-step-ahead variance forecasts have been produced, i.e., a forecast has been constructed for each day of the next trading week. The forecasted variance for the whole week would simply be the sum of the five daily variance forecasts. If the standard deviation is the required volatility estimate rather than the variance, simply take the square root of the variance forecasts. Note also, however, that standard deviations are not additive. Hence, if daily standard deviations are the required volatility measure, they must be squared to turn them to variances. Then the variances would be added and the square root taken to obtain a weekly standard deviation.

9.17 Testing Non-Linear Restrictions or Testing Hypotheses About Non-Linear Models

The usual *t*- and *F*-tests are still valid in the context of non-linear models, but they are not flexible enough. For example, suppose that it is of interest to test a hypothesis that $\alpha_1\beta = 1$. Now that the model class has been extended to non-linear models, there is no reason to suppose that relevant restrictions are only linear.

Under OLS estimation, the *F*-test procedure works by examining the degree to which the *RSS* rises when the restrictions are imposed. In very general terms, hypothesis testing under ML works in a similar fashion – that is, the procedure works by examining the degree to which the maximal value of the *LLF* falls upon imposing the restriction. If the *LLF* falls ‘a lot’, it would be concluded that the restrictions are not supported by the data and thus the hypothesis should be rejected.

There are three hypothesis testing procedures based on maximum likelihood principles: Wald, Likelihood ratio and Lagrange Multiplier. To illustrate briefly how each of these operates, consider a single parameter, θ to be estimated, and denote the *ML* estimate as $\hat{\theta}$ and a restricted estimate as $\tilde{\theta}$. Denoting the maximised value of the *LLF* by unconstrained *ML* as $L(\hat{\theta})$ and the constrained optimum as $L(\tilde{\theta})$, the three testing procedures can be illustrated as in Figure 9.4.

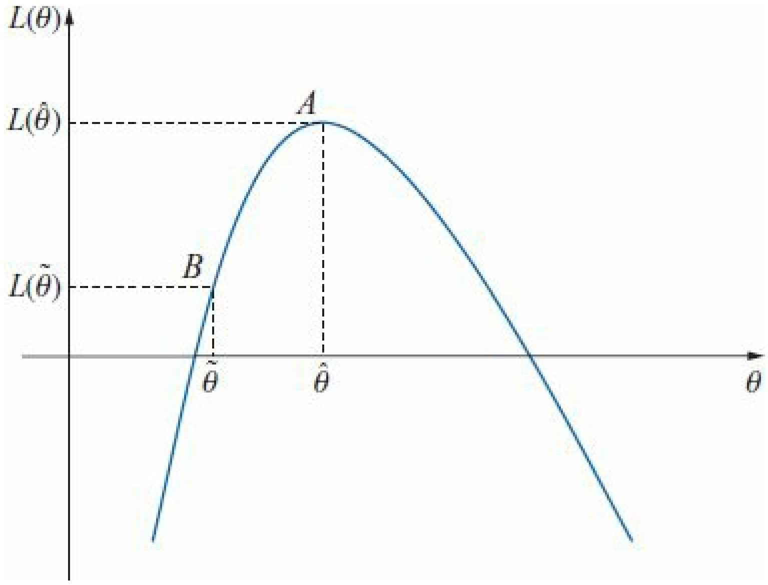


Figure 9.4 Three approaches to hypothesis testing under maximum likelihood

The tests all require the measurement of the 'distance' between the points *A* (representing the unconstrained maximised value of the log likelihood function) and *B* (representing the constrained value). The vertical distance forms the basis of the *LR* test. Twice this vertical distance is given by $2[L(\hat{\theta}) - -L(\tilde{\theta})] = 2\ln[l(\hat{\theta})/l(\tilde{\theta})]$, where *L* denotes the log-likelihood function, and *l* denotes the likelihood function. The Wald test is based on the horizontal distance between $\hat{\theta}$ and $\tilde{\theta}$, while the *LM* test compares the slopes of the curve at *A* and *B*. At *A*, the unrestricted maximum of the log-likelihood function, the slope of the curve is zero. But is it 'significantly steep' at $L(\tilde{\theta})$, i.e., at point *B* ? The steeper the curve is at *B*, the less likely the restriction is to be supported by the data.

Expressions for *LM* test statistics involve the first and second derivatives of the log-likelihood function with respect to the parameters at the constrained estimate. The first derivatives of the log-likelihood function are collectively known as the score vector, measuring the slope of the *LLF* for each possible value of the parameters. The expected values of the second derivatives comprise the information matrix, measuring the peakedness of the *LLF*, and how much higher the *LLF* value is at the optimum than in other places. This matrix of second derivatives is also used to construct the coefficient standard errors. The *LM* test involves estimating only a restricted regression, since the slope of the *LLF* at the maximum will be zero by definition. Since the restricted regression is usually easier to estimate than the unrestricted case, *LM* tests are usually the easiest of the three procedures to employ in practice. The reason that restricted regressions are usually simpler is that imposing the restrictions often means that some components in the model will be set to zero or combined under the null hypothesis, so that there are fewer parameters to estimate. The Wald test involves estimating only an unrestricted regression, and the usual OLS *t*-tests and *F*-tests are examples of Wald tests (since again, only unrestricted estimation occurs).

Of the three approaches to hypothesis testing in the maximum-likelihood framework, the likelihood ratio test is the most intuitively appealing, and therefore a deeper examination of it will be the subject of the following section; see Ghosh (1991, Section 10.3) for further details.

9.17.1 Likelihood Ratio Tests

Likelihood ratio (*LR*) tests involve estimation under the null hypothesis and under the alternative, so that two models are estimated: an unrestricted model and a model where the restrictions have been imposed. The maximised values of the *LLF* for the restricted and unrestricted cases are ‘compared’. Suppose that the unconstrained model has been estimated and that a given maximised value of the *LLF*, denoted L_u, has been achieved. Suppose also that the model has been estimated imposing the constraint(s) and a new value of the *LLF* obtained, denoted L_r. The *LR* test statistic asymptotically follows a Chi-squared distribution and is given by

$$LR = -2(L_r - L_u) \sim \chi^2(m) \quad \textbf{(9.75)}$$

where m = number of restrictions. Note that the maximised value of the log-likelihood function will always be at least as big for the unrestricted model as for the restricted model, so that $L_r \leq L_u$. This rule is intuitive and comparable to the effect of imposing a restriction on a linear model estimated by OLS, that *RRSS* ≥ *URSS*. Similarly, the equality between L_r and L_u will hold only when the restriction was already present in the data.

Note, however, that the usual *F*-test is in fact a Wald test, and not a *LR* test – that is, it can be calculated using an unrestricted model only. The *F*-test approach based on comparing *RSS* arises conveniently as a result of the OLS algebra. Example 9.3 demonstrates how a likelihood ratio test is implemented.

EXAMPLE 9.3

A GARCH model is estimated and a maximised *LLF* of 66.85 is obtained. Suppose that a researcher wishes to test whether $\beta = 0$ in (9.77)

$$y_t = \mu + \phi y_{t-1} + u_t, u_t \sim N(0, \sigma_t^2) \quad \textbf{(9.76)}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta \sigma_{t-1}^2 \quad \textbf{(9.77)}$$

The model is estimated imposing the restriction and the maximised *LLF* falls to 64.54. Is the restriction supported by the data, which would correspond to the situation where an ARCH(1) specification was sufficient? The test statistic is given by

$$LR = -2(64.54 - 66.85) = 4.62 \quad \textbf{(9.78)}$$

The test follows a $\chi^2(1) = 3.84$ at 5%, so that the null is marginally rejected. It would thus be concluded that an ARCH(1) model, with no lag of the conditional variance in the variance equation, is not quite sufficient to describe the dependence in volatility over time.

9.18 Volatility Forecasting: Some Examples and Results from the Literature

There is a vast and relatively new literature that attempts to compare the accuracies of various models for producing out-of-sample volatility forecasts. Akgiray (1989), for example, finds the GARCH model superior to ARCH, exponentially weighted moving average and historical mean models for forecasting monthly US stock index volatility. A similar result concerning the apparent superiority of GARCH is observed by West and Cho (1995) using one-step-ahead forecasts of dollar exchange rate volatility, although for longer horizons, the model behaves no better than their alternatives. Pagan and Schwert (1990) compare GARCH, EGARCH, Markov switching regime and three non-parametric models for forecasting monthly US stock return volatilities. The EGARCH followed by the GARCH models perform moderately; the remaining models produce very poor predictions. Franses and van Dijk (1996) compare three members of the GARCH family (standard GARCH, QGARCH and the GJR model) for forecasting the weekly volatility of various European stock market indices. They find that the non-linear GARCH models were unable to beat the standard GARCH model. Finally, Brailsford and Faff (1996) find GJR and GARCH models slightly superior to various simpler models for predicting Australian monthly stock index volatility. The conclusion arising from this growing body of research is that forecasting volatility is a 'notoriously difficult task' (Brailsford and Faff, 1996, p. 419), although it appears that conditional heteroscedasticity models are among the best that are currently available. In particular, more complex non-linear and non-parametric models are inferior in prediction to simpler models, a result echoed in an earlier paper by Dimson and Marsh (1990) in the context of relatively complex versus parsimonious linear models. Finally, Brooks (1998), considers whether measures of market volume can assist in improving volatility forecast accuracy, finding that they cannot.

A particularly clear example of the style and content of this class of research is given by Day and Lewis (1992). The Day and Lewis study will therefore now be examined in depth. The purpose of their paper is to consider the out-of-sample forecasting performance of GARCH and EGARCH models for predicting stock index volatility. The forecasts from these econometric models are compared with those given from an 'implied volatility'. As discussed above, implied volatility is the market's expectation of the 'average' level of volatility of an underlying asset over the life of the option that is implied by the current traded price of the option. Given an assumed model for pricing options, such as the Black–Scholes, all of the inputs to the model except for volatility can be observed directly from the market or are specified in the terms of the option contract. Thus, it is possible, using an iterative search procedure such as the Newton–Raphson method (see, for example, Watsham and Parramore, 2004), to 'back out' the volatility of the underlying asset from the option's price. An important question for research is whether implied or econometric models produce more accurate forecasts of the volatility of the underlying asset. If the options and underlying asset markets are informationally efficient, econometric volatility forecasting models based on past realised values of underlying volatility should have no incremental explanatory power for future values of volatility of the underlying asset.

On the other hand, if econometric models do hold additional information useful for forecasting future volatility, it is possible that such forecasts could be turned into a profitable trading rule.

The data employed by Day and Lewis comprise weekly closing prices (Wednesday to Wednesday, and Friday to Friday) for the S&P100 Index option and the underlying index from 11 March 1983–31 December 1989. They employ both mid-week to midweek returns and Friday to Friday returns to determine whether weekend effects have any significant impact on the latter. They argue that Friday returns contain expiration effects since implied volatilities are seen to jump on the Friday of the week of expiration. This issue is not of direct interest to this book, and consequently only the mid-week to mid-week results will be shown here.

The models that Day and Lewis employ are as follows. First, for the conditional mean of the time series models, they employ a GARCH-M specification for the excess of the market return over a risk-free proxy

$$R_{Mt} - R_{Ft} = \lambda_0 + \lambda_1 \sqrt{h_t} + u_t \tag{9.79}$$

where R_{Mt} denotes the return on the market portfolio, and R_{Ft} denotes the riskfree rate. Note that Day and Lewis denote the conditional variance by h_t^2, while this is modified to the standard h_t here. Also, the notation σ_t^2 will be used to denote implied volatility estimates. For the variance, two specifications are employed: a ‘plain vanilla’ GARCH(1,1) and an EGARCH

$$h_t = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta_1 h_{t-1} \tag{9.80}$$

or

$$\ln(h_t) = \alpha_0 + \beta_1 \ln(h_{t-1}) + \alpha_1 \left(\theta \frac{u_{t-1}}{\sqrt{h_{t-1}}} + \gamma \left[\left| \frac{u_{t-1}}{\sqrt{h_{t-1}}} \right| - \left(\frac{2}{\pi} \right)^{1/2} \right] \right) \tag{9.81}$$

One way to test whether implied or GARCH-type volatility models perform best is to add a lagged value of the implied volatility estimate (σ_{t-1}^2) to equations (9.80) and (9.81). A ‘hybrid’ or ‘encompassing’ specification would thus result. Equation (9.80) becomes

$$h_t = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta_1 h_{t-1} + \delta \sigma_{t-1}^2 \tag{9.82}$$

and equation (9.81) becomes

$$\ln(h_t) = \alpha_0 + \beta_1 \ln(h_{t-1}) + \alpha_1 \left(\theta \frac{u_{t-1}}{\sqrt{h_{t-1}}} + \gamma \left[\left| \frac{u_{t-1}}{\sqrt{h_{t-1}}} \right| - \left(\frac{2}{\pi} \right)^{1/2} \right] \right) + \delta \ln\left(\sigma_{t-1}^2\right) \qquad \textbf{(9.83)}$$

The tests of interest are given by $H_0 : \delta = 0$ in (9.82) or (9.83). If these null hypotheses cannot be rejected, the conclusion would be that implied volatility contains no incremental information useful for explaining volatility than that derived from a GARCH model. At the same time, H_0 : $\alpha_1 = 0$ and $\beta_1 = 0$ in (9.82), and H_0: $\alpha_1 = 0$ and $\beta_1 = 0$ and $\theta = 0$ and $\gamma = 0$ in (9.83) are also tested. If this second set of restrictions holds, then equations (9.82) and (9.83) collapse to

$$h_t = \alpha_0 + \delta \sigma_{t-1}^2 \qquad \textbf{(9.82')}$$

and

$$\ln(h_t) = \alpha_0 + \delta \ln(\sigma_{t-1}^2) \qquad \textbf{(9.83')}$$

These sets of restrictions on equations (9.82) and (9.83) test whether the lagged squared error and lagged conditional variance from a GARCH model contain any additional explanatory power once implied volatility is included in the specification. All of these restrictions can be tested fairly easily using a likelihood ratio test. The results of such a test are presented in Table 9.1.

Table 9.1 GARCH versus implied volatility

$$R_{Mt} - R_{Ft} = \lambda_0 + \lambda_1\sqrt{h_t} + u_t \quad (9.79)$$

$$h_t = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta_1 h_{t-1} \quad (9.80)$$

$$h_t = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta_1 h_{t-1} + \delta\sigma_{t-1}^2 \quad (9.82)$$

$$h_t = \alpha_0 + \delta\sigma_{t-1}^2 \quad (9.82')$$

Equation for variance	λ_0	λ_1	$\alpha_0 \times 10^{-4}$	α_1	β_1	δ	Log-L	χ^2
(9.80)	0.0072	0.071	5.428	0.093	0.854	–	767.321	17.77
	(0.005)	(0.01)	(1.65)	(0.84)	(8.17)			
(9.82)	0.0015	0.043	2.065	0.266	−0.068	0.318	776.204	–
	(0.028)	(0.02)	(2.98)	(1.17)	(−0.59)	(3.00)		
(9.82′)	0.0056	−0.184	0.993	–	–	0.581	764.394	23.62
	(0.001)	(−0.001)	(1.50)			(2.94)		

Notes: t-ratios in parentheses, Log-L denotes the maximised value of the log-likelihood function in each case. χ^2 denotes the value of the test statistic, which follows a $\chi^2(1)$ in the case of equation (9.82) restricted to equation (9.80), and a $\chi^2(2)$ in the case of equation (9.82) restricted to equation (9.82′).

Source: Day and Lewis (1992). Reprinted with the permission of Elsevier.

It appears from the coefficient estimates and their standard errors under the specification (9.82) that the implied volatility term (δ) is statistically significant, while the GARCH terms (α_1 and β_1) are not. However, the test statistics given in the final column are both greater than their corresponding χ^2 critical values, indicating that both GARCH and implied volatility have incremental power for modelling the underlying stock volatility. A similar analysis is undertaken in Day and Lewis that compares EGARCH with implied volatility. The results are presented here in Table 9.2.

Table 9.2 EGARCH versus implied volatility

$$R_{Mt} - R_{Ft} = \lambda_0 + \lambda_1\sqrt{h_t} + u_t \quad (9.79)$$

$$\ln(h_t) = \alpha_0 + \beta_1 \ln(h_{t-1}) + \alpha_1 \left(\theta \frac{u_{t-1}}{\sqrt{h_{t-1}}} + \gamma \left[\left|\frac{u_{t-1}}{\sqrt{h_{t-1}}}\right| - \left(\frac{2}{\pi}\right)^{1/2}\right]\right) \quad (9.81)$$

$$\ln(h_t) = \alpha_0 + \beta_1 \ln(h_{t-1}) + \alpha_1 \left(\theta \frac{u_{t-1}}{\sqrt{h_{t-1}}} + \gamma \left[\left|\frac{u_{t-1}}{\sqrt{h_{t-1}}}\right| - \left(\frac{2}{\pi}\right)^{1/2}\right]\right) + \delta \ln\left(\sigma^2_{t-1}\right) \quad (9.83)$$

$$\ln(h_t) = \alpha_0 + \delta \ln\left(\sigma^2_{t-1}\right) \quad (9.83')$$

Equation for variance	λ_0	λ_1	$\alpha_0 \times 10^{-4}$	β_1	θ	γ	δ	Log-L	χ^2
(9.81)	−0.0026	0.094	−3.62	0.529	0.273	0.357	–	776.436	8.09
	(−0.03)	(0.25)	(−2.90)	(3.26)	(−4.13)	(3.17)			
(9.83)	0.0035	−0.076	−2.28	0.373	−0.282	0.210	0.351	780.480	–
	(0.56)	(−0.24)	(−1.82)	(1.48)	(−4.34)	(1.89)	(1.82)		
(9.83′)	0.0047	−0.139	−2.76	–	–	–	0.667	765.034	30.89
	(0.71)	(−0.43)	(−2.30)				(4.01)		

Notes: t-ratios in parentheses, Log-L denotes the maximised value of the log-likelihood function in each case. χ^2 denotes the value of the test statistic, which follows a $\chi^2(1)$ in the case of equation (9.83) restricted to equation (9.81), and a $\chi^2(3)$ in the case of equation (9.83) restricted to equation (9.83′).

Source: Day and Lewis (1992). Reprinted with the permission of Elsevier.

The EGARCH results tell a very similar story to those of the GARCH specifications. Neither the lagged information from the EGARCH specification nor the lagged implied volatility terms can be suppressed, according to the likelihood ratio statistics. In specification (9.83), both the EGARCH terms and the implied volatility coefficients are marginally significant.

However, the tests given above do not represent a true test of the predictive ability of the models, since all of the observations were used in both estimating and testing the models. Hence the authors proceed to conduct an out-of-sample forecasting test. There are a total of 729 data points in their sample. They use the first 410 to estimate the models, and then make a one-step-ahead forecast of the following week's volatility. They then roll the sample forward one observation at a time, constructing a new one-step-ahead forecast at each stage.

They evaluate the forecasts in two ways. The first is by regressing the realised volatility series on the forecasts plus a constant

$$\sigma^2_{t+1} = b_0 + b_1\sigma^2_{ft} + \xi_{t+1} \quad \textbf{(9.84)}$$

where σ^2_{t+1} is the ‘actual’ value of volatility at time $t + 1$, and σ^2_{ft} is the value forecasted for it during period t. Perfectly efficient forecasts would imply $b_0 = 0$ and $b_1 = 1$. The second method is via a set of forecast encompassing tests. Essentially, these operate by regressing the realised volatility on the forecasts generated by several models. The forecast series that have significant coefficients are concluded to encompass those of models whose coefficients are not significant.

But what is volatility? In other words, with what measure of realised or ‘*ex post*’ volatility should the forecasts be compared? This is a question that received very little attention in the literature until recently. A common method employed is to assume, for a daily volatility forecasting exercise, that the relevant *ex post* measure is the square of that day’s return. For any random variable r_t, its conditional variance can be expressed as

$$\text{var}(r_t) = \text{E}[r_t - \text{E}(r_t)]^2 \tag{9.85}$$

As stated previously, it is typical, and not unreasonable for relatively high frequency data, to assume that $\text{E}(r_t)$ is zero, so that the expression for the variance reduces to

$$\text{var}(r_t) = \text{E}\left[r_t^2\right] \tag{9.86}$$

Andersen and Bollerslev (1998) argue that squared daily returns provide a very noisy proxy for the true volatility, and a much better proxy for the day’s variance would be to compute the volatility for the day from intra-daily data. For example, a superior daily variance measure could be obtained by taking hourly returns, squaring them and adding them up. The reason that the use of higher frequency data provides a better measure of *ex post* volatility is simply that it employs more information. By using only daily data to compute a daily volatility measure, effectively only two observations on the underlying price series are employed. If the daily closing price is the same one day as the next, the squared return and therefore the volatility would be calculated to be zero, when there may have been substantial intra-day fluctuations. Hansen and Lunde (2006) go further and suggest that even the ranking of models by volatility forecast accuracy could be inconsistent if the evaluation uses a poor proxy for the true, underlying volatility.

Day and Lewis use two measures of *ex post* volatility in their study (for which the frequency of data employed in the models is weekly)

(1) The square of the weekly return on the index, which they call SR

(2) The variance of the week's daily returns multiplied by the number of trading days in that week, which they call WV.

The Andersen and Bollerslev argument implies that the latter measure is likely to be superior, and therefore that more emphasis should be placed on those results.

The results for the separate regressions of realised volatility on a constant and the forecast are given in Table 9.3. The coefficient estimates for b_0 given in Table 9.3 can be interpreted as indicators of whether the respective forecasting approaches are biased. In all cases, the b_0 coefficients are close to zero. Only for the historic volatility forecasts and the implied volatility forecast when the *ex post* measure is the squared weekly return, are the estimates statistically significant. Positive coefficient estimates would suggest that on average the forecasts are too low. The estimated b_1 coefficients are in all cases a long way from unity, except for the GARCH (with daily variance *ex post* volatility) and EGARCH (with squared weekly variance as *ex post* measure) models. Finally, the R^2 values are very small (all less than 10%, and most less than 3%), suggesting that the forecast series do a poor job of explaining the variability of the realised volatility measure.

Table 9.3 Out-of-sample predictive power for weekly volatility forecasts

$\sigma^2_{t+1} = b_0 + b_1\sigma^2_{ft} + \xi_{t+1}$				(9.84)
Forecasting model	Proxy for ex post volatility	b_0	b_1	R^2
Historic	SR	0.0004 (5.60)	0.129 (21.18)	0.094
Historic	WV	0.0005 (2.90)	0.154 (7.58)	0.024
GARCH	SR	0.0002 (1.02)	0.671 (2.10)	0.039
GARCH	WV	0.0002 (1.07)	1.074 (3.34)	0.018
EGARCH	SR	0.0000 (0.05)	1.075 (2.06)	0.022
EGARCH	WV	−0.0001 (−0.48)	1.529 (2.58)	0.008
Implied volatility	SR	0.0022 (2.22)	0.357 (1.82)	0.037
Implied volatility	WV	0.0005 (0.389)	0.718 (1.95)	0.026

Notes: 'Historic' refers to the use of a simple historical average of the squared returns to forecast volatility; *t*-ratios in parentheses; SR and WV refer to the square of the weekly return on the S&P100, and the variance of the week's daily returns multiplied by the number of trading days in that week, respectively.

Source: Day and Lewis (1992). Reprinted with the permission of Elsevier.

The forecast encompassing regressions are based on a procedure due to Fair and Shiller (1990) that seeks to determine whether differing sets of forecasts contain different sets of information from one another. The test regression is of the form

$$\sigma^2_{t+1} = b_0 + b_1\sigma^2_{It} + b_2\sigma^2_{Gt} + b_3\sigma^2_{Et} + b_4\sigma^2_{Ht} + \xi_{t+1} \qquad \textbf{(9.87)}$$

with results presented in Table 9.4.

Table 9.4 Comparisons of the relative information content of out-of-sample volatility forecasts

$\sigma_{t+1}^2 = b_0 + b_1\sigma_{It}^2 + b_2\sigma_{Gt}^2 + b_2\sigma_{Gt}^2 + b_3\sigma_{Et}^2 + b_4\sigma_{Ht}^2 + \xi_{t+1}$ (9.87)						
Forecast comparisons	b_0	b_1	b_2	b_3	b_4	R^2
Implied versus GARCH	−0.00010	0.601	0.298	–	–	0.027
	(−0.09)	(1.03)	(0.42)			
Implied versus GARCH	0.00018	0.632	−0.243	–	0.123	0.038
versus Historical	(1.15)	(1.02)	(−0.28)		(7.01)	
Implied versus EGARCH	−0.00001	0.695	–	0.176	–	0.026
	(−0.07)	(1.62)		(0.27)		
Implied versus EGARCH	0.00026	0.590	−0.374	–	0.118	0.038
versus Historical	(1.37)	(1.45)	(−0.57)		(7.74)	
GARCH versus EGARCH	0.00005	–	1.070	−0.001	–	0.018
	(0.370)		(2.78)	(−0.00)		

Notes: t-ratios in parentheses; the *ex post* measure used in this table is the variance of the week's daily returns multiplied by the number of trading days in that week.

Source: Day and Lewis (1992). Reprinted with the permission of Elsevier.

The sizes and significances of the coefficients in Table 9.4 are of interest. The most salient feature is the lack of significance of most of the forecast series. In the first comparison, neither the implied nor the GARCH forecast series have statistically significant coefficients. When historical volatility is added, its coefficient is positive and statistically significant. An identical pattern emerges when forecasts from implied and EGARCH models are compared: that is, neither forecast series is significant, but when a simple historical average series is added, its coefficient is significant. It is clear from this, and from the last row of Table 9.4, that the asymmetry term in the EGARCH model has no additional explanatory power compared with that embodied in the symmetric GARCH model. Again, all of the R^2 values are very low (less than 4%).

The conclusion reached from this study (which is broadly in line with many others) is that within sample, the results suggest that implied volatility contains extra information not contained in the GARCH/EGARCH specifications. But the out-of-sample results suggest that predicting volatility is a difficult task!

9.19 Stochastic Volatility Models Revisited

Autoregressive models were discussed above in Section 9.6 and these are special cases of a more general class of models known as stochastic volatility (SV) models. It is a common misconception that GARCH-type specifications are sorts of stochastic volatility models. However, as the name suggests, stochastic volatility models differ from GARCH principally in that the conditional variance equation of a GARCH specification is completely deterministic given all information available up to that of the previous period. In other words, there is no error term in the variance equation of a GARCH model, only in the mean equation.

Stochastic volatility models contain a second error term, which enters into the conditional variance equation. The autoregressive volatility specification is simple to understand and simple to estimate, because it requires that we have an observable measure of volatility which is then simply used as any other variable in an autoregressive model. However, the term 'stochastic volatility' is usually associated with a different formulation, a possible example of which would be

$$y_t = \mu + u_t\sigma_t, u_t \sim N(0, 1) \quad \textbf{(9.88)}$$

$$\ln\left(\sigma_t^2\right) = \alpha_0 + \beta_1 \ln\left(\sigma_{t-1}^2\right) + \sigma_\eta \eta_t \quad \textbf{(9.89)}$$

where η_t is another N(0,1) random variable that is independent of u_t. Here the volatility is latent rather than observed, and so is modelled indirectly.

Stochastic volatility models are closely related to the financial theories used in the options pricing literature. Early work by Black and Scholes (1973) had assumed that volatility is constant through time. Such an assumption was made largely for simplicity, although it could hardly be considered realistic. One unappealing side-effect of employing a model with the embedded assumption that volatility is fixed, is that options deep in-the-money and far out-of-the-money are underpriced relative to actual traded prices. This empirical observation provided part of the genesis for stochastic volatility models, where the logarithm of an unobserved variance process is modelled by a linear stochastic specification, such as an autoregressive model. The primary advantage of stochastic volatility models is that they can be viewed as discrete time approximations to the continuous time models employed in options pricing frameworks (see, for example, Hull and White, 1987). However, such models are hard to estimate. For reviews of (univariate) stochastic volatility models, see Taylor (1994), Ghysels, Harvey and Renault (1995) or Shephard (1996) and the references therein.

While stochastic volatility models have been widely employed in the mathematical options pricing literature, they have not been popular in empirical discrete-time financial applications, probably owing to the complexity involved in the process of estimating the model parameters (see Harvey, Ruiz and Shephard, 1994). So, while GARCH-type models are further from their continuous time theoretical underpinnings than stochastic volatility, they are much simpler to estimate using maximum likelihood. A relatively simple modification to the maximum likelihood procedure used for GARCH model estimation is not available, and hence stochastic volatility models are not discussed further here.

9.19.1 Higher Moment Models

Research over the past two decades has moved from examination purely of the first moment of financial time series (i.e., estimating models for the returns themselves), to consideration of the *second moment* (models for the variance). While this clearly represents a large step forward in the analysis of financial data, it is also evident that conditional variance specifications are not able to fully capture all of the relevant time series properties. For example, GARCH models with normal (0,1) standardised disturbances cannot generate sufficiently fat tails to model the leptokurtosis that is actually observed in financial asset returns series. One proposed approach to this issue has been to suggest that the standardised disturbances are drawn from a Student's *t* distribution rather than a normal. However, there is also no reason to suppose that the fatness of tails should be constant over time, which it is forced to be by the GARCH-*t* model.

Another possible extension would be to use a conditional model for the third or fourth moments of the distribution of returns (i.e., the skewness and kurtosis, respectively). Under such a specification, the conditional skewness or kurtosis of the returns could follow a GARCH-type process that allows it to vary through time. Harvey and Siddique (1999, 2000) have developed an autoregressive conditional skewness model, while a conditional kurtosis model was proposed in Brooks *et al.* (2005). Such models could have many other applications in finance, including asset allocation (portfolio selection), option pricing, estimation of risk premia, and so on.

An extension of the analysis to moments of the return distribution higher than the second has also been undertaken in the context of the capital asset pricing model, where the conditional co-skewness and co-kurtosis of the asset's returns with the market's are accounted for (e.g., Hung, Shackleton and Xu, 2004). A recent study by Brooks, Černý and Miffre (2012) proposed a utility-based framework for the determination of optimal hedge ratios that can allow for the impact of higher moments on the hedging decision in the context of hedging commodity exposures with futures contracts.

9.19.2 Tail Models

It is widely known that financial asset returns do not follow a normal distribution, but rather they are almost always *leptokurtic*, or *fat-tailed*. This observation has several implications for econometric modelling. First, models and inference procedures are required that are robust to non-normal error distributions. Second, the riskiness of holding a particular security is probably no longer appropriately measured by its variance alone. In a risk management context, assuming normality when returns are fat-tailed will result in a systematic underestimation of the riskiness of the portfolio. Consequently, several approaches have been employed to systematically allow for the leptokurtosis in financial data, including the use of a Student's *t* distribution.

Arguably the simplest approach is the use of a mixture of normal distributions. It can be seen that a mixture of normal distributions with different variances will lead to an overall series that is leptokurtic. Second, a Student's *t* distribution can be used, with the usual degrees of freedom parameter estimated using maximum likelihood along with other parameters of the model. The degrees of freedom estimate will control the fatness of the tails fitted from the model. Other probability distributions can also be employed, such as the 'stable' distributions that fall under the general umbrella of extreme value theory – see Section 14.3 of Chapter 14 for a detailed presentation of this class of models.

9.20 Forecasting Covariances and Correlations

A major limitation of the volatility models examined above is that they are entirely univariate in nature – that is, they model the conditional variance of each series entirely independently of all other series. This is potentially an important limitation for two reasons. First, to the extent that there may be 'volatility spillovers' between markets or assets (a tendency for volatility to change in one market or asset following a change in the volatility of another), the univariate model will be misspecified. For instance, using a multivariate model will allow us to determine whether the volatility in one market leads or lags the volatility in others.

Second, it is often the case in finance that the covariances between series are of interest, as well as the variances of the individual series themselves. The calculation of hedge ratios, portfolio value at risk estimates, CAPM betas, and so on, all require covariances as inputs.

Multivariate GARCH models can potentially overcome both of these deficiencies with their univariate counterparts. Multivariate extensions to GARCH models can be used to forecast the volatilities of the component series, just as with univariate models and since the volatilities of financial time series often move together, a joint approach to modelling may be more efficient than treating each separately. In addition, because multivariate models give estimates for the conditional covariances as well as the conditional variances, they have a number of other potentially useful applications.

Several papers have investigated the forecasting ability of various models incor-porating correlations. Siegel (1997), for example, finds that implied correlation forecasts from traded options encompass all information embodied in the historical returns (although he does not consider EWMA- or GARCH-based models). Walter and Lopez (2000), on the other hand, find that implied correlation is generally less useful for predicting the future correlation between the underlying assets' returns than forecasts derived from GARCH models. Finally, Gibson and Boyer (1998) find that a diagonal GARCH and a Markov switching approach provide better correlation forecasts than simpler models in the sense that the latter produce smaller profits when the forecasts are employed in a trading strategy.

9.21 Covariance Modelling and Forecasting in Finance: Some Examples

9.21.1 The Estimation of Conditional Betas

The CAPM beta for asset *i* is defined as the ratio of the covariance between the market portfolio return and the asset return, to the variance of the market portfolio return. Betas are typically constructed using a set of historical data on market variances and covariances. However, like most other problems in finance, beta estimation conducted in this fashion is backward-looking, when investors should really be concerned with the beta that will prevail in the future over the time that the investor is considering holding the asset. Multivariate GARCH models provide a simple method for estimating conditional (or time-varying) betas. Then forecasts of the covariance between the asset and the market portfolio returns and forecasts of the variance of the market portfolio are made from the model, so that the beta is a forecast, whose value will vary over time

$$\beta_{i,t} = \frac{\sigma_{im,t}}{\sigma_{m,t}^2} \tag{9.90}$$

where $\beta_{i,t}$ is the time-varying beta estimate at time t for stock i, $\sigma_{im,t}$ is the covariance between market returns and returns to stock i at time t and $\sigma_{m,t}^2$ is the variance of the market return at time t.

9.21.2 Dynamic Hedge Ratios

Although there are many techniques available for reducing and managing risk, the simplest and perhaps the most widely used, is hedging with futures contracts. A hedge is achieved by taking opposite positions in spot and futures markets simultaneously, so that any loss sustained from an adverse price movement in one market should to some degree be offset by a favourable price movement in the other. The ratio of the number of units of the futures asset that are purchased relative to the number of units of the spot asset is known as the *hedge ratio*. Since risk in this context is usually measured as the volatility of portfolio returns, an intuitively plausible strategy might be to choose that hedge ratio which minimises the variance of the returns of a portfolio containing the spot and futures position; this is known as the *optimal hedge ratio*. The optimal value of the hedge ratio may be determined in the usual way, following Hull (2017) by first defining:

ΔS = change in spot price S, during the life of the hedge ΔF = change in futures price, F, during the life of the hedge σ_s = standard deviation of $\Delta S \sigma_F$ = standard deviation of $\Delta F p$ = correlation coefficient between ΔS and $\Delta F h$ = hedge ratio

For a short hedge (i.e., long in the asset and short in the futures contract), the change in the value of the hedger's position during the life of the hedge will be given by ($\Delta S - h\,\Delta F$), while for a long hedge, the appropriate expression will be ($h\Delta F - \Delta S$).

The variances of the two hedged portfolios (long spot and short futures or long futures and short spot) are the same. These can be obtained from

$$\text{var}(h\Delta F - \Delta S)$$

Remembering the rules for manipulating the variance operator, this can be written

$$\text{var}(\Delta S) + \text{var}(h\Delta F) - 2\text{cov}(\Delta S, h\Delta F)$$

or

$$\text{var}(\Delta S) + h^2\text{var}(\Delta F) - 2h\text{cov}(\Delta S, \Delta F)$$

Hence the variance of the change in the value of the hedged position is given by

$$v = \sigma_s^2 + h^2\sigma_F^2 - 2hp\sigma_s\sigma_F \qquad \textbf{(9.91)}$$

Minimising this expression w.r.t. *h* would give

$$h = p\frac{\sigma_s}{\sigma_F} \qquad \textbf{(9.92)}$$

Again, according to this formula, the optimal hedge ratio is time-invariant, and would be calculated using historical data. However, what if the standard deviations are changing over time? The standard deviations and the correlation between movements in the spot and futures series could be forecast from a multivariate GARCH model, so that the expression above is replaced by

$$h_t = p_t\frac{\sigma_{s,t}}{\sigma_{F,t}} \qquad \textbf{(9.93)}$$

Various models are available for covariance or correlation forecasting, and several will be discussed below, which are grouped into simple models, multivariate GARCH models, and specific correlation models.

9.22 Simple Covariance Models

9.22.1 Historical Covariance and Correlation

In exactly the same fashion as for volatility, the historical covariance or correlation between two series can be calculated in the standard way using a set of historical data.

9.22.2 Implied Covariance Models

Implied covariances can be calculated using options whose payoffs are dependent on more than one underlying asset. The relatively small number of such options that exist limits the circumstances in which implied covariances can be calculated. Examples include rainbow options, 'crack-spread' options for different grades of oil, and currency options. In the latter case, the implied variance of the cross-currency returns *xy* is given by

$$\tilde{\sigma}^2(xy) = \tilde{\sigma}^2(x) + \tilde{\sigma}^2(y) - 2\tilde{\sigma}(x, y) \quad \textbf{(9.94)}$$

where $\tilde{\sigma}^2(x)$ and $\tilde{\sigma}^2(y)$ are the implied variances of the *x* and *y* returns, respectively, and $\tilde{\sigma}(x, y)$ is the implied covariance between *x* and *y*. By substituting the observed option implied volatilities of the three currencies into equation (9.94), the implied covariance is obtained via

$$\tilde{\sigma}(x, y) = \frac{\tilde{\sigma}^2(x) + \tilde{\sigma}^2(y) - \tilde{\sigma}^2(xy)}{2} \quad \textbf{(9.95)}$$

So, for instance, if the implied covariance between USD/DEM and USD/JPY is of interest, then the implied variances of the returns of USD/DEM and USD/JPY, as well as the returns of the cross-currency DEM/JPY, are required so as to obtain the implied covariance using equation (9.94).

9.22.3 Exponentially Weighted Moving Average Model for Covariances

Again, as for the case of single series volatility modelling, a EWMA specification is available that gives more weight in the calculation of covariance to recent observations than the estimate based on the simple average. The EWMA model estimates for variances and covariances at time *t* in the bivariate setup with two returns series *x* and *y* may be written as

$$h_{ij,t} = \lambda h_{ij,t-1} + (1 - \lambda)x_{t-1}y_{t-1} \quad \textbf{(9.96)}$$

where $i \neq j$ for the covariances and $i = j$; $x = y$ for the variance specifications. As for the univariate case, the fitted values for *h* also become the forecasts for subsequent periods. $\lambda(0 < \lambda < 1)$ again denotes the decay factor determining the relative weights attached to recent versus less recent observations. this parameter could be estimated (for example, by maximum likelihood), but is often set arbitrarily (– for example, Riskmetrics use a decay factor of 0.97 for monthly data but 0.94 when the data are of daily frequency).

This equation can be rewritten as an infinite order function of only the returns by successively substituting out the covariances

$$h_{ij,t} = (1-\lambda)\sum_{i=0}^{\infty}\lambda^{i}x_{t-i}y_{t-i} \qquad \textbf{(9.97)}$$

While the EWMA model is probably the simplest way to allow for time-varying variances and covariances, the model is a restricted version of an integrated GARCH (IGARCH) specification, and it does not guarantee the fitted variance-covariance matrix to be positive definite. As a result of the parallel with IGARCH, EWMA models also cannot allow for the observed mean reversion in the volatilities or covariances of asset returns that is particularly prevalent at lower frequencies of observation.

9.23 Multivariate GARCH Models

Multivariate GARCH models are in spirit very similar to their univariate counterparts, except that the former also specify equations for how the covariances move over time and are therefore by their nature inherently more complex to specify and estimate. Several different multivariate GARCH formulations have been proposed in the literature, the most popular of which are the *VECH*, the diagonal *VECH* and the *BEKK* models. Each of these and several others is discussed in turn below; for a more detailed discussion, see Kroner and Ng (1998). In each case, there are *N* assets, whose return variances and covariances are to be modelled.

9.23.1 The VECH model

As with univariate GARCH models, the conditional mean equation may be parameterised in any way desired, although it is worth noting that, since the conditional variances are measured about the mean, misspecification of the latter is likely to imply misspecification of the former. To introduce some notation, suppose, that y_t $(y_{1t}\ y_{2t}\ \dots\ y_{Nt})$, is an $N \times 1$ vector of time-series observations, C is an $N(N + 1)/2$ column vector of conditional variance and covariance intercepts, and A and B are square parameter matrices of order $N(N + 1)/2$. A common specification of the *VECH* model, initially due to Bollerslev, Engle and Wooldridge (1988), is

$$VECH(H_t) = C + AVECH(\Xi_{t-1}\Xi'_{t-1}) + BVECH(H_{t-1})$$
$$\Xi_t|\psi_{t-1} \sim N(0, H_t), \quad \textbf{(9.98)}$$

where H_t is a $N \times N$ conditional variance–covariance matrix, Ξ_t is a $N \times 1$ innovation (disturbance) vector, ψ_{t-1} represents the information set at time $t - 1$, and *VECH* $(\cdot)$ denotes the column-stacking operator applied to the upper portion of the symmetric matrix. In the bivariate case (i.e., $N = 2$), C will be a 3×1 parameter vector, and A and B will be 3×3 parameter matrices.

The unconditional variance matrix for the *VECH* will be given by $C[I - A - B]^{-1}$, where I is an identity matrix of order $N(N + 1)/2$. Stationarity of the *VECH* model requires that the eigenvalues of $[A + B]$ are all less than one in absolute value.

In order to gain a better understanding of how the *VECH* model works, the elements for $N = 2$ are written out below. Define

$$H_t = \begin{bmatrix} h_{11t} & h_{12t} \\ h_{21t} & h_{22t} \end{bmatrix}, \Xi_t = \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix}, C = \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix},$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix},$$

The *VECH* operator takes the ‘upper triangular’ portion of a matrix, and stacks each element into a vector with a single column. For example, in the case of $VECH(H_t)$, this becomes

$$VECH(H_t) = \begin{bmatrix} h_{11t} \\ h_{22t} \\ h_{12t} \end{bmatrix}$$

where h_{iit} represent the conditional variances at time t of the two-asset return series ($i = 1, 2$) used in the model, and h_{ijt} ($i \neq j$) represent the conditional covariances between the asset returns. In the case of $VECH(\Xi_t\Xi'_t)$, this can be expressed as

$$
\begin{aligned}
VECH(\Xi_t\Xi_t') &= VECH\left(\begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix}\begin{bmatrix} u_{1t} & u_{2t} \end{bmatrix}\right) \\
&= VECH\begin{pmatrix} u_{1t}^2 & u_{1t}u_{2t} \\ u_{1t}u_{2t} & u_{2t}^2 \end{pmatrix} \\
&= \begin{bmatrix} u_{1t}^2 \\ u_{2t}^2 \\ u_{1t}u_{2t} \end{bmatrix}
\end{aligned}
$$

The *VECH* model in full is given by

$$
\begin{aligned}
h_{11t} = c_{11} + a_{11}u_{1t-1}^2 + a_{12}u_{2t-1}^2 + a_{13}u_{1t-1}u_{2t-1} + b_{11}h_{11t-1} \\
+ b_{12}h_{22t-1} + b_{13}h_{12t-1}
\end{aligned} \tag{9.99}
$$

$$
\begin{aligned}
h_{22t} = c_{21} + a_{21}u_{1t-1}^2 + a_{22}u_{2t-1}^2 + a_{23}u_{1t-1}u_{2t-1} + b_{21}h_{11t-1} \\
+ b_{22}h_{22t-1} + b_{23}h_{12t-1}
\end{aligned} \tag{9.100}
$$

$$
\begin{aligned}
h_{12t} = c_{31} + a_{31}u_{1t-1}^2 + a_{32}u_{2t-1}^2 + a_{33}u_{1t-1}u_{2t-1} + b_{31}h_{11t-1} \\
+ b_{32}h_{22t-1} + b_{33}h_{12t-1}
\end{aligned} \tag{9.101}
$$

Thus, it is clear that the conditional variances and conditional covariances depend on the lagged values of all of the conditional variances of, and conditional covariances between, all of the asset returns in the series, as well as the lagged squared errors and the error cross-products. This unrestricted model is highly parameterised, and it is challenging to estimate. For $N = 2$ there are 21 parameters (C has 3 elements, A and B each have 9 elements), while for $N = 3$ there are 78, and $N = 4$ implies 210 parameters!

9.23.2 The Diagonal VECH Model

As the number of assets employed increases, estimation of the *VECH* model can quickly become infeasible. Hence the *VECH* model's conditional variance–covariance matrix has been restricted to the form developed by Bollerslev, Engle and Wooldridge (1988), in which A and B are assumed to be diagonal. This restriction implies that there are no direct volatility spillovers from one series to another, which considerably reduces the number of parameters to be estimated to nine in the bivariate case (now A and B each have three elements) and 18 for a trivariate system (i.e., if $N = 3$). The model, known as a diagonal *VECH*, is now characterised by

$$
h_{ij,t} = \omega_{ij} + \alpha_{ij}u_{i,t-1}u_{j,t-1} + \beta_{ij}h_{ij,t-1} \quad \text{for } i,j = 1,2
$$

(9.102)

where ω_{ij}, α_{ij} and β_{ij} are parameters.

The diagonal *VECH* multivariate GARCH model could also be expressed as an infinite order multivariate ARCH model, where the covariance is expressed as a geometrically declining weighted average of past cross products of unexpected returns, with recent observations carrying higher weights. An alternative solution to the dimensionality problem would be to use orthogonal GARCH (see, for example, Van der Weide, 2002) or factor GARCH models (see Engle, Ng and Rothschild, 1990). A disadvantage of the *VECH* model is that there is no guarantee of a positive semi-definite covariance matrix.

A variance–covariance or correlation matrix must always be ‘positive semi-definite’, and in the case where all the returns in a particular series are all the same so that their variance is zero is disregarded, then the matrix will be positive definite. Among other things, this means that the variance–covariance matrix will have all positive numbers on the leading diagonal, and will be symmetrical about this leading diagonal. These properties are intuitively appealing as well as important from a mathematical point of view, for variances can never be negative, and the covariance between two series is the same irrespective of which of the two series is taken first, and positive definiteness ensures that this is the case.

A positive definite correlations matrix is also important for many applications in finance – for example, from a risk management point of view. It is this property which ensures that, whatever the weight of each series in the asset portfolio, an estimated value-at-risk is always positive. Fortunately, this desirable property is automatically a feature of time-invariant correlations matrices which are computed directly using actual data. An anomaly arises when either the correlation matrix is estimated using a non-linear optimisation procedure (as multivariate GARCH models are), or when modified values for some of the correlations are used by the risk manager. The resulting modified correlation matrix may or may not be positive definite, depending on the values of the correlations that are put in, and the values of the remaining correlations. If, by chance, the matrix is not positive definite, the upshot is that for some weightings of the individual assets in the portfolio, the estimated portfolio variance could be negative.

9.23.3 The BEKK model

The *BEKK* model (Engle and Kroner, 1995) addresses the difficulty with *VECH* of ensuring that the H matrix is always positive definite.[1] It is represented by

$$H_t = W'W + A'H_{t-1}A + B'\Xi_{t-1}\Xi'_{t-1}B \quad \textbf{(9.103)}$$

where A, and B are $N \times N$ matrices of parameters and W is an upper triangular matrix of parameters. The positive definiteness of the covariance matrix is ensured owing to the quadratic nature of the terms on the equation's RHS.

9.23.4 Model Estimation for Multivariate GARCH

Under the assumption of conditional normality, the parameters of the multivariate GARCH models of any of the above specifications can be estimated by maximising the log-likelihood function

$$\ell(\theta) = -\frac{TN}{2}\ln 2\pi - \frac{1}{2}\sum_{t=1}^{T}\left(\ln|H_t| + \Xi'_t H_t^{-1}\Xi_t\right) \quad \textbf{(9.104)}$$

where θ denotes all the unknown parameters to be estimated, N is the number of assets (i.e., the number of series in the system) and T is the number of observations and all other notation is as above. The maximum-likelihood estimate for θ is asymptotically normal, and thus traditional procedures for statistical inference are applicable. Further details on maximum-likelihood estimation in the context of multivariate GARCH models are beyond the scope of this book. But suffice to say that the additional complexity and extra parameters involved compared with univariate models make estimation a computationally more difficult task, although the principles are essentially the same.

9.24 Direct Correlation Models

The *VECH* and *BEKK* models specify the dynamics of the covariances between a set of series, and the correlations between any given pair of series at each point in time can be constructed by dividing the conditional covariances by the product of the conditional standard deviations. A subtly different approach would be to model the dynamics for the correlations directly – Bauwens, Laurent and Rombouts (2006) term these 'non-linear combinations of univariate GARCH models' for reasons that will become apparent in the following sub-section.

9.24.1 The Constant Correlation Model

An alternative method for reducing the number of parameters in the MGARCH framework is to require the correlations between the disturbances, ϵ_t, (or equivalently between the observed variables, y_t) to be fixed through time. Thus, although the conditional covariances are not fixed, they are tied to the variances as proposed in the constant conditional correlation (CCC) model due to Bollerslev (1990). The conditional variances in the fixed correlation model are identical to those of a set of univariate GARCH specifications (although they are estimated jointly)

$$h_{ii,t} = c_i + a_i \epsilon_{i,t-i}^2 + b_i h_{ii,t-1}, \qquad i = 1, \ldots, N \tag{9.105}$$

The off-diagonal elements of H_t, $h_{ij,t}$ ($i \neq j$), are defined indirectly via the correlations, denoted ρ_{ij}

$$h_{ij,t} = \rho_{ij} h_{ii,t}^{1/2} h_{jj,t}^{1/2}, \qquad i, j = 1, \ldots, N, i < j \tag{9.106}$$

Is it empirically plausible to assume that the correlations are constant through time? Several tests of this assumption have been developed, including a test based on the information matrix due to Bera and Kim (2002) and a Lagrange Multiplier test due to Tse (2000). The conclusions reached appear dependent on which test is used, but there seems to be non-negligible evidence against constant correlations, particularly in the context of stock returns.

9.24.2 The Dynamic Conditional Correlation Model

Several different formulations of the dynamic conditional correlation (DCC) model are available, but a popular specification is due to Engle (2002). The model is related to the CCC formulation described above, but where the correlations are allowed to vary over time. Define the variance–covariance matrix, H_t, as

$$H_t = D_t R_t D_t \tag{9.107}$$

where D_t is a diagonal matrix containing the conditional standard deviations (i.e., the square roots of the conditional variances from univariate GARCH model estimations on each of the N individual series) on the leading diagonal; R_t is the conditional correlation matrix. Forcing R_t to be time-invariant would lead back to the constant conditional correlation model.

Numerous explicit parameterisations of R_t are possible, including an exponential smoothing approach discussed in Engle (2002). More generally, a model of the MGARCH form could be specified as

$$Q_t = S \circ (\iota\iota' - A - B) + A \circ u_{t-1}u'_{t-1} + B \circ Q_{t-1} \quad \textbf{(9.108)}$$

where S is the unconditional correlation matrix of the vector of standardised residuals (from the first stage estimation – see below), $u_t = D_t^{-1}\epsilon_t$. ι is a vector of ones, and Q_t is an $N \times N$ symmetric positive definite variance-covariance matrix. $\circ$ denotes the *Hadamard* or element-by-element matrix multiplication procedure. This specification for the intercept term simplifies estimation and reduces the number of parameters to be estimated, but is not necessary. Engle (2002) proposes a GARCH-esque formulation for dynamically modelling Q_t with the conditional correlation matrix, R_t, then constructed as

$$R_t = diag\{Q_t^*\}^{-1} Q_t diag\{Q_t^*\}^{-1} \quad \textbf{(9.109)}$$

where $diag(\cdot)$ denotes a matrix comprising the main diagonal elements of $(\cdot)$ and Q^* is a matrix that takes the square roots of each element in Q. This operation is effectively taking the covariances in Q_t and dividing them by the product of the appropriate standard deviations in Q_t^* to create a matrix of correlations.

A slightly different form of the DCC was proposed by Tse and Tsui (2002), and equation (9.108) could also be simplified by specifying A and B each as single scalars so that all the conditional correlations would follow the same process.

The model may be estimated in one single stage using maximum likelihood, although this will still be a difficult exercise in the context of large systems. Consequently, Engle advocates a two-stage estimation procedure where each variable in the system is first modelled separately as a univariate GARCH process. A joint log-likelihood function for this stage could be constructed, which would simply be the sum (over N) of all of the log-likelihoods for the individual GARCH models. Then, in the second stage, the conditional likelihood is maximised with respect to any unknown parameters in the correlation matrix. The log-likelihood function for the second stage estimation will be of the form

$$\ell(\theta_2|\theta_1) = \sum_{t=1}^{T} \left(\ln |R_t| + u_t' R_t^{-1} u_t \right) \quad \textbf{(9.110)}$$

where θ_1 denotes all the unknown parameters that were estimated in the first stage and θ_2 denotes all those to be estimated in the second stage. Estimation using this two-step procedure will be consistent but inefficient as a result of any parameter uncertainty from the first stage being carried through to the second.

9.25 Extensions to the Basic Multivariate GARCH Model

Numerous extensions to the univariate specification have been proposed, and many of these carry over to the multivariate case. For example, conditional variance or covariance terms can be included in the conditional mean equation (see Bollerslev, Engle and Wooldridge, 1988, for instance). In the context of financial applications, where the y_t are returns, the parameters on these variables can be loosely interpreted as risk premia.

9.25.1 Asymmetric Multivariate GARCH

Asymmetric models have become very popular in empirical applications, where the conditional variances and/or covariances are permitted to react differently to positive and negative innovations of the same magnitude. In the multivariate context, this is usually achieved in the Glosten, Jagannathan and Runkle (1993) framework, rather than the EGARCH specification of Nelson (1991). Kroner and Ng (1998), for example, suggest the following extension to the BEKK formulation (with obvious related modifications for the *VECH* or diagonal *VECH* models)

$$H_t = W'W + A'H_{t-1}A + B'\Xi_{t-1}\Xi_{t-1}'B + D'z_{t-1}z_{t-1}'D \quad \textbf{(9.111)}$$

where z_{t-1} is an N-dimensional column vector with elements taking the value $-\epsilon_{t-1}$ if the corresponding element of ϵ_{t-1} is negative and zero otherwise. The asymmetric properties of time-varying covariance matrix models are analysed by Kroner and Ng (1998), who identify three possible forms of asymmetric behaviour. First, the covariance matrix displays own variance asymmetry if the conditional variance of one series is affected by the sign of the innovation in that series. Second, the covariance matrix displays cross variance asymmetry if the conditional variance of one series is affected by the sign of the innovation of another series. Finally, if the conditional covariance is sensitive to the sign of the innovation in return for either series, then the model is said to display covariance asymmetry.

9.25.2 Alternative Distributional Assumptions

As was the case for stochastic volatility and univariate GARCH models, an assumption of (multivariate) conditional normality cannot generate sufficiently fat tails to accurately model the distributional properties of financial data. A better approximation to the actual distributions of (especially financial) time series can be obtained using a Student's *t* distribution. Such a model can still be estimated using maximum likelihood but with a different (and more complex) likelihood function. The standard formulation will involve estimating, as part of the process, a single degree of freedom parameter which applies to all of the series in the system. An additional potential drawback of this approach is that the tail fatness embodied in the degrees of freedom parameter is fixed over time. Brooks *et al.* (2005) propose a model where both of these limitations are removed. However, some identifying restrictions are still required. A further issue is the extent to which the unconditional distribution of the shocks is skewed. If this is the case, then a model based on the Student's *t* will be inadequate, and an alternative such as the multivariate skew Student's *t* of Bauwens and Laurent (2002) must be used.

Although many other extensions of the basic models may be conceived of, such as periodic or seasonal MGARCH, the range of specifications employed in the existing literature is narrower than for the corresponding univariate models. A major drawback for even the more parsimonious of the models above is that they are too highly parameterised, and yet many potential applications in economics and finance are in the context of high dimensional systems (such as asset allocation among a number of stocks). Thus, an important innovation was the development of orthogonal and factor models referenced above. Both have the same fundamental idea that by forcing some structure on the variance-covariance matrix, a simplification can be achieved.

9.26 A Multivariate GARCH Model for the CAPM

with Time-Varying Covariances

Bollerslev, Engle and Wooldridge (1988) estimate a multivariate GARCH model for returns to US Treasury bills, gilts and stocks. The data employed comprised calculated quarterly excess holding period returns for six-month US Treasury bills, twenty-year US Treasury bonds and a Center for Research in Security Prices record of the return on the New York Stock Exchange (NYSE) value-weighted index. The data run from 1959Q1 to 1984Q2 – a total of 102 observations.

A multivariate GARCH-M model of the diagonal *VECH* type is employed, with coefficients estimated by maximum likelihood, and the Berndt *et al.* (1974) algorithm is used. The coefficient estimates are easiest presented in the following equations for the conditional mean and variance equations, respectively

$$\begin{vmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \end{vmatrix} = \begin{vmatrix} \underset{(0.032)}{0.070} \\ \underset{(1.030)}{-4.342} \\ \underset{(0.710)}{-3.117} \end{vmatrix} + \underset{(0.160)}{0.499} \sum_j \omega_{jt-1} \begin{vmatrix} h_{1jt} \\ h_{2jt} \\ h_{3jt} \end{vmatrix} + \begin{vmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{vmatrix} \qquad \textbf{(9.112)}$$

$$\begin{vmatrix} h_{11t} \\ h_{12t} \\ h_{22t} \\ h_{13t} \\ h_{23t} \\ h_{33t} \end{vmatrix} = \begin{vmatrix} \underset{(0.004)}{0.011} \\ \underset{(0.062)}{0.176} \\ \underset{(6.372)}{13.305} \\ \underset{(0.009)}{0.018} \\ \underset{(2.820)}{5.143} \\ \underset{(1.466)}{2.083} \end{vmatrix} + \begin{vmatrix} \underset{(0.105)}{0.445\varepsilon^2_{1t-1}} \\ \underset{(0.092)}{0.233\varepsilon_{1t-1}\varepsilon_{2t-1}} \\ \underset{(0.113)}{0.188\varepsilon^2_{2t-1}} \\ \underset{(0.132)}{0.197\varepsilon_{1t-1}\varepsilon_{3t-1}} \\ \underset{(0.093)}{0.165\varepsilon_{2t-1}\varepsilon_{3t-1}} \\ \underset{(0.066)}{0.078\varepsilon^2_{3t-1}} \end{vmatrix} + \begin{vmatrix} \underset{(0.056)}{0.466h_{11t-1}} \\ \underset{(0.052)}{0.598h_{12t-1}} \\ \underset{(0.215)}{0.441h_{22t-1}} \\ \underset{(0.361)}{-0.362h_{13t-1}} \\ \underset{(0.338)}{-0.348h_{23t-1}} \\ \underset{(0.333)}{0.469h_{33t-1}} \end{vmatrix} \qquad \textbf{(9.113)}$$

Source: Bollerslev, Engle and Wooldridge (1988). Reprinted with the permission of University of Chicago Press.

where y_{jt} are the returns, ω_{jt-1} are a set vector of value weights at time $t - 1$, $i = 1, 2, 3$, refers to bills, bonds and stocks, respectively and standard errors are given in parentheses. Consider now the implications of the signs, sizes and significances of the coefficient estimates in equations (9.112) and (9.113). The coefficient of 0.499 in the conditional mean equation gives an aggregate measure of relative risk aversion, also interpreted as representing the market trade-off between return and risk. This conditional variance-in-mean coefficient gives the required additional return as compensation for taking an additional unit of variance (risk). The intercept coefficients in the conditional mean equation for bonds and stocks are very negative and highly statistically significant. The authors argue that this is to be expected since favourable tax treatments for investing in longer-term assets encourages investors to hold them even at relatively low rates of return.

The dynamic structure in the conditional variance and covariance equations is strongest for bills and bonds, and very weak for stocks, as indicated by their respective statistical significances. In fact, none of the parameters in the conditional variance or covariance equations for the stock return equations is significant at the 5% level. The unconditional covariance between bills and bonds is positive, while that between bills and stocks, and between bonds and stocks, is negative. This arises since, in the latter two cases, the lagged conditional covariance parameters are negative and larger in absolute value than those of the corresponding lagged error cross-products.

Finally, the degree of persistence in the conditional variance (given by $\alpha_1 + \beta$), which embodies the degree of clustering in volatility, is relatively large for the bills equation, but surprisingly small for bonds and stocks, given the results of other relevant papers in this literature.

9.27 Estimating a Time-Varying Hedge Ratio for FTSE Stock Index Returns

A paper by Brooks, Henry and Persand (2002) compared the effectiveness of hedging on the basis of hedge ratios derived from various multivariate GARCH specifications and other, simpler techniques. Some of their main results are discussed below.

9.27.1 Background

There has been much empirical research into the calculation of optimal hedge ratios. The general consensus is that the use of multivariate GARCH (MGARCH) models yields superior performances, evidenced by lower portfolio volatilities, than either time-invariant or rolling OLS hedges. Cecchetti, Cumby and Figlewski (1988), Myers and Thompson (1989) and Baillie and Myers (1991), for example, argue that commodity prices are characterised by time-varying covariance matrices. As news about spot and futures prices arrives to the market in discrete bunches, the conditional covariance matrix, and hence the optimal hedging ratio, becomes time-varying. Baillie and Myers (1991) and Kroner and Sultan (1993), *inter alia*, employ MGARCH models to capture time-variation in the covariance matrix and to estimate the resulting hedge ratio.

9.27.2 Notation

Let S_t and F_t represent the logarithms of the stock index and stock index futures prices, respectively. The actual return on a spot position held from time $t - 1$ to t is $\Delta S_t = S_t - S_{t-1}$ similarly, the actual return on a futures position is $\Delta F_t = F_t - F_{t-1}$. However at time $t - 1$ the expected return, $E_{t-1}(R_t)$, of the portfolio comprising one unit of the stock index and β units of the futures contract may be written as

$$E_{t-1}(R_t) = E_{t-1}(\Delta S_t) - \beta_{t-1} E_{t-1}(\Delta F_t) \quad \textbf{(9.114)}$$

where β_{t-1} is the hedge ratio determined at time $t - 1$, for employment in period t. The variance of the expected return, $h_{p,t}$, of the portfolio may be written as

$$h_{p,t} = h_{s,t} + \beta_{t-1}^2 h_{F,t} - 2\beta_{t-1} h_{SF,t} \quad \textbf{(9.115)}$$

where $h_{p,t}$, $h_{s,t}$ and $h_{F,t}$ represent the conditional variances of the portfolio and the spot and futures positions, respectively and $h_{SF,t}$ represents the conditional covariance between the spot and futures position. β_{t-1}^*, the optimal number of futures contracts in the investor's portfolio, i.e., the optimal hedge ratio, is given by

$$\beta_{t-1}^* = -\frac{h_{SF,t}}{h_{F,t}} \quad \textbf{(9.116)}$$

If the conditional variance–covariance matrix is time-invariant (and if S_t and F_t are not cointegrated) then an estimate of β^*, the constant optimal hedge ratio, may be obtained from the estimated slope coefficient b in the regression

$$\Delta S_t = a + b\Delta F_t + u_t \quad \textbf{(9.117)}$$

The OLS estimate of the optimal hedge ratio could be given by $b = h_{SF}/h_F$.

9.27.3 Data and Results

The data employed in the Brooks, Henry and Persand (2002) study comprise 3,580 daily observations on the FTSE 100 stock index and stock index futures contract spanning the period 1 January 1985–9 April 1999. Several approaches to estimating the optimal hedge ratio are investigated.

The hedging effectiveness is first evaluated in-sample, that is, where the hedges are constructed and evaluated using the same set of data. The out-of-sample hedging effectiveness for a one-day hedging horizon is also investigated by forming one-step-ahead forecasts of the conditional variance of the futures series and the conditional covariance between the spot and futures series. These forecasts are then translated into hedge ratios using equation (9.116). The hedging performance of a *BEKK* formulation is examined, and also a *BEKK* model including asymmetry terms (in the same style as GJR models). The returns and variances for the various hedging strategies are presented in Table 9.5.

Table 9.5 Hedging effectiveness: summary statistics for portfolio returns

In-sample				
(1)	Unhedged $\beta = 0$ (2)	Naive hedge $\beta = -1$ (3)	Symmetric time-varying hedge $\beta_t = \frac{h_{FS,t}}{h_{F,t}}$ (4)	Asymmetric time-varying hedge $\beta_t = \frac{h_{FS,t}}{h_{F,t}}$ (5)
Return	0.0389 {2.3713}	−0.0003 {−0.0351}	0.0061 {0.9562}	0.0060 {0.9580}
Variance	0.8286	0.1718	0.1240	0.1211
Out-of-sample				
	Unhedged $\beta = 0$	Naive hedge $\beta = -1$	Symmetric time-varying hedge $\beta_t = \frac{h_{FS,t}}{h_{F,t}}$	Asymmetric time-varying hedge $\beta_t = \frac{h_{FS,t}}{h_{F,t}}$
Return	0.0819 {1.4958}	−0.0004 {0.0216}	0.0120 {0.7761}	0.0140 {0.9083}
Variance	1.4972	0.1696	0.1186	0.1188

Note: t-ratios displayed as {.}.

Source: Brooks, Henry and Persand (2002).

The simplest approach, presented in column (2), is that of no hedge at all. In this case, the portfolio simply comprises a long position in the cash market. Such an approach is able to achieve significant positive returns in sample, but with a large variability of portfolio returns. Although none of the alternative strategies generate returns that are significantly different from zero, either in-sample or out-of-sample, it is clear from columns (3)–(5) of Table 9.5 that any hedge generates significantly less return variability than none at all.

The 'naive' hedge, which takes one short futures contract for every spot unit, but does not allow the hedge to time-vary, generates a reduction in variance of the order of 80% in-sample and nearly 90% out-of-sample relative to the unhedged position. Allowing the hedge ratio to be time-varying and determined from a symmetric multivariate GARCH model leads to a further reduction as a proportion of the unhedged variance of 5% and 2% for the in-sample and holdout sample, respectively. Allowing for an asymmetric response of the conditional variance to positive and negative shocks yields a very modest reduction in variance (a further 0.5% of the initial value) in-sample, and virtually no change out-of-sample.

Figure 9.5 graphs the time-varying hedge ratio from the symmetric and asymmetric MGARCH models (source: Brooks, Henry and Persand, 2002). The optimal hedge ratio is never greater than 0.96 futures contracts per index contract, with an average value of 0.82 futures contracts sold per long index contract. The variance of the estimated optimal hedge ratio is 0.0019. Moreover the optimal hedge ratio series obtained through the estimation of the asymmetric GARCH model appears stationary. An ADF test of the null hypothesis $\beta^*_{t-1} \sim I(1)$ (i.e., that the optimal hedge ratio from the asymmetric *BEKK* model contains a unit root) was strongly rejected by the data (ADF statistic = −5.7215, 5% Critical value = −2.8630). The time-varying hedge requires the sale (purchase) of fewer futures contracts per long (short) index contract and hence would save the firm wishing to hedge a short exposure money relative to the time-invariant hedge. One possible interpretation of the better performance of the dynamic strategies over the naive hedge is that the dynamic hedge uses short-run information, while the naive hedge is driven by long-run considerations and an assumption that the relationship between spot and futures price movements is 1:1.

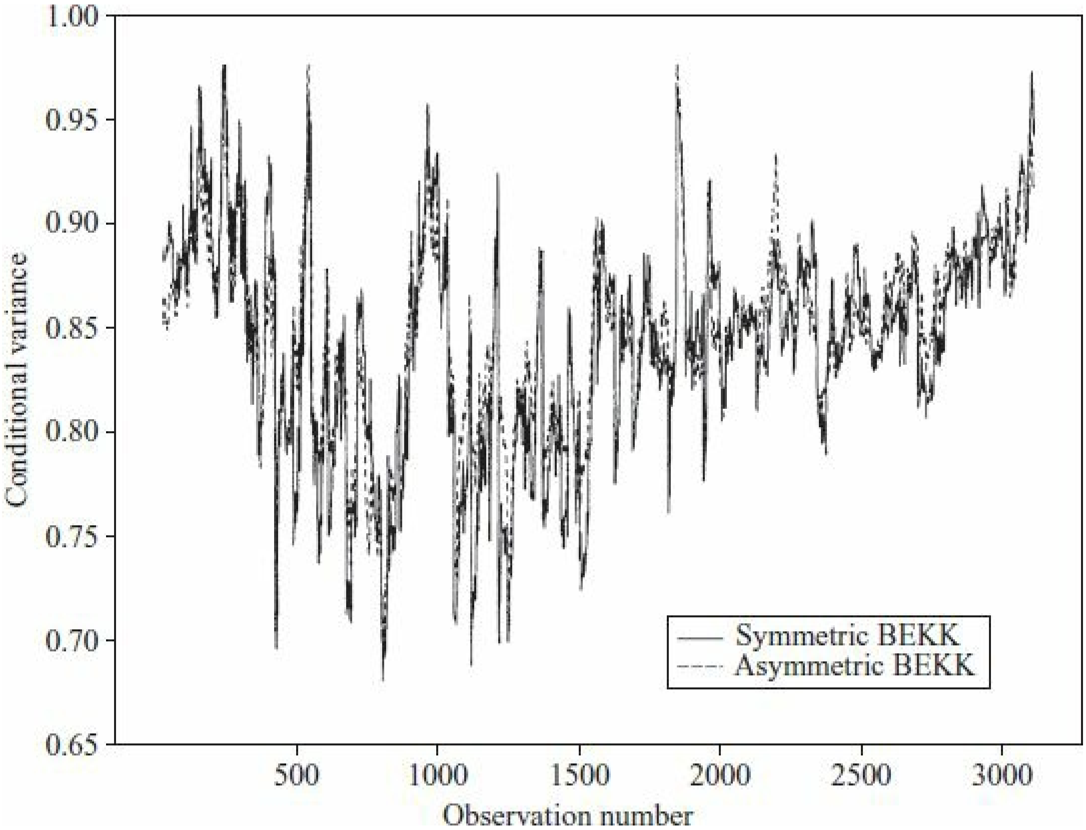


Figure 9.5 Time-varying hedge ratios derived from symmetric and asymmetric *BEKK* models for FTSE returns

Source. Brooks, Henry and Persand (2002).

Brooks, Henry and Persand also investigate the hedging performances of the various models using a modern risk management approach. They find, once again, that the time-varying hedge results in a considerable improvement, but that allowing for asymmetries results in only a very modest incremental reduction in hedged portfolio risk.

9.28 Multivariate Stochastic Volatility Models

As in the univariate case, while the term ‘stochastic volatility’ is commonly used to describe models from the multivariate GARCH family, strictly they do not fit well under this umbrella because the conditional variance and covariance equations are deterministic given the information set up to the previous period. That is, there is no additional source of noise in the conditional variance (or covariance) equation of a multivariate GARCH model.

The multivariate stochastic volatility (MSV) model was initially proposed by Harvey, Ruiz and Shephard (1994) and the notation here will closely follow theirs. Let y_t be the elements of an $N \times 1$ vector of observations at time t on a series i, with time-varying variance σ_i^2, defined as

$$y_{it} = \epsilon_{it}(\exp\{h_{it}\})^{1/2}, \qquad i = 1, \ldots, N; t = 1, \ldots, T \qquad \textbf{(9.118)}$$

where $\epsilon = (\epsilon_{1t}, \ldots, \epsilon_{Nt})$ is a vector of disturbances with zero mean and covariance matrix Σ_ϵ and where

$$h_{it} = \ln(\sigma_{it}^2) \qquad \textbf{(9.119)}$$

This covariance matrix, Σ_ϵ is defined to have unity on the leading diagonal (and it is therefore also a correlation matrix), while its off-diagonal elements are denoted ρ_{ij}.

Under the stochastic volatility model, the h_{it} can be specified to evolve as an autoregressive (AR) process of order P

$$h_{it} = \gamma_i + \sum_{p=1}^{P} \psi_{ip} h_{i,t-p} + \eta_{it} \qquad i = 1, \ldots, N \qquad \textbf{(9.120)}$$

$\eta_t = (\eta_{1t}, \ldots, \eta_{Nt})$ is a vector of disturbances to the conditional variance having zero mean and covariance matrix Σ_η. It is usually further assumed that ϵ_{it} and η_{it} are mutually independent and that each is multivariate normally distributed. Usually, $P = 1$ is deemed sufficient so that the variance dynamics for each series in the system are AR(1). Moving average terms or even exogenous variables could be added to the variance specification but rarely are in practice.

It is worth noting that in this model, the correlations ρ_{ij} between the mean equation disturbances are required to be fixed over time. Thus the covariances across the N series evolve as functions of the variances rather than independently of them. This formulation parallels the constant conditional correlation multivariate GARCH model of Bollerslev (1990) discussed above, and represents an important limitation of the model. It does, however, imply that MSV models are highly parsimonious, and the number of parameters scales directly with the number of variables in the system. For example, in the context of a bivariate MSV model, there are eight parameters to estimate.[2]

Harvey, Ruiz and Shephard (1994) propose estimating the model using quasimaximum likelihood (QML) via the Kalman filter. However, Danielsson (1998) argues that their QML approach results in inefficient estimation. An alternative approach to estimating MSV models is to make use of Bayesian Markov Chain Monte Carlo (MCMC) methods, as proposed by Jacquier, Polson and Rossi (1995).[3]

KEY CONCEPTS

The key terms to be able to define and explain from this chapter are

- non-linearity
- conditional variance
- maximum likelihood
- lagrange multiplier test
- asymmetry in volatility
- constant conditional correlation
- diagonal VECH
- news impact curve
- volatility clustering
- GARCH model
- Wald test
- likelihood ratio test
- GJR specification
- exponentially weighted moving average
- BEKK model
- GARCH-in-mean

Appendix 9.1 Parameter Estimation Using Maximum Likelihood

For simplicity, this appendix will consider by way of illustration the bivariate regression case with homoscedastic errors (i.e., assuming that there is no ARCH and that the variance of the errors is constant over time). Suppose that the linear regression model of interest is of the form

$$y_t = \beta_1 + \beta_2 x_t + u_t \quad \textbf{(9A.1)}$$

Assuming that $u_t \sim N(0, \sigma^2)$, then $y_t \sim N(\beta_1 + \beta_2 x_t, \sigma^2)$ so that the probability density function for a normally distributed random variable with this mean and variance is given by

$$f(y_t \mid \beta_1 + \beta_2 x_t, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\frac{(y_t - \beta_1 - \beta_2 x_t)^2}{\sigma^2}\right\} \tag{9A.2}$$

The probability density is a function of the data given the parameters. Successive values of y_t would trace out the familiar bell-shaped curve of the normal distribution. Since the ys are iid, the joint probability density function (pdf) for all the ys can be expressed as a product of the individual density functions

$$\begin{aligned} &f(y_1, y_2, \ldots, y_T \mid \beta_1 + \beta_2 x_1, \beta_1 + \beta_2 x_2, \ldots, \beta_1 + \beta_2 x_T, \sigma^2) \\ &= f(y_1 \mid \beta_1 + \beta_2 x_2, \sigma^2) f(y_2 \mid \beta_1 + \beta_2 x_2, \sigma^2) \ldots f(y_T \mid \beta_1 + \beta_2 x_T, \sigma^2) \\ &= \prod_{t=1}^{T} f(y_t \mid \beta_1 + \beta_2 x_t, \sigma^2) \quad \text{for } t = 1, \ldots, T \end{aligned} \tag{9A.3}$$

The term on the LHS of this expression is known as the *joint density* and the terms on the RHS are known as the *marginal densities*. This result follows from the independence of the y values, in the same way as under elementary probability, for three independent events A, B and C, the probability of A, B and C all happening is the probability of A multiplied by the probability of B multiplied by the probability of C. Equation (9A.3) shows the probability of obtaining all of the values of y that did occur. Substituting into equation (9A.3) for every y_t from (9A.2), and using the result that $Ae^{x_1} \times Ae^{x_2} \times \cdots Ae^{x_T} = A^T(e^{x_1} \times e^{x_2} \times \cdots \times e^{x_T}) = A^T e^{(x_1 + x_2 + \cdots + x_T)}$, the following expression is obtained

$$\begin{aligned} &f(y_1, y_2, \ldots, y_T \mid \beta_1 + \beta_2 x_t, \sigma^2) \\ &= \frac{1}{\sigma^T(\sqrt{2\pi})^T} \exp\left\{-\frac{1}{2}\sum_{t=1}^{T}\frac{(y_t - \beta_1 - \beta_2 x_t)^2}{\sigma^2}\right\} \end{aligned} \tag{9A.4}$$

This is the joint density of all of the ys given the values of x_t, β_1, β_2 and σ^2. However, the typical situation that occurs in practice is the reverse of the above situation – that is, the x_t and y_t are given and β_1, β_2, σ^2 are to be estimated. If this is the case, then $f(\bullet)$ is known as a likelihood function, denoted $LF(\beta_1, \beta_2, \sigma^2)$, which would be written

$$LF(\beta_1, \beta_2, \sigma^2) = \frac{1}{\sigma^T(\sqrt{2\pi})^T} \exp\left\{-\frac{1}{2}\sum_{t=1}^{T}\frac{(y_t - \beta_1 - \beta_2 x_t)^2}{\sigma^2}\right\} \qquad \textbf{(9A.5)}$$

Maximum likelihood estimation involves choosing parameter values (β_1, β_2, σ^2) that maximise this function. Doing this ensures that the values of the parameters are chosen that maximise the likelihood that we would have actually observed the *y*s that we did. It is necessary to differentiate (9A.5) w.r.t. β_1, β_2, σ^2, but equation (9A.5) is a product containing *T* terms, and so would be difficult to differentiate.

Fortunately, since $\max_x f(x) = \max_x \ln(f(x))$, logs of equation (9A.3) can be taken, and the resulting expression differentiated, knowing that the same optimal values for the parameters will be chosen in both cases. Then, using the various laws for transforming functions containing logarithms, the log-likelihood function, *LLF* is obtained

$$LLF = -T\ln\sigma - \frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\frac{(y_t - \beta_1 - \beta_2 x_t)^2}{\sigma^2} \qquad \textbf{(9A.6)}$$

which is equivalent to

$$LLF = -\frac{T}{2}\ln\sigma^2 - \frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\frac{(y_t - \beta_1 - \beta_2 x_t)^2}{\sigma^2} \qquad \textbf{(9A.7)}$$

Only the first part of the RHS of equation (9A.6) has been changed in equation (9A.7) to make σ^2 appear in that part of the expression rather than σ.

Remembering the result that

$$\frac{\partial}{\partial x}(\ln(x)) = \frac{1}{x}$$

and differentiating equation (9A.7) w.r.t. β_1, β_2, σ^2, the following expressions for the first derivatives are obtained

$$\frac{\partial LLF}{\partial \beta_1} = -\frac{1}{2}\sum\frac{(y_t - \beta_1 - \beta_2 x_t).2. - 1}{\sigma^2} \qquad \textbf{(9A.8)}$$

$$\frac{\partial LLF}{\partial \beta_2} = -\frac{1}{2}\sum\frac{(y_t - \beta_1 - \beta_2 x_t).2. - x_t}{\sigma^2} \qquad \textbf{(9A.9)}$$

$$\frac{\partial LLF}{\partial \sigma^2} = -\frac{T}{2}\frac{1}{\sigma^2} + \frac{1}{2}\sum\frac{(y_t - \beta_1 - \beta_2 x_t)^2}{\sigma^4} \qquad \textbf{(9A.10)}$$

Setting equations (9A.8)–(9A.10) to zero to minimise the functions, and placing hats above the parameters to denote the maximum likelihood estimators, from equation (9A.8)

$$\sum (y_t - \hat{\beta}_1 - \hat{\beta}_2 x_t) = 0 \quad \textbf{(9A.11)}$$
$$\sum y_t - \sum \hat{\beta}_1 - \sum \hat{\beta}_2 x_t = 0 \quad \textbf{(9A.12)}$$
$$\sum y_t - T\hat{\beta}_1 - \hat{\beta}_2 \sum x_t = 0 \quad \textbf{(9A.13)}$$
$$\frac{1}{T}\sum y_t - \hat{\beta}_1 - \hat{\beta}_2 \frac{1}{T}\sum x_t = 0 \quad \textbf{(9A.14)}$$

Recall that

$$\frac{1}{T}\sum y_t = \bar{y}_t$$

the mean of *y* and similarly for *x*, an estimator for $\hat{\beta}_1$ can finally be derived

$$\hat{\beta}_1 = \bar{y} - \hat{\beta}_2 \bar{x} \quad \textbf{(9A.15)}$$

From equation (9A.9)

$$\sum (y_t - \hat{\beta}_1 - \hat{\beta}_2 x_t) x_t = 0 \quad \textbf{(9A.16)}$$
$$\sum y_t x_t - \sum \hat{\beta}_1 x_t - \sum \hat{\beta}_2 x_t^2 = 0 \quad \textbf{(9A.17)}$$
$$\sum y_t x_t - \hat{\beta}_1 \sum x_t - \hat{\beta}_2 \sum x_t^2 = 0 \quad \textbf{(9A.18)}$$
$$\hat{\beta}_2 \sum x_t^2 = \sum y_t x_t - (\bar{y} - \hat{\beta}_2 \bar{x}) \sum x_t \quad \textbf{(9A.19)}$$
$$\hat{\beta}_2 \sum x_t^2 = \sum y_t x_t - T\overline{xy} + \hat{\beta}_2 T\bar{x}^2 \quad \textbf{(9A.20)}$$
$$\hat{\beta}_2 (\sum x_t^2 - T\bar{x}^2) = \sum y_t x_t - T\overline{xy} \quad \textbf{(9A.21)}$$
$$\hat{\beta}_2 = \frac{\sum y_t x_t - T\overline{xy}}{(\sum x_t^2 - T\bar{x}^2)} \quad \textbf{(9A.22)}$$

From equation (9A.10)

$$\frac{T}{\hat{\sigma}^2} = \frac{1}{\hat{\sigma}^4} \sum (y_t - \hat{\beta}_1 - \hat{\beta}_2 x_t)^2 \quad \textbf{(9A.23)}$$

Rearranging,

$$\hat{\sigma}^2 = \frac{1}{T} \sum (y_t - \hat{\beta}_1 - \hat{\beta}_2 x_t)^2 \quad \textbf{(9A.24)}$$

But the term in parentheses on the RHS of equation (9A.24) is the residual for time *t* (i.e., the actual minus the fitted value), so

$$\hat{\sigma}^2 = \frac{1}{T}\sum \hat{u}_t^2 \qquad \textbf{(9A.25)}$$

How do these formulae compare with the OLS estimators? Equations (9A.15) and (9A.22) are identical to those of OLS. So maximum likelihood and OLS will deliver identical estimates of the intercept and slope coefficients. However, the estimate of $\hat{\sigma}^2$ in equation (9A.25) is different. The OLS estimator was

$$\hat{\sigma}^2 = \frac{1}{T-k}\sum \hat{u}_t^2 \qquad \textbf{(9A.26)}$$

and it was also shown that the OLS estimator is unbiased. Therefore, the ML estimator of the error variance must be biased, although it is consistent, since as $T \rightarrow \infty$, $T - k \approx T$.

Note that the derivation above could also have been conducted using matrix rather than sigma algebra. The resulting estimators for the intercept and slope coefficients would still be identical to those of OLS, while the estimate of the error variance would again be biased. It is also worth noting that the ML estimator is consistent and asymptotically efficient. Derivation of the ML estimator for the GARCH *LLF* is algebraically difficult and therefore beyond the scope of this book.

SELF-STUDY QUESTIONS

1. (a) What stylised features of financial data cannot be explained using linear time series models?

(b) Which of these features could be modelled using a GARCH(1,1) process?

(c) Why, in recent empirical research, have researchers preferred GARCH(1,1) models to pure ARCH(p)?

(d) Describe two extensions to the original GARCH model. What additional characteristics of financial data might they be able to capture?

(e) Consider the following GARCH(1,1) model

$$y_t = \mu + u_t, \qquad u_t \sim \mathrm{N}(0, \sigma_t^2)$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta\sigma_{t-1}^2$$

If y_t is a daily stock return series, what range of values are likely for the coefficients μ, α_0, α_1 and β?

(f) Suppose that a researcher wanted to test the null hypothesis that $\alpha_1 + \beta = 1$ in the equation for part (e). Explain how this might be achieved within the maximum likelihood framework.

(g) Suppose now that the researcher had estimated the above GARCH model for a series of returns on a stock index and obtained the following parameter estimates: $\hat{\mu} = 0.0023$, $\hat{\alpha}_0 = 0.0172$, $\hat{\beta} = 0.9811$, $\hat{\alpha}_1 = 0.1251$. If the researcher has data available up to and including time T, write down a set of equations in σ_t^2 and u_t^2 their lagged values, which could be employed to produce one-, two-, and three-step-ahead forecasts for the conditional variance of y_t.

(h) Suppose now that the coefficient estimate of $\hat{\beta}$ for this model is 0.98 instead. By reconsidering the forecast expressions you derived in part (g), explain what would happen to the forecasts in this case.

2. (a) Discuss briefly the principles behind maximum likelihood.

(b) Describe briefly the three hypothesis testing procedures that are available under maximum likelihood estimation. Which is likely to be the easiest to calculate in practice, and why?

(c) OLS and maximum likelihood are used to estimate the parameters of a standard linear regression model. Will they give the same estimates? Explain your answer.

3. (a) Distinguish between the terms ‘conditional variance’ and ‘unconditional variance’. Which of the two is more likely to be relevant for producing:

i. one-step-ahead volatility forecasts

ii. twenty-step-ahead volatility forecasts.

(b) If u_t follows a GARCH(1,1) process, what would be the likely result if a regression of the form in Question 1(e) were estimated using OLS and assuming a constant conditional variance?

(c) Compare and contrast the following models for volatility, noting their strengths and weaknesses:

i. Historical volatility

ii. EWMA

iii. GARCH(1,1)

iv. Implied volatility.

4. Suppose that a researcher is interested in modelling the correlation between the returns of the NYSE and LSE markets.

(a) Write down a simple diagonal *VECH* model for this problem. Discuss the values for the coefficient estimates that you would expect.

(b) Suppose that weekly correlation forecasts for two weeks ahead are required. Describe a procedure for constructing such forecasts from a set of daily returns data for the two market indices.

(c) What other approaches to correlation modelling are available?

(d) What are the strengths and weaknesses of multivariate GARCH models relative to the alternatives that you propose in part (c)?

5. (a) What is a news impact curve? Using a spreadsheet or otherwise, construct the news impact curve for the following estimated EGARCH and GARCH models, setting the lagged conditional variance to the value of the unconditional variance (estimated from the sample data rather than the mode parameter estimates), which is 0.096

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \alpha_2 \sigma_{t-1}^2$$

$$\ln(\sigma_t^2) = \alpha_0 + \alpha_1 \frac{u_{t-1}}{\sqrt{\sigma_{t-1}^2}} + \alpha_2 \ln(\sigma_{t-1}^2) + \alpha_3 \left[\frac{|u_{t-1}|}{\sqrt{\sigma_{t-1}^2}} - \sqrt{\frac{2}{\pi}} \right]$$

	GARCH	EGARCH
μ	−0.0130	−0.0278
	(0.0669)	(0.0855)
α_0	0.0019	0.0823
	(0.0017)	(0.5728)
α_1	0.1022**	−0.0214
	(0.0333)	(0.0332)

α_2	0.9050**	0.9639**
	(0.0175)	(0.0136)
α_3	–	0.2326**
		(0.0795)

(b) In fact, the models in part (a) were estimated using daily foreign exchange returns. How can financial theory explain the patterns observed in the news impact curves?

1 The *BEKK* acronym arises from the fact that early versions of the paper also listed Baba and Krafts as co-authors.

2 This compares with nine for a diagonal *VECH* MGARCH model and 21 for the unrestricted MGARCH.

3 See Chib and Greenberg (1996) for an extensive but very technical discussion of the intricacies of the MCMC technique.

10

Switching and State Space Models

LEARNING OUTCOMES

In this chapter, you will learn how to

- Use intercept and slope dummy variables to allow for seasonal behaviour in time-series
- Motivate the use of regime switching models in financial econometrics
- Specify and explain the logic behind Markov switching models
- Compare and contrast Markov switching and threshold autoregressive models
- Describe the intuition behind the estimation of regime switching models
- Set up and interpret simple state space models
- Explain how the Kalman filter is used to estimate state space models

10.1 Motivations

Many financial and economic time series seem to undergo episodes in which the behaviour of the series changes quite dramatically compared to that exhibited previously. The behaviour of a series could change over time in terms of its mean value, its volatility, or to what extent its current value is related to its previous value. The behaviour may change once and for all, usually known as a 'structural break' in a series. Or it may change for a period of time before reverting back to its original behaviour or switching to yet another style of behaviour, and the latter is typically termed a 'regime shift' or 'regime switch'. Finally, the relationship between series or their average values may change continuously but in an at least partially predictable fashion. This chapter presents several models that can capture such time-varying behaviour.

10.1.1 What Might Cause One-Off Fundamental Changes in the Properties of a Series?

Usually, very substantial changes in the properties of a series are attributed to large-scale events, such as wars, financial panics – e.g., a 'run on a bank', significant changes in government policy, such as the introduction of an inflation target, or the removal of exchange controls, or changes in market microstructure – e.g., the 'Big Bang', when trading on the London Stock Exchange (LSE) became electronic, or a change in the market trading mechanism, such as the partial move of the LSE from a quote-driven to an order-driven system in 1997.

However, it is also true that regime shifts can occur on a regular basis and at much higher frequency. Such changes may occur as a result of more subtle factors, but still leading to statistically important modifications in behaviour. An example would be the intraday patterns observed in equity market bid–ask spreads (see Chapter 7). These appear to start with high values at the open, gradually narrowing throughout the day, before widening again at the close.

To give an illustration of the kind of shifts that may be seen to occur, Figure 10.1 gives an extreme example. As can be seen from the figure, the behaviour of the series changes markedly at around observation 500. Not only does the series become much more volatile than previously, its mean value is also substantially increased. Although this is a severe case that was generated using simulated data, clearly, in the face of such 'regime changes' a linear model estimated over the whole sample covering the change would not be appropriate. One possible approach to this problem would be simply to split the data around the time of the change and to estimate separate models on each portion. It would be possible to allow a series, y_t to be drawn from two or more different generating processes at different times. For example, if it was thought an AR(1) process was appropriate to capture the relevant features of a particular series whose behaviour changed at observation 500, say, two models could be estimated:

$$y_t = \mu_1 + \phi_1 y_{t-1} + u_{1t} \quad \text{before observation 500} \tag{10.1}$$

$$y_t = \mu_2 + \phi_2 y_{t-1} + u_{2t} \quad \text{after observation 500} \tag{10.2}$$

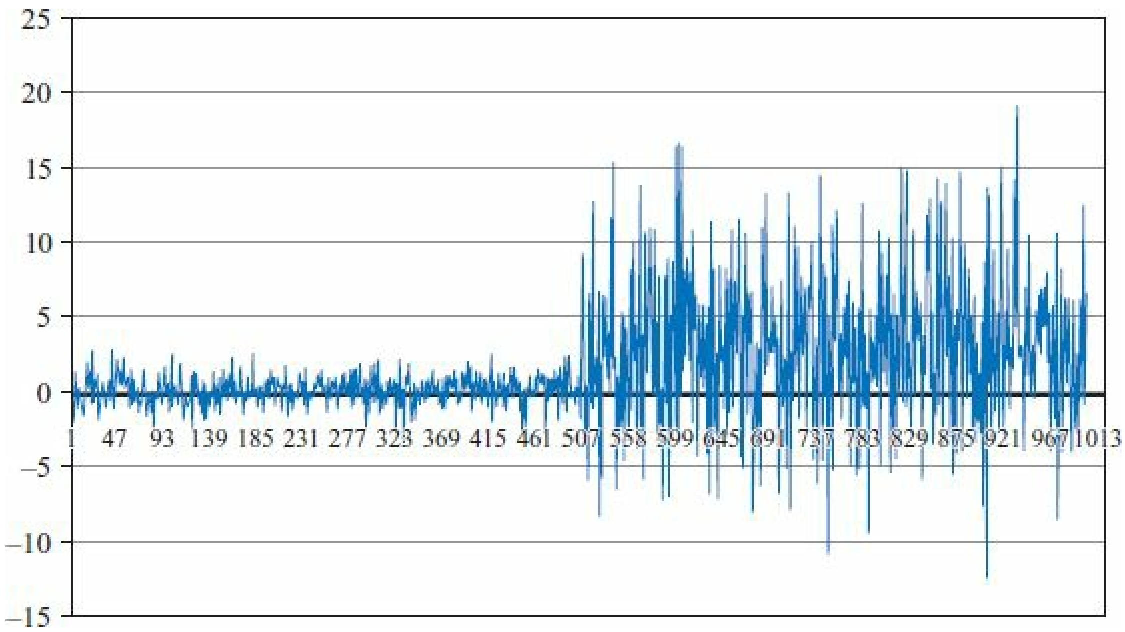


Figure 10.1 Sample time-series plot illustrating a regime shift

In the context of Figure 10.1, this would involve focusing on the mean shift only. These equations represent a very simple example of what is known as a piecewise linear model – that is, although the model is globally (i.e., when it is taken as a whole) non-linear, each of the component parts is a linear model.

This method may be valid, but it is also likely to be wasteful of information. For example, even if there were enough observations in each sub-sample to estimate separate (linear) models, there would be an efficiency loss in having fewer observations in each of two samples than if all the observations were collected together. Also, it may be the case that only one property of the series has changed – for example, the (unconditional) mean value of the series may have changed, leaving its other properties unaffected. In this case, it would be sensible to try to keep all of the observations together, but to allow for the particular form of the structural change in the model-building process. Thus, what is required is a set of models that allow all of the observations on a series to be used for estimating a model, but also that the model is sufficiently flexible to allow different types of behaviour at different points in time. Two classes of regime switching models that potentially allow this to occur are *Markov switching models* and *threshold autoregressive models*.

A first and central question to ask is: How can it be determined where the switch(es) occurs? The method employed for making this choice will depend upon the model used. A simple type of switching model is one where the switches are made deterministically using dummy variables. One important use of this in finance is to allow for 'seasonality' in financial data. In economics and finance generally, many series are believed to exhibit seasonal behaviour, which results in a certain element of partly predictable cycling of the series over time. For example, if monthly or quarterly data on consumer spending are examined, it is likely that the value of the series will rise rapidly in late November owing to Christmas-related expenditure, followed by a fall in mid-January, when consumers realise that they have spent too much before Christmas and in the January sales! Consumer spending in the UK also typically drops during the August vacation period when all of the sensible people have left the country. Such phenomena will be apparent in many series and will be present to some degree at the same time every year, whatever else is happening in terms of the long-term trend and short-term variability of the series.

10.2 Seasonalities in Financial Markets: Introduction and Literature Review

In the context of financial markets, and especially in the case of equities, a number of other 'seasonal effects' have been noted. Such effects are usually known as 'calendar anomalies' or 'calendar effects'. Examples include open- and close-of-market effects, 'the January effect', weekend effects and bank holiday effects. Investigation into the existence or otherwise of 'calendar effects' in financial markets has been the subject of a considerable amount of recent academic research. Calendar effects may be loosely defined as the tendency of financial asset returns to display systematic patterns at certain times of the day, week, month or year. One example of the most important such anomalies is the *day-of-the-week effect*, which results in average returns being significantly higher on some days of the week than others. Studies by French (1980), Gibbons and Hess (1981) and Keim and Stambaugh (1984), for example, have found that the average market close-to-close return in the US is significantly negative on Monday and significantly positive on Friday. By contrast, Jaffe and Westerfield (1985) found that the lowest mean returns for the Japanese and Australian stock markets occur on Tuesdays.

At first glance, these results seem to contradict the efficient markets hypothesis, since the existence of calendar anomalies might be taken to imply that investors could develop trading strategies which make abnormal profits on the basis of such patterns. For example, holding all other factors constant, equity purchasers may wish to sell at the close on Friday and to buy at the close on Thursday in order to take advantage of these effects. However, evidence for the predictability of stock returns does not necessarily imply market inefficiency, for at least two reasons. First, it is likely that the small average excess returns documented by the above papers would not generate net gains when employed in a trading strategy once the costs of transacting in the markets has been taken into account. Therefore, under many 'modern' definitions of market efficiency (e.g., Jensen, 1978), these markets would not be classified as inefficient. Second, the apparent differences in returns on different days of the week may be attributable to time-varying stock market risk premiums.

If any of these calendar phenomena are present in the data but ignored by the model-building process, the result is likely to be a misspecified model. For example, ignored seasonality in y_t is likely to lead to residual autocorrelation of the order of the seasonality – e.g., fifth order residual autocorrelation if y_t is a series of daily returns.

10.3 Modelling Seasonality in Financial Data

As discussed above, seasonalities at various different frequencies in financial time-series data are so well documented that their existence cannot be doubted, even if there is argument about how they can be rationalised. One very simple method for coping with this and examining the degree to which seasonality is present is the inclusion of dummy variables in regression equations. The number of dummy variables that could sensibly be constructed to model the seasonality would depend on the frequency of the data. For example, four dummy variables would be created for quarterly data, twelve for monthly data, five for daily data and so on. In the case of quarterly data, the four dummy variables would be defined as follows:

$D1_t$ = 1 in quarter 1 and zero otherwise

$D2_t$ = 1 in quarter 2 and zero otherwise

$D3_t$ = 1 in quarter 3 and zero otherwise

$D4_t$ = 1 in quarter 4 and zero otherwise

How many dummy variables can be placed in a regression model? If an intercept term is used in the regression, the number of dummies that could also be included would be one less than the ‘seasonality’ of the data. To see why this is the case, consider what happens if all four dummies are used for the quarterly series. The following gives the values that the dummy variables would take for a period during the mid-1980s, together with the sum of the dummies at each point in time, presented in the last column

		$D1$	$D2$	$D3$	$D4$	Sum
1986	Q1	1	0	0	0	1
	Q2	0	1	0	0	1
	Q3	0	0	1	0	1
	Q4	0	0	0	1	1
1987	Q1	1	0	0	0	1
	Q2	0	1	0	0	1
	Q3	0	0	1	0	1

etc.,

The sum of the four dummies would be 1 in every time period. Unfortunately, this sum is of course identical to the variable that is implicitly attached to the intercept coefficient. Thus, if the four dummy variables and the intercept were both included in the same regression, the problem would be one of perfect multicollinearity so that $(X'X)^{-1}$ would not exist and none of the coefficients could be estimated. This problem is known as the *dummy variable trap*. The solution would be either to just use three dummy variables plus the intercept, or to use the four dummy variables with no intercept.

The seasonal features in the data would be captured using either of these, and the residuals in each case would be identical, although the interpretation of the coefficients would be changed. If four dummy variables were used (and assuming that there were no explanatory variables in the regression), the estimated coefficients could be interpreted as the average value of the dependent variable during each quarter. In the case where a constant and three dummy variables were used, the interpretation of the estimated coefficients on the dummy variables would be that they represented the average deviations of the dependent variables for the included quarters from their average values for the excluded quarter, as discussed in Box 10.1.

BOX 10.1 How do dummy variables work?

The dummy variables as described above operate by *changing the intercept*, so that the average value of the dependent variable, given all of the explanatory variables, is permitted to change across the seasons. This is shown in Figure 10.2.

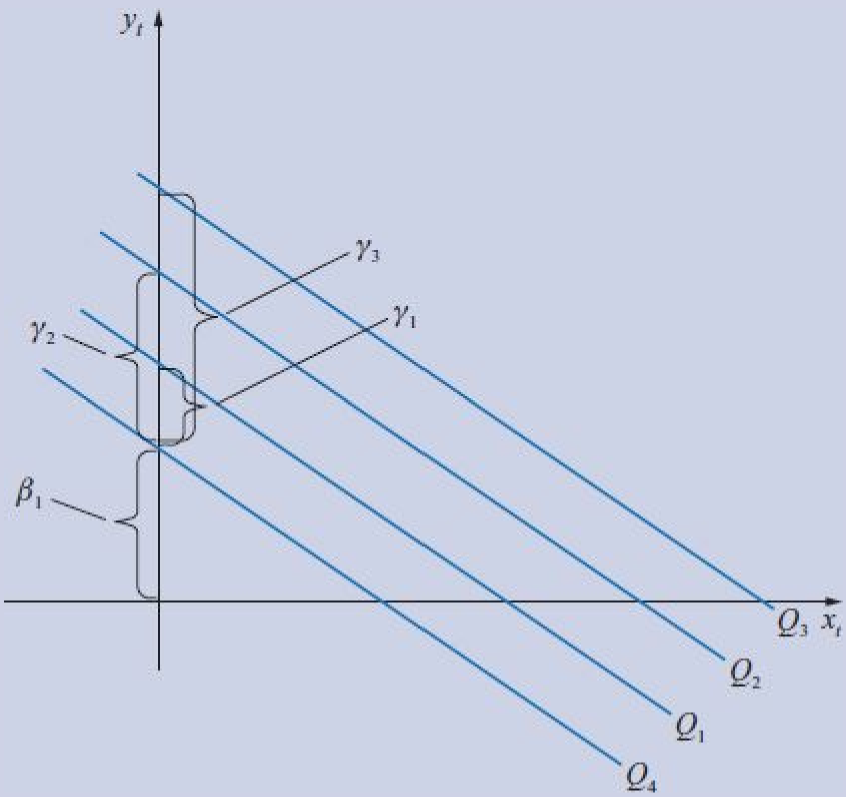


Figure 10.2 Use of intercept dummy variables for quarterly data

Consider the following regression

$$y_t = \beta_1 + \gamma_1 D1_t + \gamma_2 D2_t + \gamma_3 D3_t + \beta_2 x_{2t} + \cdots + u_t \tag{10.3}$$

During each period, the intercept will be changed. The intercept will be:

- $\hat{\beta}_1 + \hat{\gamma}_1$ in the first quarter, since $D1 = 1$ and $D2 = D3 = 0$ for all quarter 1 observations
- $\hat{\beta}_1 + \hat{\gamma}_2$ in the second quarter, since $D2 = 1$ and $D1 = D3 = 0$ for all quarter 2 observations.

- $\hat{\beta}_1 + \hat{\gamma}_3$ in the third quarter, since $D3 = 1$ and $D1 = D2 = 0$ for all quarter 3 observations
- $\hat{\beta}_1$ in the fourth quarter, since $D1 = D2 = D3 = 0$ for all quarter 4 observations.

EXAMPLE 10.1

Brooks and Persand (2001a) examine the evidence for a day-of-the-week effect in five Southeast Asian stock markets: South Korea, Malaysia, the Philippines, Taiwan and Thailand. The data, obtained from Primark Datastream, are collected on a daily close-to-close basis for all weekdays (Mondays to Fridays) falling in the period 31 December 1989 to 19 January 1996 (a total of 1,581 observations). The first regressions estimated, which constitute the simplest tests for day-of-the-week effects, are of the form

$$r_t = \gamma_1 D1_t + \gamma_2 D2_t + \gamma_3 D3_t + \gamma_4 D4_t + \gamma_5 D5_t + u_t \qquad \textbf{(10.4)}$$

where r_t is the return at time t for each country examined separately, $D1_t$ is a dummy variable for Monday, taking the value 1 for all Monday observations and zero otherwise, and so on. The coefficient estimates can be interpreted as the average sample return on each day of the week. The results from these regressions are shown in Table 10.1.

Table 10.1 Values and significances of days of the week coefficients

	Thailand	Malaysia	Taiwan	South Korea	Phil
Monday	0.49E-3	0.00322	0.00185	0.56E-3	0.0
	(0.6740)	(3.9804)**	(2.9304)**	(0.4321)	(1.4
Tuesday	−0.45E-3	−0.00179	−0.00175	0.00104	−0.9
	(−0.3692)	(−1.6834)	(−2.1258)**	(0.5955)	(−0.
Wednesday	−0.37E-3	−0.00160	0.31E-3	−0.00264	−0.4
	(−0.5005)	(−1.5912)	(0.4786)	(−2.107)**	(−0.
Thursday	0.40E-3	0.00100	0.00159	−0.00159	0.9

	(0.5468)	(1.0379)	(2.2886)**	(−1.2724)	(0.8
Friday	−0.31E-3	0.52E-3	0.40E-4	0.43E-3	0.0
	(−0.3998)	(0.5036)	(0.0536)	(0.3123)	(1.7

Notes: Coefficients are given in each cell followed by *t*-ratios in parentheses; * and** denote significance at the 5% and 1% levels, respectively.

Source: Brooks and Persand (2001a).

Briefly, the main features are as follows. Neither Thailand nor the Philippines have significant calendar effects; both Taiwan and Malaysia have significant positive Monday average returns and significant negative Tuesday returns; Taiwan has a significant Thursday effect.

Dummy variables could also be used to test for other calendar anomalies, such as the January effect, etc. as discussed above, and a given regression can include dummies of different frequencies at the same time. For example, a new dummy variable $D6_t$ could be added to equation (10.4) for 'April effects', associated with the start of the new tax year in the UK. Such a variable, even for a regression using daily data, would take the value 1 for all observations falling in April and zero otherwise.

If we choose to omit one of the dummy variables and to retain the intercept, then the omitted dummy variable becomes the reference category against which all the others are compared. For example consider a model such as the one above, but where the Monday dummy variable has been omitted

$$r_t = \alpha + \gamma_2 D2_t + \gamma_3 D3_t + \gamma_4 D4_t + \gamma_5 D5_t + u_t \quad \textbf{(10.5)}$$

The estimate of the intercept will be $\hat{\alpha}$ on Monday, $\hat{\alpha} + \hat{\gamma}_{21}$ on Tuesday and so on. $\hat{\gamma}_2$ will now be interpreted as the difference in average returns between Monday and Tuesday. Similarly, $\hat{\gamma}_3, \ldots, \hat{\gamma}_5$ can also be interpreted as the differences in average returns between Wednesday, …, Friday, and Monday.

This analysis should hopefully have made it clear that by thinking carefully about which dummy variable (or the intercept) to omit from the regression, we can control the interpretation to test naturally the hypothesis that is of most interest. The same logic can also be applied to slope dummy variables, which are described in the following section.

10.3.1 Slope Dummy Variables

As well as, or instead of, intercept dummies, slope dummy variables can also be used. These operate by changing the slope of the regression line, leaving the intercept unchanged. Figure 10.3 gives an illustration in the context of just one slope dummy (i.e., two different 'states'). Such a setup would apply if, for example, the data were bi-annual (twice yearly) or bi-weekly or observations made at the open and close of markets. Then D_t would be defined as $D_t = 1$ for the first half of the year and zero for the second half.

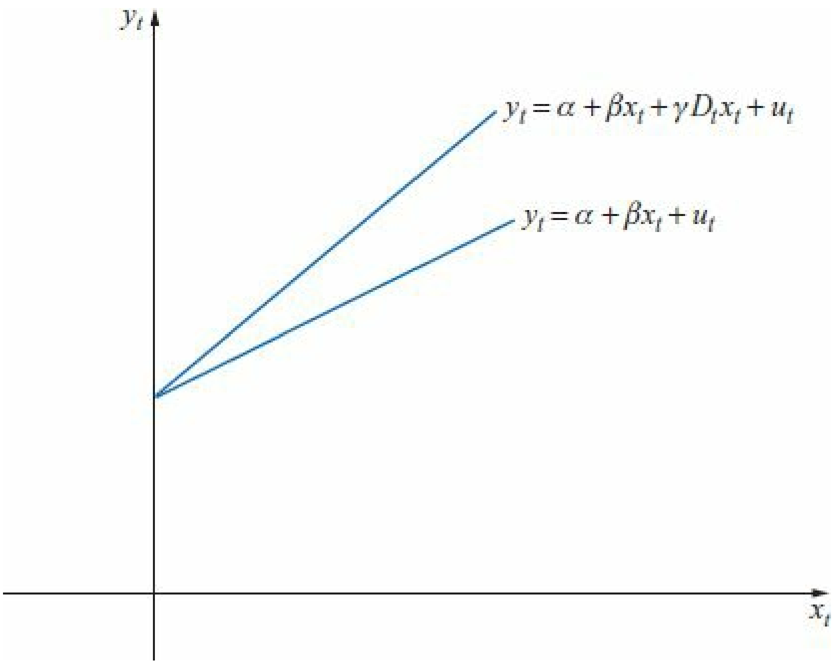


Figure 10.3 Use of slope dummy variables

A slope dummy changes the slope of the regression line, leaving the intercept unchanged. In the above case, the intercept is fixed at α, while the slope varies over time. For periods where the value of the dummy is zero, the slope will be β, while for periods where the dummy is one, the slope will be $\beta + \gamma$.

Of course, it is also possible to use more than one dummy variable for the slopes. For example, if the data were quarterly, the following setup could be used, with $D1_t \ldots D3_t$ representing quarters 1–3

$$y_t = \alpha + \beta x_t + \gamma_1 D1_t x_t + \gamma_2 D2_t x_t + \gamma_3 D3_t x_t + u_t \quad \textbf{(10.6)}$$

In this case, since there is also a term in x_t with no dummy attached, the interpretation of the coefficients on the dummies (γ_1, etc.) is that they represent the deviation of the slope for that quarter from the average slope over all quarters. On the other hand, if the four slope dummy variables were included (and not βx_t), the coefficients on the dummies would be interpreted as the average slope coefficients during each quarter. Again, it is important not to include four quarterly slope dummies and the βx_t in the regression together, otherwise perfect multicollinearity would result.

10.3.2 Interactive Dummy Variables

It is often of interest to examine how variables interact with one another. This is achieved in a regression model by including them multiplied together. Frequently, a dummy variable will be interacted either with another dummy variable or with a standard explanatory variable. In the following example, we show how a seasonal dummy variable can be interacted with the market risk factor to allow for time-varying risk.

To offer an illustration of how two dummy variables can be interacted together, consider a regression model where we are trying to explain the risk levels ($risk_i$) of the portfolios of professional fund managers according to the manager's age and their gender

$$risk_i = \beta_1 + \beta_2 DG_i + \beta_3 DA_i + \beta_4 DA_i DG_i + u_i \quad \textbf{(10.7)}$$

where DG_i is a gender dummy variable, taking the value 1 if the fund manager is female and 0 otherwise; DA_i is a dummy variable for the age of the fund manager taking the value 0 for less than 40 years old and 1 otherwise. We would, of course, usually add some additional explanatory variables to the model but none are included here for simplicity.

EXAMPLE 10.2

Returning to the example of day-of-the-week effects in Southeast Asian stock markets, although significant coefficients in equation (10.4) will support the hypothesis of seasonality in returns, it is important to note that risk factors have not been taken into account. Before drawing conclusions on the potential presence of arbitrage opportunities or inefficient markets, it is important to allow for the possibility that the market can be more or less risky on certain days than others. Hence, low (high) significant returns in equation (10.4) might be explained by low (high) risk. Brooks and Persand thus test for seasonality using the empirical market model, whereby market risk is proxied by the return on the FTA World Price Index. Hence, in order to look at how risk varies across the days of the week, interactive (i.e., slope) dummy variables are used to determine whether risk increases (decreases) on the day of high (low) returns. The equation, estimated separately using time-series data for each country can be written

$$r_t = \left(\sum_{i=1}^{5} \alpha_i D_{it} + \beta_i D_{it} RWM_t \right) + u_t \quad \textbf{(10.8)}$$

where α_i and β_i are coefficients to be estimated, D_{it} is the ith dummy variable taking the value 1 for day $t = i$ and zero otherwise, and RWM_t is the return on the world market index. In this way, when considering the effect of market risk on seasonality, both risk and return are permitted to vary across the days of the week. The results from estimation of equation (10.8) are given in Table 10.2. Note that South Korea and the Philippines are excluded from this part of the analysis, since no significant calendar anomalies were found to explain in Table 10.1.

Table 10.2 Day-of-the-week effects with the inclusion of interactive dummy variables with the risk proxy

	Thailand	Malaysia	Taiwan
Monday	0.00322 (3.3571)**	0.00185 (2.8025)**	0.544E-3 (0.3945)
Tuesday	−0.00114 (−1.1545)	−0.00122 (−1.8172)	0.00140 (1.0163)
Wednesday	−0.00164 (−1.6926)	0.25E-3 (0.3711)	−0.00263 (−1.9188)
Thursday	0.00104 (1.0913)	0.00157 (2.3515)*	−0.00166 (−1.2116)

Friday	0.31E-4 (0.03214)	−0.3752 (−0.5680)	−0.13E-3 (−0.0976)
Beta-Monday	0.3573 (2.1987)*	0.5494 (4.9284)**	0.6330 (2.7464)**
Beta-Tuesday	1.0254 (8.0035)**	0.9822 (11.2708)**	0.6572 (3.7078)**
Beta-Wednesday	0.6040 (3.7147)**	0.5753 (5.1870)**	0.3444 (1.4856)
Beta-Thursday	0.6662 (3.9313)**	0.8163 (6.9846)**	0.6055 (2.5146)*
Beta-Friday	0.9124 (5.8301)**	0.8059 (7.4493)**	1.0906 (4.9294)**

Notes: Coefficients are given in each cell followed by *t*-ratios in parentheses; * and ** denote significance at the 5% and 1%, levels respectively.

Source: Brooks and Persand (2001a).

As can be seen, significant Monday effects in the Bangkok and Kuala Lumpur stock exchanges, and a significant Thursday effect in the latter, remain even after the inclusion of the slope dummy variables which allow risk to vary across the week. The *t*-ratios do fall slightly in absolute value, however, indicating that the day-of-the-week effects become slightly less pronounced. The significant negative average return for the Taiwanese stock exchange, however, completely disappears. It is also clear that average risk levels vary across the days of the week. For example, the betas for the Bangkok stock exchange vary from a low of 0.36 on Monday to a high of over unity on Tuesday. This illustrates that not only is there a significant positive Monday effect in this market, but also that the responsiveness of Bangkok market movements to changes in the value of the general world stock market is considerably lower on this day than on other days of the week.

It is evident in equation (10.7) that the age and gender dummy variables are included both as individual terms and also interacted with each other. This allows increasing age to have a different affect on the amount of risk taken by a man compared with a woman. Some possible outcomes and their interpretations are

- $\hat{\beta}_2$ is significant but $\hat{\beta}_3$ and $\hat{\beta}_4$ are insignificant. This would suggest that there is a statistical difference between the average risk levels of men and women that does not vary with age but there is no difference between the risk levels of older and younger fund managers.
- $\hat{\beta}_2$ and $\hat{\beta}_3$ are significant but $\hat{\beta}_4$ is insignificant. This would suggest that there is a statistical difference between the risk levels of male and female fund managers, and between those who are younger and older, but the differences in the amount of risk taken by male and female managers does not vary with age (or put differently but equivalently, the differences in the amount of risk taken by younger versus older fund managers does not vary by gender).
- The other possible outcomes would be interpreted similarly.

If we assume that all three parameter estimates are non-zero, we would calculate the average levels of risk taken by each group of fund managers as follows

- $\hat{\beta}_1$ (D_{Gi} = 0 and D_{Ai} = 0, so picking out younger men)
- $\hat{\beta}_1 + \hat{\beta}_2$ (D_{Gi} = 1 and D_{Ai} = 0, so picking out younger women)
- $\hat{\beta}_1 + \hat{\beta}_3$ (D_{Gi} = 0 and D_{Ai} = 1, so picking out older men)
- $\hat{\beta}_1 + \hat{\beta}_2 + \hat{\beta}_3 + \hat{\beta}_4$ (D_{Gi} = 1 and D_{Ai} = 1, so picking out older women)

10.4 Estimating Simple Piecewise Linear Functions

The piecewise linear model is one example of a general set of models known as *spline techniques*. Spline techniques involve the application of polynomial functions in a piecewise fashion to different portions of the data. These models are widely used to fit yield curves to available data on the yields of bonds of different maturities (see, for example, Shea, 1984).

A simple piecewise linear model could operate as follows. If the relationship between two series, y and x, differs depending on whether x is smaller or larger than some threshold value x^*, this phenomenon can be captured using dummy variables. A dummy variable, D_t, could be defined, taking values

$$D_t = \begin{cases} 0 & if\ x_t < x^* \\ 1 & if\ x_t \geq x^* \end{cases} \quad \textbf{(10.9)}$$

To offer an illustration of where this may be useful, it is sometimes the case that the tick size limits vary according to the price of the asset. For example, according to George and Longstaff (1993, see also Chapter 6 of this book), the Chicago Board of Options Exchange (CBOE) limits the tick size to be \$(1/8) for options worth \$3 or more, and \$(1/16) for options worth less than \$3. This means that the minimum permissible price movements are \$(1/8) and (\$1/16) for options worth \$3 or more and less than \$3, respectively. Thus, if y is the bid–ask spread for the option, and x is the option price, used as a variable to partly explain the size of the spread, the spread will vary with the option price partly in a piecewise manner owing to the tick size limit. The model could thus be specified as

$$y_t = \beta_1 + \beta_2 x_t + \beta_3 D_t + \beta_4 D_t x_t + u_t \quad \textbf{(10.10)}$$

with D_t defined as above. Viewed in the light of the above discussion on seasonal dummy variables, the dummy in equation (10.10) is used as both an intercept and a slope dummy. An example showing the data and regression line is given by Figure 10.4.

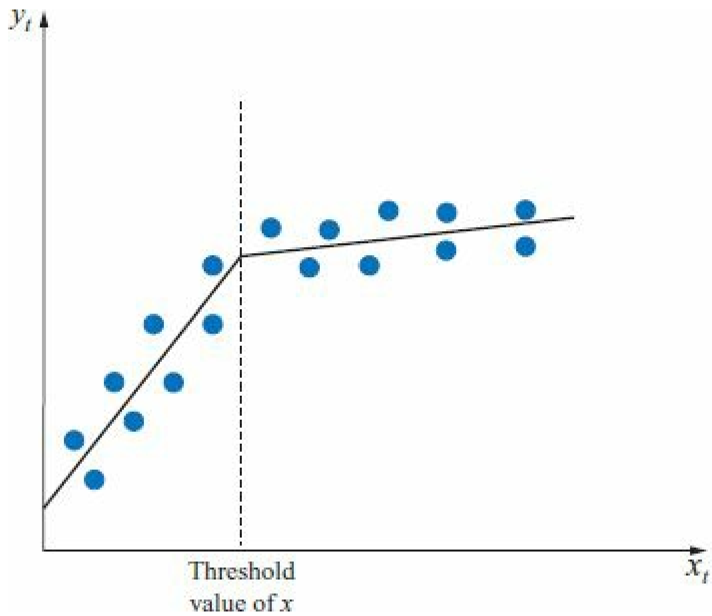


Figure 10.4 Piecewise linear model with threshold x^*

Note that the value of the threshold or 'knot' is assumed known at this stage.[1] Throughout, it is also possible that this situation could be generalised to the case where y_t is drawn from more than two regimes or is generated by a more complex model.

10.5 Markov Switching Models

Although a large number of more complex, non-linear threshold models have been proposed in the econometrics literature, only two kinds of model have had any noticeable impact in finance (aside from threshold GARCH models of the type alluded to in Chapter 8). These are the Markov regime switching model associated with Hamilton (1989, 1990), and the threshold autoregressive model associated with Tong (1983, 1990). Each of these formulations will be discussed below.

10.5.1 Fundamentals of Markov Switching Models

Under the Markov switching approach, the universe of possible occurrences is split into m states of the world, denoted s_i, $i = 1, \ldots, m$, corresponding to m regimes. In other words, it is assumed that y_t switches regime according to some unobserved variable, s_t, that takes on integer values. In the remainder of this chapter, it will be assumed that $m = 1$ or 2. So if $s_t = 1$, the process is in regime 1 at time t, and if $s_t = 2$, the process is in regime 2 at time t. Movements of the state variable between regimes are governed by a Markov process. This Markov property can be expressed as

$$P[a < y_t \leq b \mid y_1, y_2, \ldots, y_{t-1}] = P[a < y_t \leq b \mid y_{t-1}] \quad \textbf{(10.11)}$$

In plain English, this equation states that the probability distribution of the state at any time t depends only on the state at time $t - 1$ and not on the states that were passed through at times $t - 2$, $t - 3$, …Hence Markov processes are not path-dependent. The model's strength lies in its flexibility, being capable of capturing changes in the variance between state processes, as well as changes in the mean.

The most basic form of Hamilton's model, also known as 'Hamilton's filter' (see Hamilton, 1989), comprises an unobserved state variable, denoted z_t, that is postulated to evaluate according to a first order Markov process

$$\text{prob}[z_t = 1|z_{t-1} = 1] = p_{11} \tag{10.12}$$

$$\text{prob}[z_t = 2|z_{t-1} = 1] = 1 - p_{11} \tag{10.13}$$

$$\text{prob}[z_t = 2|z_{t-1} = 2] = p_{22} \tag{10.14}$$

$$\text{prob}[z_t = 1|z_{t-1} = 2] = 1 - p_{22} \tag{10.15}$$

where p_{11} and p_{22} denote the probability of being in regime 1, given that the system was in regime 1 during the previous period, and the probability of being in regime 2, given that the system was in regime 2 during the previous period, respectively. Thus $1-p_{11}$ defines the probability that y_t will change from state 1 in period $t-1$ to state 2 in period t, and $1 - p_{22}$ defines the probability of a shift from state 2 to state 1 between times $t - 1$ and t. It can be shown that under this specification, z_t evolves as an AR(1) process

$$z_t = (1 - p_{11}) + \rho z_{t-1} + \eta_t \tag{10.16}$$

where $\rho = p_{11} + p_{22} - 1$. Loosely speaking, z_t can be viewed as a generalisation of the dummy variables for one-off shifts in a series discussed above. Under the Markov switching approach, there can be multiple shifts from one set of behaviour to another. In this framework, the observed returns series evolves as given by equation (10.17)

$$y_t = \mu_1 + \mu_2 z_t + (\sigma_1^2 + \phi z_t)^{1/2} u_t \tag{10.17}$$

where $u_t \sim \text{N}(0, 1)$. The expected values and variances of the series are μ_1 and σ_1^2, respectively in state 1, and $(\mu_1 + \mu_2)$ and $\sigma_1^2 + \phi$ inrespectively, state 2. The variance in state 2 is also defined, $\sigma_2^2 = \sigma_1^2 + \phi$. The unknown parameters of the model (μ_1, μ_2, σ_1^2, σ_2^2, p_{11}, p_{22}) are estimated using maximum likelihood. Details are beyond the scope of this book, but are most comprehensively given in Engel and Hamilton (1990).

If a variable follows a Markov process, all that is required to forecast the probability that it will be in a given regime during the next period is the current period's probability and a set of transition probabilities, given for the case of two regimes by equations (10.12)–(10.15). In the general case where there are m states, the transition probabilities are best expressed in a matrix as

$$P = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1m} \\ P_{21} & P_{22} & \dots & P_{2m} \\ \dots & \dots & \dots & \dots \\ P_{m1} & P_{m2} & \dots & P_{mm} \end{bmatrix} \quad \textbf{(10.18)}$$

where P_{ij} is the probability of moving from regime i to regime j. Since, at any given time, the variable must be in one of the m states, it must be true that

$$\sum_{j=1}^{m} P_{ij} = 1 \quad \forall \quad i \quad \textbf{(10.19)}$$

A vector of current state probabilities is then defined as

$$\pi_t = [\pi_1 \quad \pi_2 \quad \dots \quad \pi_m] \quad \textbf{(10.20)}$$

where π_i is the probability that the variable y is currently in state i. Given π_t and P, the probability that the variable y will be in a given regime next period can be forecast using

$$\pi_{t+1} = \pi_t P \quad \textbf{(10.21)}$$

The probabilities for S steps into the future will be given by

$$\pi_{t+s} = \pi_t P^s \quad \textbf{(10.22)}$$

10.6 A Markov Switching Model for the Real Exchange Rate

There have been a number of applications of the Markov switching model in finance. Clearly, such an approach is useful when a series is thought to undergo shifts from one type of behaviour to another and back again, but where the 'forcing variable' that causes the regime shifts is unobservable.

One such application is to modelling the real exchange rate. As discussed in Chapter 8, purchasing power parity (PPP) theory suggests that the law of one price should always apply in the long run such that the cost of a representative basket of goods and services is the same wherever it is purchased, after converting it into a common currency. Under some assumptions, one implication of PPP is that the real exchange rate (that is, the exchange rate divided by a general price index such as the consumer price index (CPI)) should be stationary. However, a number of studies have failed to reject the unit root null hypothesis in real exchange rates, indicating evidence against the PPP theory.

It is widely known that the power of unit root tests is low in the presence of structural breaks as the ADF test finds it difficult to distinguish between a stationary process subject to structural breaks and a unit root process. In order to investigate this possibility, Bergman and Hansson (2005) estimate a Markov switching model with an AR(1) structure for the real exchange rate, which allows for multiple switches between two regimes. The specification they use is

$$y_t = \mu_{s_t} + \phi y_{t-1} + \epsilon_t \quad \textbf{(10.23)}$$

where y_t is the real exchange rate, s_t, (t = 1, 2) are the two states, and $\epsilon_t \sim N(0, \sigma^2)$.[2] The state variable s_t is assumed to follow a standard 2-regime Markov process as described above.

Quarterly observations from 1973Q2 to 1997Q4 (99 data points) are used on the real exchange rate (in units of foreign currency per US dollar) for the UK, France, Germany, Switzerland, Canada and Japan. The model is estimated using the first seventy-two observations (1973Q2–1990Q4) with the remainder retained for out-of-sample forecast evaluation. The authors use 100 times the log of the real exchange rate, and this is normalised to take a value of one for 1973Q2 for all countries. The Markov switching model estimates obtained using maximum likelihood estimation are presented in Table 10.3.

Table 10.3 Estimates of the Markov switching model for real exchange rates

Parameter	UK	France	Germany	Switzerland	Canada	Japan
μ_1	3.554 (0.550)	6.131 (0.604)	6.569 (0.733)	2.390 (0.726)	1.693 (0.230)	−0.370 (0.681)
μ_2	−5.096 (0.549)	−2.845 (0.409)	−2.676 (0.487)	−6.556 (0.775)	−0.306 (0.249)	−8.932 (1.157)
ϕ	0.928 (0.027)	0.904 (0.020)	0.888 (0.023)	0.958 (0.027)	0.922 (0.021)	0.871 (0.027)
σ^2	10.118 (1.698)	7.706 (1.293)	10.719 (1.799)	13.513 (2.268)	1.644 (0.276)	15.879 (2.665)
p_{11}	0.672	0.679	0.682	0.792	0.952	0.911
p_{22}	0.690	0.833	0.830	0.716	0.944	0.817

Note: Standard errors in parentheses.

Source: Bergman and Hansson (2005).

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As Table 10.3 shows, the model is able to separate the real exchange rates into two distinct regimes for each series, with the intercept in regime 1 (μ_1) being positive for all countries except Japan (resulting from the phenomenal strength of the yen over the sample period), corresponding to a rise in the log of the number of units of the foreign currency per US dollar, i.e., a depreciation of the domestic currency against the dollar. μ_2, the intercept in regime 2, is negative for all countries, corresponding to a domestic currency appreciation against the dollar. The probabilities of remaining within the same regime during the following period (p_{11} and p_{22}) are fairly low for the UK, France, Germany and Switzerland, indicating fairly frequent switches from one regime to another for those countries' currencies.

Interestingly, after allowing for the switching intercepts across the regimes, the AR(1) coefficient, ϕ, in Table 10.3 is a considerable distance below unity, indicating that these real exchange rates are stationary. Bergman and Hansson simulate data from the stationary Markov switching AR(1) model with the estimated parameters but they assume that the researcher conducts a standard ADF test on the artificial data. They find that for none of the cases can the unit root null hypothesis be rejected, even though clearly this null is wrong as the simulated data are stationary. It is concluded that a failure to account for time-varying intercepts (i.e., structural breaks) in previous empirical studies on real exchange rates could have been the reason for the finding that the series are unit root processes when the financial theory had suggested that they should be stationary.

Finally, the authors employ their Markov switching AR(1) model for forecasting the remainder of the exchange rates in the sample in comparison with the predictions produced by a random walk and by a Markov switching model with a random walk. They find that for all six series, and for forecast horizons up to four steps (quarters) ahead, their Markov switching AR model produces predictions with the lowest mean squared errors; these improvements over the pure random walk are statistically significant.

10.7 A Markov Switching Model for the Gilt–Equity Yield Ratio

As discussed below, a Markov switching approach is also useful for modelling the time series behaviour of the gilt–equity yield ratio (GEYR), defined as the ratio of the income yield on long-term government bonds to the dividend yield on equities. It has been suggested that the current value of the GEYR might be a useful tool for investment managers or market analysts in determining whether to invest in equities or whether to invest in gilts. Thus the GEYR is purported to contain information useful for determining the likely direction of future equity market trends. The GEYR is assumed to have a long-run equilibrium level, deviations from which are taken to signal that equity prices are at an unsustainable level. If the GEYR becomes high relative to its long-run level, equities are viewed as being expensive relative to bonds. The expectation, then, is that for given levels of bond yields, equity yields must rise, which will occur via a fall in equity prices. Similarly, if the GEYR is well below its long-run level, bonds are considered expensive relative to stocks, and by the same analysis, the price of the latter is expected to increase. Thus, in its crudest form, an equity trading rule based on the GEYR would say, ‘if the GEYR is low, buy equities; if the GEYR is high, sell equities’. The paper by Brooks and Persand (2001b) discusses the usefulness of the Markov switching approach in this context, and considers whether profitable trading rules can be developed on the basis of forecasts derived from the model.

Brooks and Persand (2001b) employ monthly stock index dividend yields and income yields on government bonds covering the period January 1975 until August 1997 (272 observations) for three countries – the UK, the US and Germany. The series used are the dividend yield and index values of the FTSE100 (UK), the S&P500 (US) and the DAX (Germany). The bond indices and redemption yields are based on the clean prices of UK government consols, and US and German ten-year government bonds.

As an example, Figure 10.5 presents a plot of the distribution of the GEYR for the US (in blue), together with a normal distribution having the same mean and variance (source: Brooks and Persand, 2001b). Clearly, the distribution of the GEYR series is not normal, and the shape suggests two separate modes: one upper part of the distribution embodying most of the observations, and a lower part covering the smallest values of the GEYR.

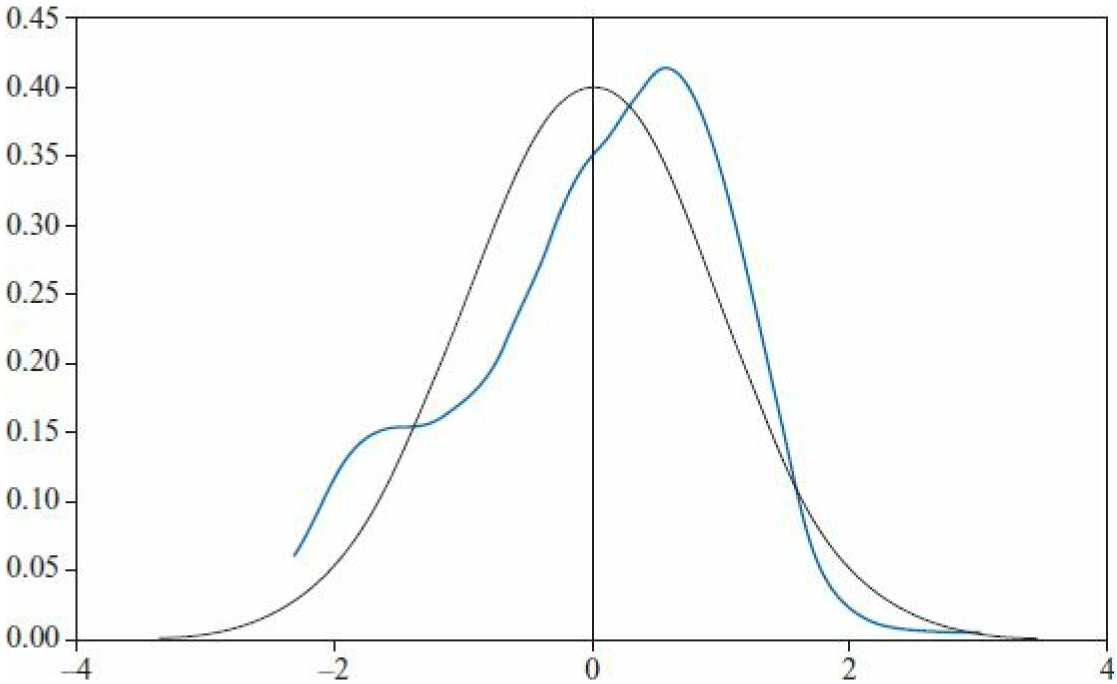

Figure 10.5 Unconditional distribution of US GEYR together with a normal distribution with the same mean and variance

Such an observation, together with the notion that a trading rule should be developed on the basis of whether the GEYR is 'high' or 'low', and in the absence of a formal econometric model for the GEYR, suggests that a Markov switching approach may be useful. Under the Markov switching approach, the values of the GEYR are drawn from a mixture of normal distributions, where the weights attached to each distribution sum to one and where movements between series are governed by a Markov process. The Markov switching model is estimated using a maximum likelihood procedure (as discussed in Chapter 9), based on GAUSS code supplied by James Hamilton. Coefficient estimates for the model are presented in Table 10.4.

Table 10.4 Estimated parameters for the Markov switching models

Statistic	μ_1 (1)	μ_2 (2)	σ_1^2 (3)	σ_2^2 (4)	p_{11} (5)	p_{22} (6)	N_1 (7)	N_2 (8)
UK	2.4293	2.0749	0.0624	0.0142	0.9547	0.9719	102	170
	(0.0301)	(0.0367)	(0.0092)	(0.0018)	(0.0726)	(0.0134)		
US	2.4554	2.1218	0.0294	0.0395	0.9717	0.9823	100	172
	(0.0181)	(0.0623)	(0.0604)	(0.0044)	(0.0171)	(0.0106)		
Germany	3.0250	2.1563	0.5510	0.0125	0.9816	0.9328	200	72
	(0.0544)	(0.0154)	(0.0569)	(0.0020)	(0.0107)	(0.0323)		

Notes: Standard errors in parentheses; N_1 and N_2 denote the number of observations deemed to be in regimes 1 and 2, respectively.

Source: Brooks and Persand (2001b).

The means and variances for the values of the GEYR for each of the two regimes are given in columns headed (1)–(4) of Table 10.4 with standard errors associated with each parameter in parentheses. It is clear that the regime switching model has split the data into two distinct samples – one with a high mean (of 2.43, 2.46 and 3.03 for the UK, US and Germany, respectively) and one with a lower mean (of 2.07, 2.12, and 2.16), as was anticipated from the unconditional distribution of returns. Also apparent is the fact that the UK and German GEYR are more variable at times when it is in the high mean regime, evidenced by their higher variance (in fact, it is around four and twenty times higher than for the low GEYR state, respectively). The number of observations for which the probability that the GEYR is in the high mean state exceeds 0.5 (and thus when the GEYR is actually deemed to be in this state) is 102 for the UK (37.5% of the total), while the figures for the US are 100 (36.8%) and for Germany 200 (73.5%). Thus, overall, the GEYR is more likely to be in the low mean regime for the UK and US, while it is likely to be high in Germany.

The columns marked (5) and (6) of Table 10.4 give the values of p_{11} and p_{22}, respectively, that is the probability of staying in state 1 given that the GEYR was in state 1 in the immediately preceding month, and the probability of staying in state 2 given that the GEYR was in state 2 previously, respectively. The high values of these parameters indicates that the regimes are highly stable with less than a 10% chance of moving from a low GEYR to a high GEYR regime and vice versa for all three series. Figure 10.6 presents a '*q*-plot', which shows the value of GEYR and probability that it is in the high GEYR regime for the UK at each point in time (source: Brooks and Persand, 2001b).

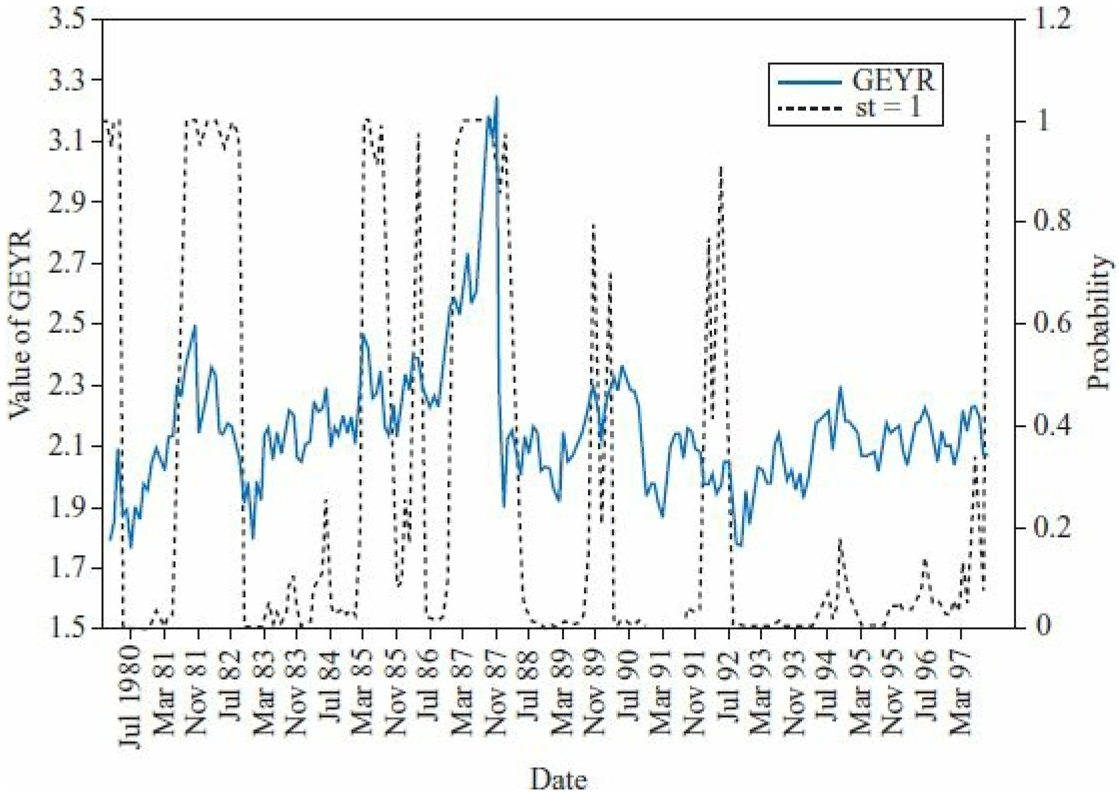

Figure 10.6 Value of GEYR and probability that it is in the high GEYR regime for the UK

As can be seen, the probability that the UK GEYR is in the 'high' regime (the dotted line) varies frequently, but spends most of its time either close to zero or close to one. The model also seems to do a reasonably good job of specifying which regime the UK GEYR should be in, given that the probability seems to match the broad trends in the actual GEYR (the full line).

Engel and Hamilton (1990) show that it is possible to give a forecast of the probability that a series y_t, which follows a Markov switching process, will be in a particular regime. Brooks and Persand (2001b) use the first sixty observations (January 1975–December 1979) for in-sample estimation of the model parameters (μ_1, μ_2, σ_1^2, p_{11}, p_{22}). Then a one step-ahead forecast is produced of the probability that the GEYR will be in the high mean regime during the next period. If the probability that the GEYR will be in the low regime during the next period is forecast to be more that 0.5, it is forecast that the GEYR will be low and hence equities are bought or held. If the probability that the GEYR is in the low regime is forecast to be less than 0.5, it is anticipated that the GEYR will be high and hence gilts are invested in or held. The model is then rolled forward one observation, with a new set of model parameters and probability forecasts being constructed. This process continues until 212 such probabilities are estimated with corresponding trading rules.

The returns for each out-of-sample month for the switching portfolio are calculated, and their characteristics compared with those of buy-and-hold equities and buy-and-hold gilts strategies. Returns are calculated as continuously compounded percentage returns on a stock (the FTSE in the UK, the S&P500 in the US, the DAX in Germany) or on a long-term government bond. The profitability of the trading rules generated by the forecasts of the Markov switching model are found to be superior in gross terms compared with a simple buy-and-hold equities strategy. In the UK context, the former yields higher average returns and lower standard deviations. The switching portfolio generates an average return of 0.69% per month, compared with 0.43% for the pure bond and 0.62% for the pure equity portfolios. The improvements are not so clear-cut for the US and Germany. The Sharpe ratio for the UK Markov switching portfolio is almost twice that of the buy-and-hold equities portfolio, suggesting that, after allowing for risk, the switching model provides a superior trading rule. The improvement in the Sharpe ratio for the other two countries is, on the contrary, only very modest.

To summarise

- The Markov switching approach can be used to model the gilt–equity yield ratio
- The resulting model can be used to produce forecasts of the probability that the GEYR will be in a particular regime
- Before transactions costs, a trading rule derived from the model produces a better performance than a buy-and-hold equities strategy, in spite of inferior predictive accuracy as measured statistically
- Net of transactions costs, rules based on the Markov switching model are not able to beat a passive investment in the index for any of the three countries studied.

10.8 Threshold Autoregressive Models

Threshold autoregressive (TAR) models are one class of non-linear autoregressive models. Such models are a relatively simple relaxation of standard linear autoregressive models that allow for a locally linear approximation over a number of states. According to Tong (1990, p. 99), the threshold principle ‘allows the analysis of a complex stochastic system by decomposing it into a set of smaller sub-systems’.

The key difference between TAR and Markov switching models is that, under the former, the state variable is assumed known and observable, while it is latent under the latter. A very simple example of a threshold autoregressive model is given by (10.24). The model contains a first order autoregressive process in each of two regimes, and there is only one threshold. Of course, the number of thresholds will always be the number of regimes minus one. Thus, the dependent variable y_t is purported to follow an autoregressive process with intercept coefficient μ_1 and autoregressive coefficient ϕ_1 if the value of the state-determining variable lagged k periods, denoted s_{t-k} is lower than some threshold value r. If the value of the state-determining variable lagged k periods, is equal to or greater than that threshold value r, y_t is specified to follow a different autoregressive process, with intercept coefficient μ_2 and autoregressive coefficient ϕ_2. The model would be written

$$y_t = \begin{cases} \mu_1 + \phi_1 y_{t-1} + u_{1t} & \text{if } s_{t-k} < r \\ \mu_2 + \phi_2 y_{t-1} + u_{2t} & \text{if } s_{t-k} \geq r \end{cases} \quad \textbf{(10.24)}$$

But what is s_{t-k}, the state-determining variable? It can be any variable that is thought to make y_t shift from one set of behaviour to another. Obviously, financial or economic theory should have an important role to play in making this decision. If $k = 0$, it is the current value of the state-determining variable that influences the regime that y is in at time t, but in many applications k is set to 1, so that the immediately preceding value of s is the one that determines the current value of y.

The simplest case for the state determining variable is where it is the variable under study, i.e., $s_{t-k} = y_{t-k}$. This situation is known as a self-exciting TAR, or a SETAR, since it is the lag of the variable y itself that determines the regime that y is currently in. The model would now be written

$$y_t = \begin{cases} \mu_1 + \phi_1 y_{t-1} + u_{1t} & \text{if } y_{t-k} < r \\ \mu_2 + \phi_2 y_{t-1} + u_{2t} & \text{if } y_{t-k} \geq r \end{cases} \quad \textbf{(10.25)}$$

The models of equations (10.24) or (10.25) can of course be extended in several directions. The number of lags of the dependent variable used in each regime may be higher than one, and the number of lags need not be the same for both regimes. The number of states can also be increased to more than two. A general threshold autoregressive model, that notationally permits the existence of more than two regimes and more than one lag, may be written

$$x_t = \sum_{j=1}^{J} I_t^{(j)} \left(\phi_0^{(j)} + \sum_{i=1}^{p_j} \phi_i^{(j)} x_{t-i} + u_t^{(j)} \right), \quad r_{j-1} \leq z_{t-d} \leq r_j \tag{10.26}$$

where $I_t^{(j)}$ is an indicator function for the *j*th regime taking the value one if the underlying variable is in state *j* and zero otherwise. z_{t-d} is an observed variable determining the switching point and $u_t^{(j)}$ is a zero-mean independently and identically distributed error process. Again, if the regime changes are driven by own lags of the underlying variable, x_t (i.e., $z_{t-d} = x_{t-d}$), then the model is a self-exciting TAR (SETAR).

It is also worth re-stating that under the TAR approach, the variable *y* is either in one regime or another, given the relevant value of *s*, and there are discrete transitions between one regime and another. This is in contrast with the Markov switching approach, where the variable *y* is in both states with some probability at each point in time. Another class of threshold autoregressive models, known as smooth transition autoregressions (STAR), allows for a more gradual transition between the regimes by using a continuous function for the regime indicator rather than an on–off switch (see Franses and van Dijk, 2000, Chapter 3).

10.9 Estimation of Threshold Autoregressive Models

Estimation of the model parameters (ϕ_i, r_j, d, p_j) is considerably more difficult than for a standard linear autoregressive process, since in general they cannot be determined simultaneously in a simple way, and the values chosen for one parameter are likely to influence estimates of the others. Tong (1983, 1990) suggests a complex non-parametric lag regression procedure to estimate the values of the thresholds (r_j) and the delay parameter (d).

Ideally, it may be preferable to endogenously estimate the values of the threshold(s) as part of the non-linear least squares (NLS) optimisation procedure, but this is not feasible. The underlying functional relationship between the variables is discontinuous in the thresholds, such that the thresholds cannot be estimated at the same time as the other components of the model. One solution to this problem that is sometimes used in empirical work is to use a grid search procedure that seeks the minimal residual sum of squares over a range of values of the threshold(s) for an assumed model.

10.9.1 Threshold Model Order (Lag Length) Determination

A simple, although far from ideal, method for determining the appropriate lag lengths for the autoregressive components for each of the regimes would be to assume that the same number of lags are required in all regimes. The lag length is then chosen in the standard fashion by determining the appropriate lag length for a linear autoregressive model, and assuming that the lag length for all states of the TAR is the same. While it is easy to implement, this approach is clearly not a good one, for it is unlikely that the lag lengths for each state when the data are drawn from different regimes would be the same as that appropriate when a linear functional form is imposed. Moreover, it is undesirable to require the lag lengths to be the same in each regime. This conflicts with the notion that the data behave differently in different states, which was precisely the motivation for considering threshold models in the first place.

An alternative and better approach, conditional upon specified threshold values, would be to employ an information criterion to select across the lag lengths in each regime simultaneously. A drawback of this approach, that Franses and van Dijk (2000) highlight, is that in practice it is often the case that the system will be resident in one regime for a considerably longer time overall than the others. In such situations, information criteria will not perform well in model selection for the regime(s) containing few observations. Since the number of observations is small in these cases, the overall reduction in the residual sum of squares as more parameters are added to these regimes will be very small. This leads the criteria to always select very small model orders for states containing few observations. A solution, therefore, is to define an information criterion that does not penalise the whole model for additional parameters in one state. Tong (1990) proposes a modified version of Akaike's information criterion (*AIC*) that weights $\hat{\sigma}^2$ for each regime by the number of observations in that regime. For the two-regime case, the modified *AIC* would be written

$$AIC\,(p_1, p_2) = T_1 \ln \hat{\sigma}_1^2 + T_2 \ln \hat{\sigma}_2^2 + 2(p_1 + 1) + 2(p_2 + 1) \qquad \textbf{(10.27)}$$

where T_1 and T_2 are the number of observations in regimes 1 and 2, respectively, p_1 and p_2 are the lag lengths and $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ are the residual variances. Similar modifications can of course be developed for other information criteria.

10.9.2 Determining the Delay Parameter, d

The delay parameter, d, can be decided in a variety of ways. It can be determined along with the lag orders for each of the regimes by an information criterion, although of course this added dimension greatly increases the number of candidate models to be estimated. In many applications, however, it is typically set to one on theoretical grounds. It has been argued (see, for example, Kräger and Kugler, 1993) that in the context of financial markets, it is most likely that the most recent past value of the state-determining variable would be the one to determine the current state, rather than that value two, three, …periods ago.

Estimation of the autoregressive coefficients can then be achieved using NLS. Further details of the procedure are discussed in Franses and van Dijk (2000, Chapter 3).

10.10 Specification Tests in the Context of Markov Switching and Threshold Autoregressive Models: A Cautionary Note

In the context of both Markov switching and TAR models, it is of interest to determine whether the threshold models represent a superior fit to the data relative to a comparable linear model. A tempting, but incorrect, way to examine this issue would be to do something like the following: estimate the desired threshold model and the linear counterpart, and compare the residual sums of squares using an F-test. However, such an approach is not valid in this instance owing to unidentified nuisance parameters under the null hypothesis. In other words, the null hypothesis for the test would be that the additional parameters in the regime switching model were zero so that the model collapsed to the linear specification, but under the linear model, there is no threshold. The upshot is that the conditions required to show that the test statistics follow a standard asymptotic distribution do not apply. Hence analytically derived critical values are not available, and critical values must be obtained via simulation for each individual case. Hamilton (1994) provides substitute hypotheses for Markov switching model evaluation that can validly be tested using the standard hypothesis testing framework, while Hansen

(1996) offers solutions in the context of TAR models.

This chapter will now examine two applications of TAR modelling in finance: one to the modelling of exchange rates within a managed floating environment, and one to arbitrage opportunities implied by the difference between spot and futures prices for a given asset. For a (rather technical) general survey of several TAR applications in finance, see Yadav, Pope and Paudyal (1994).

10.11 A SETAR Model for the French franc–German mark Exchange Rate

During the 1990s, European countries which were part of the Exchange Rate Mechanism (ERM) of the European Monetary System (EMS), were required to constrain their currencies to remain within prescribed bands relative to other ERM currencies. This seemed to present no problem by early in the new millennium since European Monetary Union (EMU) was already imminent and conversion rates of domestic currencies into euros were already known. However, in the early 1990s, the requirement that currencies remain within a certain band around their central parity forced central banks to intervene in the markets to effect either an appreciation or a depreciation in their currency. A study by Chappell *et al.* (1996) considered the effect that such interventions might have on the dynamics and time series properties of the French franc–German mark (hereafter FRF–DEM) exchange rate. ‘Core currency pairs’, such as the FRF–DEM were allowed to move up to ±2.25% either side of their central parity within the ERM. The study used daily data from 1 May 1990 until 30 March 1992. The first 450 observations are used for model estimation, with the remaining 50 being retained for out-of-sample forecasting.

A SETAR model was employed to allow for different types of behaviour according to whether the exchange rate is close to the ERM boundary. The argument is that, close to the boundary, the respective central banks will be required to intervene in opposite directions in order to drive the exchange rate back towards its central parity. Such intervention may be expected to affect the usual market dynamics that ensure fast reaction to news and the absence of arbitrage opportunities.

Let E_t denote the log of the FRF–DEM exchange rate at time t. Chappell *et al.* (1996) estimate two models: one with two thresholds and one with one threshold. The former was anticipated to be most appropriate for the data at hand since exchange rate behaviour is likely to be affected by intervention if the exchange rate comes close to either the ceiling or the floor of the band. However, over the sample period employed, the mark was never a weak currency, and therefore the FRF–DEM exchange rate was either at the top of the band or in the centre, never close to the bottom. Therefore, a model with one threshold is more appropriate since any second estimated threshold was deemed likely to be spurious.

The authors show, using DF and ADF tests, that the exchange rate series is not stationary. Therefore, a threshold model in the levels is not strictly valid for analysis. However, they argue that an econometrically valid model in first difference would lose its intuitive interpretation, since it is the *value* of the exchange rate that is targeted by the monetary authorities, not its change. In addition, if the currency bands are working effectively, the exchange rate is constrained to lie within them, and hence in some senses of the word, it must be stationary, since it cannot wander without bound in either direction. The model orders for each regime are determined using *AIC*, and the estimated model is given in Table 10.5.

Table 10.5 SETAR model for FRF–DEM

Model	For regime	Number of observations
$\hat{E}_t = 0.0222 + 0.9962E_{t-1}$ (0.0458) (0.0079)	$E_{t-1} < 5.8306$	344
$\hat{E}_t = 0.3486 + 0.4394E_{t-1} + 0.3057E_{t-2} + 0.1951E_{t-3}$ (0.2391) (0.0889) (0.1098) (0.0866)	$E_{t-1} \geq 5.8306$	103

Source: Chappell *et al.* (1996). Reprinted with permission of John Wiley and Sons.

As can be seen, the two regimes comprise a random walk with drift under normal market conditions, where the exchange rate lies below a certain threshold, and an AR(3) model corresponding to much slower market adjustment when the exchange rate lies on or above the threshold. The (natural log of) the exchange rate's central parity over the period was 5.8153, while the (log of the) ceiling of the band was 5.8376. The estimated threshold of 5.8306 is approximately 1.55% above the central parity, while the ceiling is 2.25% above the central parity. Thus, the estimated threshold is some way below the ceiling, which is in accordance with the authors' expectations since the central banks are likely to intervene before the exchange rate actually hits the ceiling.

Forecasts are then produced for the last fifty observations using the threshold model estimated above, the SETAR model with two thresholds, a random walk and an AR(2) (where the model order was chosen by in-sample minimisation of *AIC*). The results are presented here in Table 10.6.

Table 10.6 FRF–DEM forecast accuracies

	Steps ahead				
	1	2	3	5	10
	Panel A: mean squared forecast error				
Random walk	1.84E-07	3.49E-07	4.33E-07	8.03E-07	1.83E-06
AR(2)	3.96E-07	1.19E-06	2.33E-06	6.15E-06	2.19E-05
One-threshold SETAR	1.80E-07	2.96E-07	3.63E-07	5.41E-07	5.34E-07
Two-threshold SETAR	1.80E-07	2.96E-07	3.63E-07	5.74E-07	5.61E-07
	Panel B: Median squared forecast error				
Random walk	7.80E-08	1.04E-07	2.21E-07	2.49E-07	1.00E-06
AR(2)	2.29E-07	9.00E-07	1.77E-06	5.34E-06	1.37E-05
One-threshold SETAR	9.33E-08	1.22E-07	1.57E-07	2.42E-07	2.34E-07
Two-threshold SETAR	1.02E-07	1.22E-07	1.87E-07	2.57E-07	2.45E-07

Source: Chappell *et al.* (1996). Reprinted with permission of John Wiley and Sons.

For the FRF–DEM exchange rate, the one-threshold SETAR model is found to give lower mean squared errors than the other three models for one-, two-, three-, five- and ten-step-ahead forecasting horizons. Under the median squared forecast error measure, the random walk is marginally superior to the one threshold SETAR one and two steps ahead, while it has regained its prominence by three steps ahead.

However, in a footnote, the authors also argue that the SETAR model was estimated and tested for nine other ERM exchange rate series, but in every one of these other cases, the SETAR models produced less accurate forecasts than a random walk model. A possible explanation for this phenomenon is given in Section 10.13 (p. 477).

Brooks (2001) extends the work of Chappell *et al.* to allow the conditional variance of the exchange rate series to be drawn from a GARCH process which itself contains a threshold, above which the behaviour of volatility is different to that below. He finds that the dynamics of the conditional variance are quite different from one regime to the next, and that models allowing for different regimes can provide superior volatility forecasts compared to those which do not.

10.12 Threshold Models and the Dynamics of the FTSE 100 Index and Index Futures Markets

One of the examples given in Chapter 8 discussed the implications for the effective functioning of spot and futures markets of a lead–lag relationship between the two series. If the two markets are functioning effectively, it was also shown that a cointegrating relationship between them would be expected.

If stock and stock index futures markets are functioning properly, price movements in these markets should be best described by a first order vector error correction model (VECM) with the error correction term being the price differential between the two markets (the basis). The VECM could be expressed as

$$\begin{bmatrix} \Delta f_t \\ \Delta s_t \end{bmatrix} = \begin{bmatrix} \pi_{11} \\ \pi_{21} \end{bmatrix} [f_{t-1} - s_{t-1}] + \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix} \tag{10.28}$$

where Δf_t and Δs_t are changes in the log of the futures and spot prices, respectively, π_{11} and π_{21} are coefficients describing how changes in the spot and futures prices occur as a result of the basis. Writing these two equations out in full, the following would result

$$f_t - f_{t-1} = \pi_{11}[f_{t-1} - s_{t-1}] + u_{1t} \tag{10.29}$$

$$s_t - s_{t-1} = \pi_{21}[f_{t-1} - s_{t-1}] + u_{2t} \tag{10.30}$$

Subtracting equation (10.30) from equation (10.29) would give the following expression

$$(f_t - f_{t-1}) - (s_t - s_{t-1}) = (\pi_{11} - \pi_{21})[f_{t-1} - s_{t-1}] + (u_{1t} - u_{2t}) \tag{10.31}$$

which can also be written as

$$(f_t - s_t) - (f_{t-1} - s_{t-1}) = (\pi_{11} - \pi_{21})[f_{t-1} - s_{t-1}] + (u_{1t} - u_{2t}) \tag{10.32}$$

or, using the result that $b_t = f_t - s_t$

$$b_t - b_{t-1} = (\pi_{11} - \pi_{21})b_{t-1} + \varepsilon_t \tag{10.33}$$

where $\varepsilon_t = u_{1t} - u_{2t}$. Adding b_{t-1} to both sides

$$b_t = (\pi_{11} - \pi_{21} - 1)b_{t-1} + \varepsilon_t \tag{10.34}$$

If the first order *VECM* is appropriate, then it is not possible to identify structural equations for returns in stock and stock index futures markets with the obvious implications for predictability and the two markets are indeed efficient. Hence, for efficient markets and no arbitrage, there should be only a first order autoregressive process describing the basis and no further patterns. Recent evidence suggests, however, that there are more dynamics present than should be in effectively functioning markets. In particular, it has been suggested that the basis up to three trading days prior carries predictive power for movements in the FTSE 100 cash index, suggesting the possible existence of unexploited arbitrage opportunities. The paper by Brooks and Garrett (2002) analyses whether such dynamics can be explained as the result of different regimes within which arbitrage is not triggered and outside of which arbitrage will occur. The rationale for the existence of different regimes in this context is that the basis (adjusted for carrying costs if necessary), which is very important in the arbitrage process, can fluctuate within bounds determined by transaction costs without actually triggering arbitrage. Hence an autoregressive relationship between the current and previous values of the basis could arise and persist over time within the threshold boundaries since it is not profitable for traders to exploit this apparent arbitrage opportunity. Hence there will be thresholds within which there will be no arbitrage activity but once these thresholds are crossed, arbitrage should drive the basis back within the transaction cost bounds. If markets are functioning effectively then irrespective of the dynamics of the basis within the thresholds, once the thresholds have been crossed the additional dynamics should disappear.

The data used by Brooks and Garrett (2002) are the daily closing prices for the FTSE 100 stock index and stock index futures contract for the period January 1985–October 1992. The October 1987 stock market crash occurs right in the middle of this period, and therefore Brooks and Garrett conduct their analysis on a ‘pre-crash’ and a ‘post-crash’ sample as well as the whole sample. This is necessary since it has been observed that the normal spot/futures price relationship broke down around the time of the crash (see Antoniou and Garrett, 1993). Table 10.7 shows the coefficient estimates for a linear AR(3) model for the basis.

Table 10.7 Linear AR(3) model for the basis

$b_t = \phi_0 + \phi_1 b_{t-1} + \phi_2 b_{t-2} + \phi_3 b_{t-3} + \varepsilon_t$			
Parameter	**Whole sample**	**Pre-crash sample**	**Post-crash sample**
ϕ_1	0.7051**	0.7174**	0.6791**
	(0.0225)	(0.0377)	(0.0315)
ϕ_2	0.1268**	0.0946*	0.1650**
	(0.0274)	(0.0463)	(0.0378)
ϕ_3	0.0872**	0.1106**	0.0421
	(0.0225)	(0.0377)	(0.0315)

Notes: Figures in parentheses are heteroscedasticity-robust standard errors; * and** denote significance at the 5% and 1% levels, respectively.

Source: Brooks and Garrett (2002).

The results for the whole sample suggest that all of the first three lags of the basis are significant in modelling the current basis. This result is confirmed (although less strongly) for the pre-crash and post-crash sub-samples. Hence, a linear specification would seem to suggest that the basis is to some degree predictable, indicating possible arbitrage opportunities.

In the absence of transactions costs, deviations of the basis away from zero in either direction will trigger arbitrage. The existence of transactions costs, however, means that the basis can deviate from zero without actually triggering arbitrage. Thus, assuming that there are no differential transactions costs, there will be upper and lower bounds within which the basis can fluctuate without triggering arbitrage. Brooks and Garrett (2002) estimate a SETAR model for the basis, with two thresholds (three regimes) since these should correspond to the upper and lower boundaries within which the basis can fluctuate without causing arbitrage. Under efficient markets, profitable arbitrage opportunities will not be present when $r_0 \leq b_{t-1} < r_1$ where r_0 and r_1 are the thresholds determining which regime the basis is in. If these thresholds are interpreted as transactions costs bounds, when the basis falls below the lower threshold (r_0), the appropriate arbitrage transaction is to buy futures and short stock. This applies in reverse when the basis rises above r_1. When the basis lies within the thresholds, there should be no arbitrage transactions. Three lags of the basis enter into each equation and the thresholds are estimated using a grid search procedure. The one-period lag of the basis is chosen as the state-determining variable. The estimated model for each sample period is given in Table 10.8.

Table 10.8 A two-threshold SETAR model for the basis

$$b_t = \begin{cases} \phi_0^1 + \sum_{i=1}^{3} \phi_i^1 b_{t-i} + \varepsilon_t^1 \text{ if } b_{t-1} < r_0 \\ \phi_0^2 + \sum_{i=1}^{3} \phi_i^2 b_{t-i} + \varepsilon_t^2 \text{ if } r_0 \le b_{t-1} < r_1 \\ \phi_0^3 + \sum_{i=1}^{3} \phi_i^3 b_{t-i} + \varepsilon_t^3 \text{ if } b_{t-1} \ge r_1 \end{cases}$$

	$b_{t-1} < r_0$	$r_0 \le b_{t-1} < r_1$	$b_{t-1} \ge r_1$
Panel A: whole sample			
ϕ_1	0.5743**	−0.6395	0.8380**
	(0.0415)	(0.7549)	(0.0512)
ϕ_2	0.2088**	−0.0594	0.0439
	(0.0401)	(0.0846)	(0.0462)
ϕ_3	0.1330**	0.2267**	0.0415
	(0.0355)	(0.0811)	(0.0344)
$\hat{r}_0$		0.0138	
$\hat{r}_1$		0.0158	
Panel B: pre-crash sample			
ϕ_1	0.4745**	0.4482*	0.8536**
	(0.0808)	(0.1821)	(0.0720)
ϕ_2	0.2164**	0.2608**	−0.0388
	(0.0781)	(0.0950)	(0.0710)
ϕ_3	0.1142	0.2309**	0.0770
	(0.0706)	(0.0834)	(0.0531)
$\hat{r}_0$			

$\hat{r}_1$		0.0052 0.0117	
Panel C: post-crash sample			
ϕ_1	0.5019**	0.7474**	0.8397**
	(0.1230)	(0.1201)	(0.0533)
ϕ_2	0.2011*	0.2984**	0.0689
	(0.0874)	(0.0691)	(0.0514)
ϕ_3	0.0434	0.1412	0.0461
	(0.0748)	(0.0763)	(0.0400)
$\hat{r}_n$ $\hat{r}_1$		0.0080 0.0140	

Notes: Figures in parentheses are heteroscedasticity-robust standard errors,* and** denote significance at the 5% and at 1% levels, respectively.

Source: Brooks and Garrett (2002).

The results show that, to some extent, the dependence in the basis is reduced when it is permitted to be drawn from one of three regimes rather than a single linear model. For the post-crash sample, and to some extent for the whole sample and the pre-crash sample, it can be seen that there is considerably slower adjustment, evidenced by the significant second and third order autoregressive terms, between the thresholds than outside them. There still seems to be some evidence of slow adjustment below the lower threshold, where the appropriate trading strategy would be to go long the futures and short the stock. Brooks and Garrett (2002) attribute this in part to restrictions on and costs of short-selling the stock that prevent adjustment from taking place more quickly. Short-selling of futures contracts is easier and less costly, and hence there is no action in the basis beyond an AR(1) when it is above the upper threshold.

Such a finding is entirely in accordance with expectations, and suggests that, once allowance is made for reasonable transactions costs, the basis may fluctuate with some degree of flexibility where arbitrage is not profitable. Once the basis moves outside the transactions costs-determined range, adjustment occurs within one period as the theory predicted.

10.13 A Note on Regime Switching Models and Forecasting Accuracy

Several studies have noted the inability of threshold or regime switching models to generate superior out-of-sample forecasting accuracy than linear models or a random walk in spite of their apparent ability to fit the data better in sample. A possible reconciliation is offered by Dacco and Satchell (1999), who suggest that regime switching models may forecast poorly owing to the difficulty of forecasting the regime that the series will be in. Thus, any gain from a good fit of the model within the regime will be lost if the model forecasts the regime wrongly. Such an argument could apply to both the Markov switching and TAR classes of models.

10.14 State Space Models and the Kalman Filter

10.14.1 Introduction to the State Space Formulation

The vast majority of introductory-level textbooks avoid covering state space models and the Kalman filter. In some ways, this is not surprising since the material involved is inherently more complex than nearly all of the remaining subject matter covered in such books. The lack of treatment of this topic is unfortunate, however, since it provides a clever way to deal with relationships between variables that change over time, and it also provides a natural extension of the standard ARMA models discussed here in Chapter 5. I have taken up the challenge to cover this material and have tried to keep the notation consistent with that used in the rest of the book, but its intrinsic complexity means that I have elected to focus on the simplest model specifications and the intuitive aspects; readers are then signposted to further details and derivations in more advanced texts and research papers.

The state space formulation is traditionally rooted in engineering control problems, especially guidance navigation for remote controlled vehicles and trajectory analysis, but has become increasingly common in economic and financial applications. However, presenting the models is fraught with confusion right from the start as a result of differences in the situation and the common usages between engineering and economics/finance. In the former discipline, the parameters are known while the state variable is usually unobservable and to be estimated as part of the process; in finance, it is sometimes the reverse: the state variable may be an observed explanatory factor while the parameters are to be estimated. In keeping consistent with the most likely usages of the model by readers of this text, I adopt the latter convention.

Kalman filter approaches have several desirable properties. First, they are fast since state space models embody the Markov property and boil down to a set of recursions. They are thus amenable to situations where we have large amounts of data. Second, by their nature and as we will see below, they are good at dealing with unknown structural breaks and regime changes in variables whereas standard linear models are likely to be thrown off course, resulting in biased parameter estimates and inaccurate forecasts in such circumstances. A small simulation has shown that whole-period OLS experiences severe problems in the presence of structural breaks; while rolling window OLS does cope with a structural break, fitted values of the series from the Kalman filter react more quickly.[3] Third, since the Kalman filter construction of the state vector is essentially via OLS, it preserves the estimator's desirable properties of being consistent, unbiased and efficient (under assumptions). The Kalman filter can also handle missing data points in a fairly straightforward manner (see Brockwell and Davis, 1991), and finally, the way the models are constructed as recursions means they can be used for out-of-sample forecasting quite naturally.

Let us begin by introducing some relevant notation and terminology. As usual, let y_t denote the observations on some variable that we wish to model. In state space notation, there are (at least) two equations. The simplest meaningful setup (although not using matrices) is

$$y_t = \mu_t + u_t \tag{10.35}$$

$$\mu_{t+1} = T_t \mu_t + \eta_t \tag{10.36}$$

The first equation is known as the *measurement equation*, while the second is the *transition equation* or *state equation* with the transition being governed by T_t. The latter can be further simplified by assuming that T_t is fixed at some value, especially one. Then μ_t effectively follows a random walk.

It is possible to derive a state space model in vector form, so that y_t is a $K \times 1$ vector of observed variables rather than a scalar at each point in time, but we focus on the latter case here due to its tractability. Furthermore, we assume that u_t and η_t – sometimes known as the *measurement noise* and *observation noise* (or *process noise*), respectively – are normal independent and identically distributed processes with constant variances of σ_u^2 and σ_η^2, respectively. Thus it is also assumed that $E[u_t \eta_t] = 0$.

We might use the above simple pair of equations, sometimes known as the *stochastic level* or *local level* model, to describe the movement of an asset price, where μ defines the fundamental value, which moves over time according to a random walk, plus random noise (u_t) around the fundamental value. Thus the actual price is subject to two separate sources of noise (error): noise concerning the fundamental value (η_t) and additional noise around that (u_t).

Within this setup, the variances of the disturbances in the transition equation, σ_η^2, play an important role. If we allow a variable within the model to change over time, it will to some degree,[4] but it is important to consider whether it changes to a meaningful extent. Thus we can examine σ_η^2 to investigate this. Since ηt is the only element in the transition equation driving μ_t to change over time, if σ_η^2 is zero or very close to zero, this implies that μ_t is effectively constant and does not need to be allowed to vary over time so that a constant drift model would be preferable.

The way that we evaluate the size of σ_η^2 is to compute its ratio to the disturbance variance in the measurement equation – i.e., we construct the ratio of the parameters, σ_η^2/σ_u^2. Even if we had a more complex model, for example with a time-varying slope or cyclical component, we would still use the same broad approach.

Any ARMA(p,q) model can also be expressed in a state space form (in fact, each model has many different state space representations). Harvey (1989) sets up a structural time-series model based on the state space formulation where y_t is made up of trend, cycle, seasonal and irregular/random components. The trend is as in the above specification, the cyclical part is described by a function of sine and cosine terms, and the seasonality (if distinct from the cyclical part) can be captured by a set of dummy variables as discussed earlier in this chapter or with trigonometric functions. Finally, the measurement equation can be augmented with autoregressive or exogenous variables as required.

We can think of this setup in equations (10.35) and (10.36) as the simplest possible *time-varying parameters* model. Here it is just the mean of y which is changing over time. A more useful and general setup would be one where the slope parameter was allowed to vary over time. A possible model could be

$$y_t = \alpha + \beta_t x_t + u_t \tag{10.37}$$

$$\beta_{t+1} = \beta_t + \eta_t \tag{10.38}$$

Here, β_t is the hidden or unobservable state at time t. This now looks like a more familiar bivariate regression model but with a time-varying slope (and thus this is sometimes called a *stochastic slope* model). If we let y_t be the excess return on a stock or portfolio and x_t be the excess return on a proxy for the market portfolio, this could be a representation of a CAPM with a time-varying beta, which will be the subject of an example below. A further generalisation would be to allow the (Jensen's) alpha from the model (what was called μ in the first specification above) to also vary over time.

Time-varying parameter models are an appealing extension of those with fixed parameters. In general, there are many situations where we might expect parameters to change (probably slowly) over time. For example, the effects of improving technology, improving efficiency and reduced production costs (or on the other hand gradually increasing scarcity of resources) could lead the relationships between variables to slowly change over time. As an illustration, if we were interested in the relationship between the size of a petrol car engine (x) and its fuel consumption (y), the slope estimate ought to be positive but declining over time as engines have gradually become more fuel economical.

10.14.2 Parameter Estimation for State Space Models

What is the link between the Kalman filter and state space models? The former is a recursive approach to calculating the state vector for time t conditional upon information available up to and including time t and thus in essence it constitutes the core approach used to estimate state space models. It was first described in Kalman (1960) and Kalman and Bucy (1961).

In previous chapters, we thought of the scalars denoting fixed intercepts and slopes, μ and β, respectively, as being parameters. However, in the state space setup, they are no longer fixed but rather varying over time and so they are better thought of as additional variables.

So what are the parameters in this case? Recall from Chapter 9 that in maximum likelihood estimation the parameter vector (which we termed θ) not only includes the intercept and slope of the regression equation but also the variance of the errors, σ^2. In the state space models presented above, there are two innovation terms (u_t and η_t) and hence two innovation variances, σ_u^2 and σ_η^2. These are the two parameters to be estimated – in the present context, they are known as *hyperparameters* – so called because they are estimated before a final sweep of the Kalman filter and so at that point they are fixed.

Suppose that we wish to estimate the second of the above state space models (i.e., equations (10.37) and (10.38)), where the slope parameter is allowed to vary. Intuitively, the way the estimation works is that the filter predicts the new state at time t based on the previous state and the new information arriving at time t. It constructs a prediction error for y, which is minimised using OLS. Then a proportional updating term is used to correct for the previous error and a new set of states is constructed. Parameter estimation would be achieved in several steps, as discussed in Box 10.2.

BOX 10.2 Parameter estimation using the Kalman filter

1. Specify an initial value of β, β_1, which is assumed to follow a normal distribution with mean b and variance P. Since no prior information is usually available, the convention is to set b to its expected value of zero and P to an arbitrarily large positive number such as 1000000. The reason for the latter is simply that usually the state variable β_t is integrated of order one (I(1)) and so has an infinite variance. The requirement for initial values of the state vector and of the error variances to be specified, which as can be seen are usually set in a rather arbitrary way, is a disadvantage of the Kalman filter approach.
2. Beginning with t = 2, apply the Kalman filter to β_t so as to provide an estimate for β_{t+1}, which we might call $\hat{\beta}_{t+1}$ and with that compute an estimate for y_{t+1}, denoted $\hat{y}_{t+1}$. Compare this estimate with the actual value, y_{t+1}, and compute the prediction error as $y_{t+1} - \hat{y}_{t+1}$.
3. Make an adjustment to the value of $\hat{\beta}_{t+1}$ according to a fixed fraction of this prediction error, termed the *Kalman gain*. This process is effectively choosing the Kalman gain and therefore the $\hat{\beta}$ series to to minimise σ_η^2. It is easy to see from this step, where prior estimates are updated with new information to form posterior estimates, why the Kalman filter has a Bayesian

interpretation.

4. Use this adjusted estimate for β_{t+1} to construct an initial estimate for β_{t+2}. The recursive procedure above continues until the estimate for β_T has been established.

5. Maximum likelihood estimation is then employed to estimate the hyperparameters. This involves using the initial parameter values and the associated estimates of the residuals from equations (10.37) and (10.38), using the Kalman filter recursions repeatedly and using the partial derivatives of the log-likelihood function with respect to the parameters in order to recursively reduce the error variances until convergence is achieved.

6. Once the final values of the hyperparameters are obtained, the Kalman filter is employed once again to construct the final sets of series of unobservable variables (e.g., μ_t and/or β_t) and ensuring that σ_u^2 and σ_η^2 are minimised. Since this is now conducted on the whole sample, it is actually a smoother rather than a filter.

A useful distinction to draw is between the Kalman filter and the Kalman smoother. While they are similar approaches that use the same recursions, the filter moves forwards through the data while the smoother moves backwards. The Kalman filter finds the expected value of the state vector conditional upon information available only up to and including time *t*. The Kalman smoother, by contrast, uses all information in the sample (i.e., before and after *t*) in estimating the state vector. Thus, the smoother is used for post-processing.

The fitted outputs from the filter and smoother would be very similar for the observations towards the end of the sample but the smoother values could be considerably different near the start of the sample. By definition, since the smoother is using all of the data from the outset, it must produce a fitted series and error variance estimates ($\hat{\sigma}_u^2$ and $\hat{\sigma}_\eta^2$) that are at least as good as those from the filter. To produce real time forecasts would imply the use of the filter using known information up to that point. If, on the other hand, the purpose of the exercise was to capture the dynamics of an existing series with perfect hindsight and no interest in real time forecasting, it would make sense to then use the smoother to obtain better fitted values.

10.14.3 Example: Time-Varying Beta Estimation

Numerous applications of state space modelling in finance now exist. An obvious one is to capture time-varying betas in the context of asset pricing models. We will examine one such illustration shortly, but there are also more recent studies, including that by Swinkels and Van Der Sluis (2006), which examines the potential impact of changing fund manager investment style on portfolio returns. Other areas where the Kalman filter is used include the estimation of time-varying optimal hedge ratios and in arbitrage strategy rule implementation via pairs trading. Finally, the approach has also been widely implemented in the context of term structure modelling (i.e., estimating the yield curve) – see Babbs and Nowman (1999) or Prokopczuk and Wu (2013) for illustrations.

We will now examine in detail the research by Black, Fraser and Power (1992). Although now rather old, this was an important study as it was one of the first to employ a time-varying parameter model in the context of estimating the CAPM, and almost certainly the first such study applied to UK data. The entire paper is focused on this approach and thus a lot of detail is given on the method and results; more recent studies on time-varying beta estimation such as that by Hollstein and Prokopczuk (2016) cover a range of techniques and thus present less detail on the Kalman filter although they are nonetheless worth reading for a discussion of more contemporary techniques and evidence.

The objective of Black, Fraser and Power is to examine the performance of UK unit trust managers after allowing for their level of market risk according to the CAPM. The core approach used is that of separate time-series regressions for each fund and the calculation of Jensen's alpha along the lines of the earlier work discussed in Sections 3.11 and 3.12 of this book.

Previous studies, including Jensen's (1968) original research, estimated the CAPM beta as a parameter in an OLS regression. This assumes that the beta is constant over time, and thus computes the abnormal performance (alpha) relative to a fixed sensitivity to market risk. However, Black *et al.* highlight several reasons why in reality beta will probably vary over time. First, fund managers are likely to turn over their portfolios, buying some stocks and selling others. To the extent that the stocks they buy have different betas to the stocks they sell, the betas of the overall portfolios would also change. Second, managers may attempt to make use of market timing, where they switch significant proportions of their assets under management into cash or other less risky assets during times of expected turbulence. This would again cause the portfolio betas to vary over time.

Moreover, as Black *et al*. note, even if the fund manager conducts little active portfolio turnover, its beta will still vary over time as the betas of the individual stocks change along with the underlying firms' activities (for example, if the firms whose stocks the fund managers are holding borrow significant amounts of money, establish new product lines, take over other businesses, etc.) Finally, as the relative values of the constituent stocks change, it will cause those whose prices have risen to have more weight in the portfolio and those whose prices have fallen to have less, unless the weights are continuously rebalanced.

Thus the motivation for using an approach allowing for time-varying betas when investigating the performance of unit trusts is strong, and Black *et al*. do this via a states pace model with time-varying beta coefficients that follow a random walk. In essence, their setup is identical to that in equations (10.37) and (10.38) above. Note that they allow the slopes to vary over time but the intercepts in each case are fixed for the whole sample.

Their sample is based on a random selection of 30 unit trusts over the January 1980–December 1989 period, with data obtained from Datastream on their prices and dividends, which are then combined to form total returns. The FTSE All-Share index is used as a proxy for the market portfolio and the three-month Treasury bill yield is used as a proxy for the risk-free rate of return. The portfolio and market returns are turned into excess return form by subtracting the monthly risk-free return in each case, so that if there is no abnormal performance the intercept estimate would be expected to be zero.

Due to space constraints here and to avoid repetition, I only report the results for the first 15 unit trusts in Black *et al*.'s alphabetically-ordered list. The unit trust results I present are the following (with short-hand codes/ mnemonics in parentheses): Abbey General Trust (ABUT); Aetna Exempt Unit Trust (AEEA); Aetna Income & Growth Trust (AEIA); Aetna Smaller Companies Growth Unit Trust (AESA); Allied Dunbar Accumulator Trust (ALAT); Allied Dunbar Asset Value Trust (ALUT); Barclays Unicorn Capital Trust (BACU); Barclays Unicorn General Trust (BAGU); Barclays Unicorn Growth Accumulator Trust (BART); Confederation Growth Fund (CFGF); Fidelity Special Situations Trust (FISS); GT UK Capital Fund (GCCA); Gartmore Practical Investment Fund (GFMP); Hill Samuel Capital Trust (HSUT); Kleinwort Benson General Trust (KBGA).

To save space and to facilitate comparisons, I combine the results from Black *et al.*'s Tables 3, 4 and 6 into a single Table 10.9 here. Considering first the OLS results in columns (2) and (3) of my table, seven of the 15 unit trusts that I report (and ten of the 30 overall) have positive and statistically significant intercepts and thus they 'beat the market' after allowing for their levels of market risk. The betas are, as expected, mostly between 0.8 and 1.1, indicating that the portfolios have on average approximately the same degree of movements as the FTSE All-Share index.

Table 10.9 Unit trust performance with time-varying beta estimation

Unit trust Code	OLS estimates		Time-varying parameter estimates		
	α	β	σ_η^2/σ_u^2	α	mean of β_t
(1)	(2)	(3)	(4)	(5)	(6)
ABUT	0.103E-02 (0.169E-02)	0.910 (0.305E-01)	0.551 (0.766)	0.100E-02 (0.119E-02)	0.897 -
AEEA	0.319E-02 (0.178E-02)	0.834 (0.353E-01)	3.036 (2.358)	0.392E-02* (0.125E-02)	0.792 -
AEIA	0.327E-02* (0.156E-02)	0.848 (0.248E-01)	0.000 -	0.399E-02* (0.261E-11)	0.849 -
AESA	−0.671E-03 (0.325E-02)	0.922 (0.783E-01)	0.000 -	−0.269E-03 (0.233E-02)	0.922 -
ALAT	0.406E-02* (0.163E-02)	0.982 (0.282E-01)	2.889 (3.118)	0.428E-02* (0.112E-02)	0.882 -
ALUT	0.631E-02 (0.758E-02)	1.105 (0.145)	0.000 -	0.693E-02 (0.557E-02)	1.103 -
BACU	0.922E-03 (0.158E-02)	0.898 (0.400E-01)	5.425 (3.921)	0.806E-03 (0.102E-02)	0.897 -
BAGU	0.423E-02* (0.205E-02)	0.862 (0.472E-01)	0.103 (0.286)	0.458E-02* (0.145E-02)	0.859 -
BART	0.155E-02 (0.186E-02)	0.919 (0.573E-01)	25.555 (21.650)	0.206E-02 (0.120E-02)	0.846 -
CFGF	0.592E-02* (0.211E-02)	0.833 (0.584E-01)	1.027 (1.267)	0.623E-02* (0.149E-02)	0.801 -
FISS	0.878E-02* (0.368E-02)	0.994 (0.839)	0.000 -	0.923E-02* (0.266E-02)	0.987 -
GCCA	0.456E-02* (0.223E-02)	1.011 (0.508E-01)	0.429 (0.469)	0.498E-02* (0.158E-02)	0.993 -
GFMP	0.485E-02* (0.235E-02)	0.722 (0.507E-01)	0.310 (0.478)	0.467E-02* (0.167E-02)	0.710 -
HSUT	0.184E-02 (0.242E-02)	0.880 (0.841E-01)	13.245 (11.888)	0.207E-02 (0.151E-02)	0.830 -
KBGA	0.444E-02 (0.265E-02)	0.897 (0.113)	4.264 (3.673)	0.416E-02* (0.161E-02)	0.834 -

Notes: Column (1) of the table gives the mnemonics; columns (2) and (3) give the intercept and slope estimates (alpha and beta, respectively) for the OLS estimated models with fixed betas; column (4) gives the ratio of the transition equation error variance to the measurement equation residual variance; columns (5) and (6) give the intercepts and the average beta estimates from the time-varying parameters model. Heteroscedasticity-consistent standard errors are given in parentheses and a single asterisk denotes significance at the 10% level for the hyperparameter ratio and at the 5% level for the intercept estimates (the OLS slope estimates are significantly different from zero at the 1% level in all cases).

Source: Black, Fraser and Power (1992). Reprinted with permission from Elsevier.

Column (4) of Table 10.9 shows the ratio of the hyperparameters – that is, the ratio of σ_η^2 to σ_u^2. The values for four of the ratios that I report – AEIA, AESA, ALUT and FISS (and for eight of the total sample of 30 unit trusts) are all exactly zero, suggesting that the exposures to market risk are precisely constant over time in those cases. On the other hand, for some of the portfolios (e.g., BART), the ratio is very high, indicating a large amount of variation in β_t over time, although for none of the series I report (and for only three out of the other 15) is the ratio significantly different from zero. Overall, this presents a mixed picture of the extent to which β_t varies over time.

Columns (5) and (6) of Table 10.9 show the time-invariant alphas and the averages of the time-varying betas from the state space model for each unit trust. These figures can be directly compared with those of columns two and three in the table, respectively. The results now show even more evidence of fund manager skill: nine of the 15 unit trusts in Table 10.9 (21 out of 30 overall) have positive and statistically significant alphas and therefore positive abnormal returns. From OLS to time-varying parameters, in most cases, the parameter estimates have increased and the standard errors declined, resulting in increased *t*-ratios. Also, in most cases, the average of the time-varying beta estimates is slightly smaller than the corresponding OLS estimate, although they are not markedly altered.

While not reported here, Black *et al*. also conduct Dickey-Fuller unit root tests on the time-varying beta series, which they argue can be interpreted as tests for the stability of the betas. They find that for only three of the 30 portfolios is the unit root null hypothesis rejected, indicating mean reversion of the betas in those cases and considerable instability among the others.

In conclusion, the findings of the paper suggest that in some cases, different (and arguably incorrect) inferences about the fund manager's ability would be made if the model does not allow the risk of the fund to vary over time when it does in reality. The results are also indicative that this set of UK-based unit trust managers experienced considerably better performance than was the case for comparable studies that were conducted in the US and many were able to outperform a passive investment strategy on a risk-adjusted basis.

10.14.4 Further Reading on State Space Models

Technical but very comprehensive and detailed presentations of the mechanics of Kalman filtering for a wide variety of state space models are given in Durbin and Koopman (2001), Harvey (1989) and Hamilton (1994). A more accessible treatment, which I drew on to some extent in writing some of the above is due to Jalles (2009). A challenging discussion of the application of the Kalman filter to estimate stochastic volatility models is given in Harvey *et al.* (1994).

KEY CONCEPTS

The key terms to be able to define and explain from this chapter are

- seasonality
- slope dummy variable
- regime switching
- self-exciting TAR
- Markov process
- Kalman filter
- Kalman gain
- hyperparameter
- transition equation
- intercept dummy variable
- dummy variable trap
- threshold autoregression (TAR)
- delay parameter
- transition probability
- Kalman smoother
- state space model
- measurement equation
- stochastic level model

SELF-STUDY QUESTIONS

1. A researcher is attempting to form an econometric model to explain daily movements of stock returns. A colleague suggests that she might want to see whether her data are influenced by daily seasonality.

(a) How might she go about doing this?

(b) The researcher estimates a model with the dependent variable as the daily returns on a given share traded on the London stock exchange, and various macroeconomic variables and accounting ratios as independent variables. She attempts to estimate this model, together with five daily dummy variables (one for each day of the week), and a constant term, using an econometric software package. The package then tells her that it cannot estimate the parameters of the model. Explain what has probably happened, and how she can fix it.

(c) A colleague estimates instead the following model for asset returns, r_t is as follows (with standard errors in parentheses)

$$\begin{aligned}\hat{r}_t = \; & \underset{(0.0146)}{0.0034} - \underset{(0.0068)}{0.0183D1_t} + \underset{(0.0231)}{0.0155D2_t} - \underset{(0.0179)}{0.0007D3_t} \\ & - \underset{(0.0193)}{0.0272D4_t} + \textit{other variables}\end{aligned}$$

The model is estimated using 500 observations. Is there significant evidence of any 'day-of-the-week effects' after allowing for the effects of the other variables?

(d) Distinguish between intercept dummy variables and slope dummy variables, giving an example of each.

(e) A financial researcher suggests that many investors rebalance their portfolios at the end of each financial year to realise losses and consequently reduce their tax liabilities. Develop a procedure to test whether this behaviour might have an effect on equity returns.

2. (a) What is a switching model? Describe briefly and distinguish between threshold autoregressive models and Markov switching models. How would you decide which of the two model classes is more appropriate for a particular application?

(b) Describe the following terms as they are used in the context of Markov switching models

(i) The Markov property

(ii) A transition matrix.

(c) What is a SETAR model? Discuss the issues involved in estimating such a model.

(d) What problem(s) may arise if the standard information criteria presented in Chapter 6 were applied to the determination of the orders of each equation in a TAR model? How do suitably modified criteria overcome this problem?

(e) A researcher suggests a reason that many empirical studies find that PPP does not hold is the existence of transactions costs and other rigidities in the goods markets. Describe a threshold model procedure that may be used to evaluate this proposition in the context of a single good.

(f) A researcher estimates a SETAR model with one threshold and three lags in both regimes using maximum likelihood. He then estimates a linear AR(3) model by maximum likelihood and proceeds to use a likelihood ratio test to determine whether the non-linear threshold model is necessary. Explain the flaw in this approach.

(g) 'Threshold models are more complex than linear autoregressive models. Therefore, the former should produce more accurate forecasts since they should capture more relevant features of the data.' Discuss.

3. A researcher suggests that the volatility dynamics of a set of daily equity returns are different

- on Mondays relative to other days of the week
- if the previous day's return volatility was bigger than 0.1% relative to when the previous day's return volatility was less than 0.1%.

Describe models that could be used to capture these reported features of the data.

4. (a) Explain the link between state space models and the Kalman filter

(b) What is the difference between the Kalman filter and the Kalman smoother?

(c) In the context of a state space model, how can we determine whether allowing the parameters to vary over time is necessary?

(d) Suppose that we wish to estimate a CAPM-style model for fund manager performance where both the Jensen's alpha and the market beta vary over time. Using equations, describe the model that we would estimate

(e) If the Kalman filter is implemented using a set of recursive formulae, why is it still necessary to use maximum likelihood estimation?

1 Note also, that the model presented in equation (10.9) will allow both the intercept and slope to vary below and above the threshold, but it will not guarantee that the two segments will meet at the knot. In order to ensure this, we would need a slightly different formulation of the equation such as $y_t = \beta_1 + \beta_2 x_t + \beta_3 D_t(x_t - x^*) + u_t$. I am grateful to Kyoung Gook Park for pointing this out.

2 The authors also estimate models that allow ϕ and σ^2 to vary across the states, but the restriction that the parameters are the same across the two states cannot be rejected and hence the values presented in the study assume that they are constant.

3 See Renzi-Ricci, G. (July 2016) Estimating equity betas: what can a time-varying approach add? A comparison of ordinary least squares and the Kalman filter, *Nera Economic Consulting* www.nera.com.

4 Akin to the problem that a parameter estimate for a particular variable will never be exactly zero in a regression model even if the variable is completely irrelevant for explaining the dependent variable.

11

Panel Data

LEARNING OUTCOMES

In this chapter, you will learn how to

- Describe the key features of panel data and outline the advantages and disadvantages of working with panels rather than other structures
- Explain the intuition behind seemingly unrelated regressions and propose examples of where they may be usefully employed
- Contrast the fixed effect and random effect approaches to panel model specification, determining which is the more appropriate in particular cases
- Estimate and interpret the results from panel unit root and cointegration tests

11.1 Introduction: What Are Panel Techniques and Why are They Used?

The situation often arises in financial modelling where we have data comprising both time-series and cross-sectional elements, and such a dataset would be known as a panel of data or longitudinal data. A panel of data will embody information across both time and space. Importantly, a panel keeps the same individuals or objects (henceforth we will call these 'entities') and measures some quantity about them over time.[1] This chapter will present and discuss the important features of panel analysis, and will describe the techniques used to model such data.

Econometrically, the setup we may have is as described in the following equation

$$y_{it} = \alpha + \beta x_{it} + u_{it} \tag{11.1}$$

where y_{it} is the dependent variable, α is the intercept term, β is a $k \times 1$ vector of parameters to be estimated on the explanatory variables, and x_{it} is a $1 \times k$ vector of observations on the explanatory variables, $t = 1, \ldots, T$; $i = 1, \ldots, N$.[2]

The simplest way to deal with such data would be to estimate a pooled regression, which would involve estimating a single equation on all the data together, so that the dataset for *y* is stacked up into a single column containing all the cross-sectional and time series observations, and similarly all of the observations on each explanatory variable would be stacked up into single columns in the *x* matrix. Then this equation would be estimated in the usual fashion using OLS.

While this is indeed a simple way to proceed, and requires the estimation of as few parameters as possible, it has some severe limitations. Most importantly, pooling the data in this way implicitly assumes that the average values of the variables and the relationships between them are constant over time and across all of the cross-sectional units in the sample. We could, of course, estimate separate time series regressions for each of objects or entities, but this is likely to be a sub-optimal way to proceed since this approach would not take into account any common structure present in the series of interest. Alternatively, we could estimate separate cross-sectional regressions for each of the time periods, but again this may not be wise if there is some common variation in the series over time. If we are fortunate enough to have a panel of data at our disposal, there are important advantages to making full use of this rich structure

- First, and perhaps most importantly, we can address a broader range of issues and tackle more complex problems with panel data than would be possible with pure time series or pure cross-sectional data alone.
- Second, it is often of interest to examine how variables, or the relationships between them, change dynamically (over time). To do this using pure time series data would often require a long run of data simply to get a sufficient number of observations to be able to conduct any meaningful hypothesis tests. But by combining cross-

sectional and time series data, one can increase the number of degrees of freedom, and thus the power of the test, by employing information on the dynamic behaviour of a large number of entities at the same time. The additional variation introduced by combining the data in this way can also help to mitigate problems of multicollinearity that may arise if time series are modelled individually.

- Third, as will become apparent below, by structuring the model in an appropriate way, we can remove the impact of certain forms of omitted variables bias in regression results.

11.2 What Panel Techniques Are Available?

One approach to making more full use of the structure of the data would be to use the *seemingly unrelated regression* (SUR) framework initially proposed by Zellner (1962). This has been used widely in finance where the requirement is to model several closely related variables over time.[3] A SUR is so called because the dependent variables may seem unrelated across the equations at first sight, but a more careful consideration would allow us to conclude that they are in fact related after all. One example would be the flow of funds (i.e. net new money invested) to portfolios (mutual funds) operated by two different investment banks. The flows could be related since they are, to some extent, substitutes (if the manager of one fund is performing poorly, investors may switch to the other). The flows are also related because the total flow of money into all mutual funds will be affected by a set of common factors (for example, related to people's propensity to save for their retirement). Although we could entirely separately model the flow of funds for each bank, we may be able to improve the efficiency of the estimation by capturing at least part of the common structure in some way. Under the SUR approach, one would allow for the contemporaneous relationships between the error terms in the two equations for the flows to the funds in each bank by using a generalised least squares (GLS) technique. The idea behind SUR is essentially to transform the model so that the error terms become uncorrelated. If the correlations between the error terms in the individual equations had been zero in the first place, then SUR on the system of equations would have been equivalent to running separate OLS regressions on each equation. This would also be the case if all of the values of the explanatory variables were the same in all equations – for example, if the equations for the two funds contained only macroeconomic variables.

However, the applicability of the technique is limited because it can be employed only when the number of time series observations, T, per cross-sectional unit i is at least as large as the total number of such units, N. A second problem with SUR is that the number of parameters to be estimated in total is very large, and the variance–covariance matrix of the errors (which will be a phenomenal $NT \times NT$) also has to be estimated. For these reasons, the more flexible full panel data approach is much more commonly used.

There are broadly two classes of panel estimator approaches that can be employed in financial research: *fixed effects* models and *random effects* models. The simplest types of fixed effects models allow the intercept in the regression model to differ cross-sectionally but not over time, while all of the slope estimates are fixed both cross-sectionally and over time. This approach is evidently more parsimonious than a SUR (where each cross-sectional unit would have different slopes as well), but it still requires the estimation of $(N + k)$ parameters.[4]

A first distinction we must draw is between a *balanced panel* and an *unbalanced panel*. A balanced panel has the same number of time series observations for each cross-sectional unit (or equivalently but viewed the other way around, the same number of cross-sectional units at each point in time), whereas an unbalanced panel would have some cross-sectional elements with fewer observations or observations at different times to others. The same techniques are used in both cases, and while the presentation below implicitly assumes that the panel is balanced, missing observations should be automatically accounted for by the software package used to estimate the model.

11.3 The Fixed Effects Model

To see how the fixed effects model works, we can take equation (11.1) above, and decompose the disturbance term, u_{it}, into an individual specific effect, μ_i, and the 'remainder disturbance', v_{it}, that varies over time and entities (capturing everything that is left unexplained about y_{it}).

$$u_{it} = \mu_i + v_{it} \quad \textbf{(11.2)}$$

So we could rewrite equation (11.1) by substituting in for u_{it} from (11.2) to obtain

$$y_{it} = \alpha + \beta x_{it} + \mu_i + v_{it} \quad \textbf{(11.3)}$$

We can think of μ_i as encapsulating all of the variables that affect y_{it} cross-sectionally but do not vary over time – for example, the sector that a firm operates in, a person's gender, or the country where a bank has its headquarters, etc. This model could be estimated using dummy variables, which would be termed the least squares dummy variable (LSDV) approach

$$y_{it} = \beta x_{it} + \mu_1 D1_i + \mu_2 D2_i + \mu_3 D3_i + \cdots + \mu_N DN_i + v_{it} \quad \textbf{(11.4)}$$

where $D1_i$ is a dummy variable that takes the value 1 for all observations on the first entity (e.g., the first firm) in the sample and zero otherwise, $D2_i$ is a dummy variable that takes the value 1 for all observations on the second entity (e.g., the second firm) and zero otherwise, and so on. Notice that we have removed the intercept term (α) from this equation to avoid the 'dummy variable trap' described in Chapter 10 where we have perfect multicollinearity between the dummy variables and the intercept. When the fixed effects model is written in this way, it is relatively easy to see how to test for whether the panel approach is really necessary at all. This test would be a slightly modified version of the Chow test described in Chapter 5, and would involve incorporating the restriction that all of the intercept dummy variables have the same parameter (i.e. $H_0 : \mu_1 = \mu_2 = \cdots = \mu_N$). If this null hypothesis is not rejected, the data can simply be pooled together and OLS employed. If this null is rejected, however, then it is not valid to impose the restriction that the intercepts are the same over the cross-sectional units and a panel approach must be employed.

Now the model given by equation (11.4) has $N + k$ parameters to estimate, which would be a challenging problem for any regression package when N is large. In order to avoid the necessity to estimate so many dummy variable parameters, a transformation is made to the data to simplify matters. This transformation, known as the *within transformation*, involves subtracting the time-mean of each entity away from the values of the variable.[5] So define $\overline{y}_i = \frac{1}{T}\sum_{t=1}^{T} y_{it}$ as the time-mean of the observations on y for cross-sectional unit i, and similarly calculate the means of all of the explanatory variables. Then we can subtract the time-means from each variable to obtain a regression containing demeaned variables only. Note that again, such a regression does not require an intercept term since now the dependent variable will have zero mean by construction. The model containing the demeaned variables is

$$y_{it} - \overline{y}_i = \beta(x_{it} - \overline{x}_i) + u_{it} - \overline{u}_i \quad \textbf{(11.5)}$$

which we could write as

$$\ddot{y}_{it} = \beta\ddot{x}_{it} + \ddot{u}_{it} \quad \textbf{(11.6)}$$

where the double dots above the variables denote the demeaned values.

An alternative to this demeaning would be to simply run a cross-sectional regression on the time-averaged values of the variables, which is known as the *between estimator*.[6] A further possibility is that instead, the first difference operator could be applied to equation (11.1) so that the model becomes one for explaining the change in y_{it} rather than its level. When differences are taken, any variables that do not change over time (i.e., the μ_i) will again cancel out. Differencing and the within transformation will produce identical estimates in situations where there are only two time periods; when there are more, the choice between the two approaches will depend on the assumed properties of the error term. Wooldridge (2010) describes this issue in considerable detail.

Equation (11.6) can now be routinely estimated using OLS on the pooled sample of demeaned data, but we do need to be aware of the number of degrees of freedom which this regression will have. Although estimating the equation will use only k degrees of freedom from the NT observations, it is important to recognise that we also used a further N degrees of freedom in constructing the demeaned variables (i.e., we lost a degree of freedom for every one of the N explanatory variables for which we were required to estimate the mean). Hence the number of degrees of freedom that must be used in estimating the standard errors in an unbiased way and when conducting hypothesis tests is $NT - N - k$. Any software packages used to estimate such models should take this into account automatically.

The regression on the time-demeaned variables will give identical parameters and standard errors as would have been obtained directly from the LSDV regression, but without the hassle of estimating so many parameters! A major disadvantage of this process, however, is that we lose the ability to determine the influences of all of the variables that affect y_{it} but do not vary over time.

11.4 Time-Fixed Effects Models

It is also possible to have a time-fixed effects model rather than an entity-fixed effects model. We would use such a model where we thought that the average value of y_{it} changes over time but not cross-sectionally. Hence with time-fixed effects, the intercepts would be allowed to vary over time but would be assumed to be the same across entities at each given point in time. We could write a time-fixed effects model as

$$y_{it} = \alpha + \beta x_{it} + \lambda_t + v_{it} \tag{11.7}$$

where λ_t is a time-varying intercept that captures all of the variables that affect y_{it} and that vary over time but are constant cross-sectionally. An example would be where the regulatory environment or tax rate changes part-way through a sample period. In such circumstances, this change of environment may well influence y, but in the same way for all firms, which could be assumed to all be affected equally by the change.

Time variation in the intercept terms can be allowed for in exactly the same way as with entity-fixed effects. That is, a least squares dummy variable model could be estimated

$$y_{it} = \beta x_{it} + \lambda_1 D1_t + \lambda_2 D2_t + \lambda_3 D3_t + \cdots + \lambda_T DT_t + v_{it} \tag{11.8}$$

where $D1_t$, for example, denotes a dummy variable that takes the value 1 for the first time period and zero elsewhere, and so on.

The only difference is that now, the dummy variables capture time variation rather than cross-sectional variation. Similarly, in order to avoid estimating a model containing all T dummies, a within transformation can be conducted to subtract the cross-sectional averages from each observation

$$y_{it} - \overline{y}_t = \beta(x_{it} - \overline{x}_t) + u_{it} - \overline{u}_t \tag{11.9}$$

where $\overline{y}_t = \frac{1}{N}\sum_{i=1}^{N} y_{it}$ as the mean of the observations on y across the entities for each time period. We could write this equation as

$$\ddot{y}_{it} = \beta \ddot{x}_{it} + \ddot{u}_{it} \tag{11.10}$$

where the double dots above the variables denote the demeaned values (but now cross-sectionally rather than temporally demeaned).

Finally, it is possible to allow for both entity-fixed effects and time-fixed effects within the same model. Such a model would be termed a two-way error component model, which would combine equations (11.3) and (11.8), and the LSDV equivalent model would contain both cross-sectional and time dummies

$$\begin{aligned} y_{it} = \beta x_{it} + \mu_1 D1_i + \mu_2 D2_i + \mu_3 D3_i + \cdots + \mu_N DN_i + \lambda_1 D1_t \\ + \lambda_2 D2_t + \lambda_3 D3_t + \cdots + \lambda_T DT_t + v_{it} \end{aligned} \tag{11.11}$$

However, the number of parameters to be estimated would now be $k + N + T$, and the within transformation in this two-way model would be more complex.

11.5 Investigating Banking Competition Using a Fixed Effects Model

The UK retail banking sector has been subject to a considerable change in structure over the past thirty years as a result of deregulation, merger waves and new technology. The relatively high concentration of market share in retail banking among a modest number of fairly large banks, combined with apparently phenomenal profits that appear to be recurrent, have led to concerns that competitive forces in British banking are not sufficiently strong.[7] This is argued to go hand in hand with restrictive practices, barriers to entry and poor value for money for consumers. A study by Matthews, Murinde and Zhao (2007) investigates competitive conditions in the UK between 1980 and 2004 using the ‘new empirical industrial organisation’ approach pioneered by Panzar and Rosse (1982, 1987). The model posits that if the market is *contestable*, entry to and exit from the market will be easy (even if the concentration of market share among firms is high), so that prices will be set equal to marginal costs. The technique used to examine this conjecture is to derive testable restrictions upon the firm’s reduced form revenue equation.

The empirical investigation consists of deriving an index (the Panzar–Rosse H-statistic) of the sum of the elasticities of revenues to factor costs (input prices). If this lies between 0 and 1, we have monopolistic competition or a partially contestable equilibrium, whereas $H < 0$ would imply a monopoly and $H = 1$ would imply perfect competition or perfect contestability. The key point is that if the market is characterised by perfect competition, an increase in input prices will not affect the output of firms, while it will under monopolistic competition. The model Matthews *et al*. investigate is given by

$$lnREV_{it} = \alpha_0 + \alpha_1 lnPL_{it} + \alpha_2 lnPK_{it} + \alpha_3 lnPF_{it} + \beta_1 lnRISKASS_{it} + \beta_2 lnASSET_{it} + \beta_3 lnBR_{it} + \gamma_1 GROWTH_t + \mu_i + v_{it} \tag{11.12}$$

where 'REV_{it}' is the ratio of bank revenue to total assets for firm i at time t ($i = 1, \ldots, N$; $t = 1, \ldots, T$); '*PL*' is personnel expenses to employees (the unit price of labour); '*PK*' is the ratio of capital assets to fixed assets (the unit price of capital); and '*PF*' is the ratio of annual interest expenses to total loanable funds (the unit price of funds). The model also includes several variables that capture time-varying bank-specific effects on revenues and costs, and these are '*RISKASS*', the ratio of provisions to total assets; '*ASSET*' is bank size, as measured by total assets; '*BR*' is the ratio of the bank's number of branches to the total number of branches for all banks. Finally, '$GROWTH_t$' is the rate of growth of GDP, which obviously varies over time but is constant across banks at a given point in time; μ_i are bank-specific fixed effects and v_{it} is an idiosyncratic disturbance term. The contestability parameter, H, is given as $\alpha_1 + \alpha_2 + \alpha_3$.

Unfortunately, the Panzar–Rosse approach is valid only when applied to a banking market in long-run equilibrium. Hence the authors also conduct a test for this, which centres on the regression

$$lnROA_{it} = \alpha_0' + \alpha_1' lnPL_{it} + \alpha_2' lnPK_{it} + \alpha_3' lnPF_{it} + \beta_1' lnRISKASS_{it} + \beta_2' lnASSET_{it} + \beta_3' lnBR_{it} + \gamma_1' GROWTH_t + \eta_i + w_{it} \tag{11.13}$$

The explanatory variables for the equilibrium test regression in equation (11.13) are identical to those of the contestability regression in equation (11.12), but the dependent variable is now the log of the return on assets ('lnROA'). Equilibrium is argued to exist in the market if $\alpha_1' + \alpha_2' + \alpha_3' = 0$.

The UK market is argued to be of particular international interest as a result of its speed of deregulation and the magnitude of the changes in market structure that took place over the sample period and therefore the study by Matthews *et al*. focuses exclusively on the UK. They employ a fixed effects panel data model which allows for differing intercepts across the banks, but assumes that these effects are fixed over time. The fixed effects approach is a sensible one given the data analysed here since there is an unusually large number of years (twenty-five) compared with the number of banks (twelve), resulting in a total of 219 bank-years (observations). The data employed in the study are obtained from banks' annual reports and the Annual Abstract of Banking Statistics from the British Bankers Association. The analysis is conducted for the whole sample period, 1980–2004, and for two sub-samples, 1980–91 and 1992–2004. The results for tests of equilibrium are given first, in Table 11.1.

Table 11.1 Tests of banking market equilibrium with fixed effects panel models

Variable	1980–2004	1980–91	1992–2004
Intercept	0.0230*** (3.24)	0.1034* (1.87)	0.0252 (2.60)
ln*PL*	−0.0002 (0.27)	0.0059 (1.24)	0.0002 (0.37)
ln*PK*	−0.0014* (1.89)	−0.0020 (1.21)	−0.0016* (1.81)
ln*PF*	−0.0009 (1.03)	−0.0034 (1.01)	0.0005 (0.49)
ln*RISKASS*	−0.6471*** (13.56)	−0.5514*** (8.53)	−0.8343*** (5.91)
ln*ASSET*	−0.0016*** (2.69)	−0.0068** (2.07)	−0.0016** (2.07)
ln*BR*	−0.0012* (1.91)	0.0017 (0.97)	−0.0025 (1.55)
GROWTH	0.0007*** (4.19)	0.0004 (1.54)	0.0006* (1.71)
R^2 within	0.5898	0.6159	0.4706
$H_0 : \eta_i = 0$	$F(11, 200) =$ 7.78***	$F(9, 66) = 1.50$	$F(11, 117) =$ 11.28***
$H_0 : E = 0$	$F(1, 200) =$ 3.20*	$F(1, 66) = 0.01$	$F(1, 117) =$ 0.28

Notes: *t*-ratios in parentheses; *, ** and *** denote significance at the 10%, 5% and 1% levels, respectively.

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Elsevier.

The null hypothesis that the bank fixed effects are jointly zero ($H_0 : \eta_i = 0$) is rejected at the 1% significance level for the full sample and for the second sub-sample but not at all for the first sub-sample. Overall, however, this indicates the usefulness of the fixed effects panel model that allows for bank heterogeneity. The main focus of interest in Table 11.1 is the equilibrium test, and this shows slight evidence of disequilibrium (*E* is significantly different from zero at the 10% level) for the whole sample, but not for either of the individual sub-samples. Thus the conclusion is that the market appears to be sufficiently in a state of equilibrium that it is valid to continue to investigate the extent of competition using the Panzar–Rosse methodology. The results of this are presented in Table 11.2.[8]

Table 11.2 Tests of competition in banking with fixed effects panel models

Variable	**1980–2004**	**1980–91**	**1992–2004**
Intercept	−3.083 (1.60)	1.1033** (2.06)	−0.5455 (1.57)
ln*PL*	−0.0098 (0.54)	0.164*** (3.57)	−0.0164 (0.64)
ln*PK*	0.0025 (0.13)	0.0026 (0.16)	−0.0289 (0.91)
ln*PF*	0.5788*** (23.12)	0.6119*** (18.97)	0.5096*** (12.72)
ln*RISKASS*	2.9886** (2.30)	1.4147** (2.26)	5.8986 (1.17)
ln*ASSET*	−0.0551*** (3.34)	−0.0963*** (2.89)	−0.0676** (2.52)
ln*BR*	0.0461*** (2.70)	0.00094 (0.57)	0.0809 (1.43)
GROWTH	−0.0082* (1.91)	−0.0027 (1.17)	−0.0121 (1.00)
R^2 within	0.9209	0.9181	0.8165

$H_0 : \eta_i = 0$	$F(11, 200) =$ 23.94***	$F(9, 66) =$ 21.97***	$F(11, 117) =$ 11.95***
$H_0 : H = 0$	$F(1, 200) =$ 229.46***	$F(1, 66) =$ 205.89***	$F(1, 117) =$ 71.25***
$H_1 : H = 1$	$F(1, 200) =$ 128.99***	$F(1, 66) =$ 16.59***	$F(1, 117) =$ 94.76***
H	0.5715	0.7785	0.4643

Notes: t-ratios in parentheses; *, ** and ***, denote significance at the 10%, 5% and 1% levels, respectively. The final set of asterisks in the table was added by the present author.

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The value of the contestability parameter, H, which is the sum of the input elasticities, is given in the last row of Table 11.2 and falls in value from 0.78 in the first sub-sample to 0.46 in the second, suggesting that the degree of competition in UK retail banking weakened over the period. However, the results in the two rows above that show that the null hypotheses $H = 0$ and $H = 1$ can both be rejected at the 1% significance level for both sub-samples, showing that the market is best characterised by monopolistic competition rather than either perfect competition (perfect contestability) or pure monopoly. As for the equilibrium regressions, the null hypothesis that the fixed effects dummies (η_i) are jointly zero is strongly rejected, vindicating the use of the fixed effects panel approach and suggesting that the base levels of the dependent variables differ.

Finally, the additional bank control variables all appear to have intuitively appealing signs. The risk assets variable has a positive sign, so that higher risks lead to higher revenue per unit of total assets; the asset variable has a negative sign and is statistically significant at the 5% level or below in all three periods, suggesting that smaller banks are relatively more profitable; the effect of having more branches is to reduce profitability; and revenue to total assets is largely unaffected by macroeconomic conditions – if anything, the banks appear to have been more profitable when GDP was growing more slowly.

11.6 The Random Effects Model

An alternative to the fixed effects model described above is the random effects model, which is sometimes also known as the error components model. As with fixed effects, the random effects approach proposes different intercept terms for each entity and again these intercepts are constant over time, with the relationships between the explanatory and explained variables assumed to be the same both cross-sectionally and temporally.

However, the difference is that under the random effects model, the intercepts for each cross-sectional unit are assumed to arise from a common intercept α (which is the same for all cross-sectional units and over time), plus a random variable ϵ_i that varies cross-sectionally but is constant over time. ϵ_i measures the random deviation of each entity's intercept term from the 'global' intercept term α. We can write the random effects panel model as

$$y_{it} = \alpha + \beta x_{it} + \omega_{it}, \quad \omega_{it} = \epsilon_i + v_{it} \tag{11.14}$$

where x_{it} is still a $1 \times k$ vector of explanatory variables, but unlike the fixed effects model, there are no dummy variables to capture the heterogeneity (variation) in the cross-sectional dimension. Instead, this occurs via the ϵ_i terms. Note that this framework requires the assumptions that the new cross-sectional error term, ϵ_i, has zero mean, is independent of the individual observation error term (v_{it}), has constant variance σ_ϵ^2 and is independent of the explanatory variables (x_{it}).

The parameters (α and the β vector) are estimated consistently but inefficiently by OLS, and the conventional formulae would have to be modified as a result of the cross-correlations between error terms for a given cross-sectional unit at different points in time. Instead, a generalised least squares procedure is usually used. The transformation involved in this GLS procedure is to subtract a weighted mean of the y_{it} over time (i.e., part of the mean rather than the whole mean, as was the case for fixed effects estimation). Define the 'quasi-demeaned' data as $y_{it}^* = y_{it} - \theta\bar{y}_i$ and $x_{it}^* = x_{it} - \theta\bar{x}_i$, where $\bar{y}_i$ and $\bar{x}_i$ are the means over time of the observations on y_{it} and x_{it}, respectively.[9] θ will be a function of the variance of the observation error term, σ_v^2, and of the variance of the entity-specific error term, σ_ϵ^2

$$\theta = 1 - \frac{\sigma_v}{\sqrt{T\sigma_\epsilon^2 + \sigma_v^2}} \tag{11.15}$$

This transformation will be precisely that required to ensure that there are no cross-correlations in the error terms, but fortunately it should automatically be implemented by standard software packages.

Just as for the fixed effects model, with random effects it is also conceptually no more difficult to allow for time variation than it is to allow for cross-sectional variation. In the case of time variation, a time period-specific error term is included

$$y_{it} = \alpha + \beta x_{it} + \omega_{it}, \quad \omega_{it} = \epsilon_t + v_{it} \tag{11.16}$$

and again, a two-way model could be envisaged to allow the intercepts to vary both cross-sectionally and over time. Box 11.1 discusses the choice between fixed effects and random effects models.

BOX 11.1 Fixed or random effects?

It is often said that the random effects model is more appropriate when the entities in the sample can be thought of as having been randomly selected from the population, but a fixed effect model is more plausible when the entities in the sample effectively constitute the entire population (for instance, when the sample comprises all of the stocks traded on a particular exchange).

More technically, the transformation involved in the GLS procedure under the random effects approach will not remove the explanatory variables that do not vary over time, and hence their impact on y_{it} can be enumerated. Also, since there are fewer parameters to be estimated with the random effects model (no dummy variables or within transformation to perform) and therefore degrees of freedom are saved, the random effects model should produce more efficient estimation than the fixed effects approach.

However, the random effects approach has a major drawback which arises from the fact that it is valid only when the composite error term ω_{it} is uncorrelated with all of the explanatory variables. This assumption is more stringent than the corresponding one in the fixed effects case, because with random effects we thus require both ϵ_i and v_{it} to be independent of all of the x_{it}. This can also be viewed as a consideration of whether any unobserved omitted variables (that were allowed for by having different intercepts for each entity) are uncorrelated with the included explanatory variables. If they are uncorrelated, a random effects approach can be used; otherwise the fixed effects model is preferable.

A test for whether this assumption is valid for the random effects estimator is based on a slightly more complex version of the Hausman test described in Section 7.6. If the assumption does not hold, the parameter estimates will be biased and inconsistent. To see how this arises, suppose that we have only one explanatory variable, x_{2it}, that varies positively with y_{it} and also with the error term, ω_{it}. The estimator will ascribe all of any increase in y to x when in reality some of it arises from the error term, resulting in biased coefficients.

11.7 Panel Data Application to Credit Stability of Banks in Central and Eastern Europe

Banking has become increasingly global over the past two decades, with domestic markets in many countries being increasingly penetrated by foreign-owned competitors. Foreign participants in the banking sector may improve competition and efficiency to the benefit of the economy that they enter, and they may have a stabilising effect on credit provision since they will probably be better diversified than domestic banks and will therefore be more able to continue to lend when the host economy is performing poorly. But it is also argued that foreign banks may alter the credit supply to suit their own aims rather than those of the host economy, and they may act more pro-cyclically than local banks, since they have alternative markets to withdraw their credit supply to when host market activity falls. Moreover, worsening conditions in the home country may force the repatriation of funds to support a weakened parent bank.

There may be differences in policies for credit provision dependent upon the nature of the formation of the subsidiary abroad. If the subsidiary's existence results from a take-over of a domestic bank, it is likely that the subsidiary will continue to operate the policies of, and in the same manner as, and with the same management as, the original separate entity, albeit in a diluted form. However, when the foreign bank subsidiary results from the formation of an entirely new startup operation (a 'greenfield investment'), the subsidiary is more likely to reflect the aims and objectives of the parent institution from the outset, and may be more willing to rapidly expand credit growth in order to obtain a sizeable foothold in the credit market as quickly as possible.

A study by de Haas and van Lelyveld (2006) employs a panel regression using a sample of around 250 banks from ten Central and East European countries to examine whether domestic and foreign banks react differently to changes in home or host economic activity and banking crises.

The data cover the period 1993–2000 and are obtained from BankScope. The core model is a random effects panel regression of the form

$$gr_{it} = \alpha + \beta_1 Takeover_{it} + \beta_2 Greenfield_i + \beta_3 Crisis_{it} + \beta_4 Macro_{it} + \beta_5 Contr_{it} + (\mu_i + \epsilon_{it}) \tag{11.17}$$

where the dependent variable, 'gr_{it}', is the percentage growth in the credit of bank i in year t; '$Takeover_{it}$' is a dummy variable taking the value 1 for foreign banks resulting from a takeover at time t and zero otherwise; '$Greenfield_i$' is a dummy taking the value 1 if bank i is the result of a foreign firm making a new banking investment rather than taking over an existing one; '*crisis*' is a dummy variable taking the value 1 if the host country for bank i was subject to a banking disaster in year t. '*Macro*|' is a vector of variables capturing the macroeconomic conditions in the home country (the lending rate and the change in GDP for the home and host countries, the host country inflation rate, and the differences in the home and host country GDP growth rates and the differences in the home and host country lending rates). '*Contr*' is a vector of bank-specific control variables that may affect the dependent variable irrespective of whether it is a foreign or domestic bank, and these are: '*weaknessparentbank*', defined as loan loss provisions made by the parent bank; '*solvency*', the ratio of equity to total assets; '*liquidity*', the ratio of liquid assets to total assets; '*size*', the ratio of total bank assets to total banking assets in the given country; '*profitability*', return on assets; and '*efficiency*', net interest margin. α and the βs are parameters (or vectors of parameters in the cases of β_4 and β_5), $\mu_i \sim IID(0, \sigma_\mu^2)$ is the unobserved random effect that varies across banks but not over time, and $\epsilon_{it} \sim IID(0, \sigma_\epsilon^2)$ is an idiosyncratic error term, $i = 1, \ldots, N$; $t = 1, \ldots, T_i$.

de Haas and van Lelyveld discuss the various techniques that could be employed to estimate such a model. OLS is considered to be inappropriate since it does not allow for differences in average credit market growth rates at the bank level. A model allowing for entity-specific effects (i.e. a fixed effects model that effectively allowed for a different intercept for each bank) would have been preferable to OLS (used to estimate a pooled regression), but is ruled out on the grounds that there are many more banks than time periods and thus too many parameters would be required to be estimated. They also argue that these bank-specific effects are not of interest to the problem at hand, which leads them to select the random effects panel model, that essentially allows for a different error structure for each bank. A Hausman test is conducted and shows that the random effects model is valid since the bank-specific effects (μ_i) are found, 'in most cases not to be significantly correlated with the explanatory variables'.

The results of the random effects panel estimation are presented in Table 11.3. Five separate regressions are conducted, with the results displayed in columns 2–6 of the table.[10] The regression is conducted on the full sample of banks and separately on the domestic and foreign bank sub-samples. The specifications allow in separate regressions for differences between host and home variables (denoted 'I', columns 2 and 5) and the actual values of the variables rather than the differences (denoted 'II', columns 3 and 6).

Table 11.3 Results of random effects panel regression for credit stability of Central and East European banks

Explanatory variables	Full sample I	Full sample II	Domestic banks	Foreign banks I	Foreign banks II
Takeover	−11.58 (1.26)	−5.65 (0.29)			
Greenfield	14.99 (1.29)	29.59 (1.55)		12.39 (0.88)	8.11 (0.65)
Crisis	−19.79*** (4.30)	−14.42*** (2.93)	−19.36*** (3.43)	0.31 (0.03)	−4.13 (0.33)
Host − home ΔGDP	8.08*** (4.18)			8.86*** (4.11)	
Host		6.68*** (7.39)	6.74*** (6.98)		8.64*** (2.93)
Home		−6.04* (1.89)			−8.62*** (2.78)
Host − home lending rate	1.12** (1.97)			0.85 (0.88)	
Host lending rate		0.28 (1.08)	0.34 (1.36)		1.50 (1.11)
Home lending rate		2.97*** (4.03)			1.11 (1.15)
Host inflation	−0.01 (0.37)	0.03 (1.01)	0.03 (0.12)	0.08 (0.61)	0.07 (0.44)
Weakness parent bank	−0.19*** (4.37)	−0.16*** (3.04)		−0.23*** (7.00)	−0.19*** (4.27)
Solvency	1.29*** (5.34)	1.25*** (4.77)	0.85*** (3.24)	3.33*** (5.53)	3.18*** (5.30)
Liquidity	−0.05** (2.09)	0.02 (0.78)	0.02 (0.70)	−0.53 (1.40)	−0.43 (1.14)
Size	−34.65** (1.96)	−29.14 (1.56)	−21.93 (1.16)	−108.00 (0.54)	−136.19 (0.72)

Profitability	1.09**	1.09**	1.21***	2.16	0.91
	(2.18)	(2.14)	(2.81)	(0.75)	(0.29)
Interest margin	1.66***	1.90***	2.71***	−3.42	−2.84
	(2.90)	(3.41)	(4.96)	(1.18)	(0.94)
Observations	1003	1003	770	233	233
No. of banks	247	247	184	82	82
Hausman test statistic	0.66	0.94	0.76	0.58	0.92
R^2	0.28	0.33	0.30	0.46	0.47

Notes: *t*-ratios in parentheses. Intercept and country dummy parameter estimates are not shown. Empty cells occur when a particular variable is not included in a regression.

Source: de Haas and van Lelyveld (2006). Reprinted with the permission of Elsevier.

The main result is that during times of banking disasters, domestic banks significantly reduce their credit growth rates (i.e. the parameter estimate on the *crisis* variable is negative for domestic banks), while the parameter is close to zero and not significant for foreign banks. There is a significant negative relationship between home country GDP growth, but a positive relationship with host country GDP growth and credit change in the host country. This indicates that, as the authors expected, when foreign banks have fewer viable lending opportunities in their own countries and hence a lower opportunity cost for the loanable funds, they may switch their resources to the host country. Lending rates, both at home and in the host country, have little impact on credit market share growth. Interestingly, the greenfield and takeover variables are not statistically significant (although the parameters are quite large in absolute value), indicating that the method of investment of a foreign bank in the host country is unimportant in determining its credit growth rate or that the importance of the method of investment varies widely across the sample, leading to large standard errors. A weaker parent bank (with higher loss provisions) leads to a statistically significant contraction of credit in the host country as a result of the reduction in the supply of available funds. Overall, both home-related ('push') and host-related ('pull') factors are found to be important in explaining foreign bank credit growth.

11.8 Panel Unit Root and Cointegration Tests

11.8.1 Background and Motivation

The principle of unit root testing in the panel context is very similar to that employed in single equations framework discussed in Chapter 8. We noted there that unit root tests of the Dickey–Fuller and Phillips–Perron types have low power, especially for modest sample sizes. This provides a key motivation for using a panel – the hope that more powerful versions of the tests can be employed when time series and crosssectional information is combined – as a result of the increase in sample size. Of course, it would be easier to increase the number of observations by simply increasing the length of the sample period, but this data may not be available, or may be of limited use because of structural breaks in the time series.

While the single series and panel approaches to unit root and stationarity testing appear very similar on the surface, in fact a valid construction and application of the test statistics is much more complex for panels than for single series. One complication arises since different asymptotic distributions for the test statistics may result depending on whether N is fixed and T tends to infinity, or vice versa, or both T and N increase simultaneously in a fixed ratio.

Two important issues to consider are first, that the design and interpretation of the null and alternative hypotheses needs careful thought in the panel arena and second, there may be a problem of cross-sectional dependence in the errors across the unit root testing regressions. Some of the literature refers to the early studies that assumed cross-sectional independence as 'first generation' panel unit root tests, while the more recent approaches that allow for some form of dependence are termed 'second generation' tests.

A perhaps obvious starting point for unit root tests when one has a panel of data would be to run separate regressions over time for each series but to use Zellner's SUR approach, which we might term the multivariate ADF (MADF) test. This method can only be employed if $T >> N$, and Taylor and Sarno (1998) provide an early application to tests for purchasing power parity. However, it is fair to say that technique is now rarely used, researchers preferring instead to make use of the full panel structure.

A key consideration is the dimensions of the panel – is the situation that T is large or that N is large or both? If T is large and N small, the MADF approach can be used, although as Breitung and Pesaran (2008) note, in such a situation one may question whether it is worthwhile to adopt a panel approach at all, since for sufficiently large T, separate ADF tests ought to be reliable enough to render the panel approach hardly worth the additional complexity.

11.8.2 Tests with Common Alternative Hypotheses

Levin, Lin and Chu (2002) – hereafter LLC – develop a test based on the equation

$$\Delta y_{i,t} = \alpha_i + \theta_t + \delta_i t + \rho_i y_{i,t-1} + \sum \gamma_j \Delta y_{t-j} + v_{i,t} \quad t = 1, 2, \ldots, T; i = 1, 2, \ldots, N \tag{11.18}$$

The model is very general since it allows for both entity-specific and time-specific effects through α_i and θ_t, respectively, as well as separate deterministic trends in each series through $\delta_i t$, and the lag structure to mop up autocorrelation in Δy. Of course, as for the Dickey–Fuller tests, any or all of these deterministic terms can be omitted from the regression. The null hypothesis is $H_0 : \rho_i \equiv \rho = 0 \ \forall \ i$ and the alternative is $H_1 : \rho < 0 \ \forall \ i$.

One of the reasons that unit root testing is more complex in the panel framework in practice is due to the plethora of 'nuisance parameters' in the equation which are necessary to allow for the fixed effects (i.e., the α_i, θ_t, $\delta_i t$). These nuisance parameters will affect the asymptotic distribution of the test statistics and hence LLC propose that two auxiliary regressions are run to remove their impacts. First, Δy_{it} is regressed on its lags, Δy_{it-j}, $j = 1, \ldots, p_i$ and on the exogenous variables (any or all from α_i, θ_t, and $\delta_i t$ as desired); the residuals, u_{1it} are obtained. Note that the numbers of lags of the dependent variables, p_i, need not be the same for each series in the panel. Next, the lagged level of y, y_{it-1}, is regressed on the same variables to get the residuals, u_{2it}. Then the residuals from both regressions are standardised by dividing them by the regression standard error, s_i, which is obtained from the augmented Dickey–Fuller regression in equation (11.18)

$$\tilde{u}_{1it} = u_{1it}/s_i \tag{11.19}$$

and

$$\tilde{u}_{2it} = u_{2it}/s_i \tag{11.20}$$

Thus $\tilde{u}_{1it}$ will be equivalent to Δy_{it} but with the effects of the deterministic components removed, and $\tilde{u}_{2it}$ will be equivalent to y_{it-1} but with the effects of the deterministic components removed. Finally, $\tilde{u}_{1it}$ is regressed on $\tilde{u}_{2it}$, and the slope estimate from this test regression is then used to construct a test statistic which is asymptotically distributed as a standard normal variate. The test statistic will approach this 'limiting' normal distribution as T tends to infinity and as N tends to infinity, although the convergence is faster for the former than the latter.

Breitung (2000) develops a modified version of the LLC test which does not include the deterministic terms (i.e., the fixed effects and/or a deterministic trend), and which standardises the residuals from the auxiliary regression in a more sophisticated fashion.

It should be clear that under the LLC and Breitung approaches, only evidence against the non-stationary null in one series is required before the joint null will be rejected. Breitung and Pesaran (2008) suggest that the appropriate conclusion when the null is rejected is that 'a significant proportion of the cross-sectional units are stationary'. Especially in the context of large N, this might not be very helpful since no information is provided on how many of the N series are stationary. Often, the homogeneity assumption is not economically meaningful either, since there is no theory suggesting that all of the series have the same autoregressive dynamics and thus the same value of ρ.

11.8.3 Panel Unit Root Tests with Heterogeneous Processes

The difficulty described at the end of the previous sub-section led Im, Pesaran and Shin (2003) – hereafter IPS – to propose an alternative approach where, given equation (11.18) as above, the null and alternative hypotheses are now $H_0 : \rho_i = 0 \ \forall \ i$ and $H_1 : \rho_i < 0, i = 1, 2, \ldots, N_1; \rho_i = 0, i = N_1 + 1, N_1 + 2, \ldots, N$.

So the null hypothesis still specifies all series in the panel as non-stationary, but under the alternative, a proportion of the series (N_1/N) are stationary, and the remaining proportion ($(N - N_1)/N$) are non-stationary. But it is clear that no restriction where all of the ρ are identical is imposed. The statistic for the panel test in this case is constructed by conducting separate unit root tests for each series in the panel, calculating the ADF t-statistic for each one in the standard fashion, and then taking their cross-sectional average. This average is then transformed into a standard normal variate under the null hypothesis of a unit root in all the series; IPS develop an LM-test approach as well as the more familiar t-test.[11] If the time series dimension is sufficiently large, it is then possible to run separate unit root tests on each series in order to determine the proportion for which the individual tests cause a rejection, and thus how strong is the weight of evidence against the joint null hypothesis.

It should be noted that while IPS's heterogeneous panel unit root tests are superior to the homogeneous case when N is modest relative to T, they may not be sufficiently powerful when N is large and T is small, in which case the LLC approach may be preferable.

Maddala and Wu (1999) and Choi (2001) developed a slight variant on the IPS approach based on an idea dating back to Fisher (1932), where unit root tests are again conducted separately on each series in the panel, and the p-values associated with the test statistics are then combined. If we call these p-values pv_i, $i = 1, 2, \ldots, N$, then under the null hypothesis of a unit root in each series, each pv_i will be distributed uniformly over the [0,1] interval and hence the following will hold for given N as $T \to \infty$

$$\lambda = -2\sum_{i=1}^{N} ln(pv_i) \sim \chi^2_{2N}. \quad \textbf{(11.21)}$$

The number of observations per series can differ in this case as the regressions are run separately for each series and then only their p-values are combined in the test statistic. Notice that the cross-sectional independence assumption is crucial here for this sum to follow a χ^2 distribution. Since the distribution of the ADF test statistic is non-standard and is dependent upon the inclusion of the nuisance parameters, unfortunately the p-values for inclusion in this equation must be obtained from a Monte Carlo simulation. Moreover, if the series under consideration have different lag lengths for Δy_{it} or there are different numbers of observations, each will require a separate Monte Carlo!

As well as the χ^2 statistic, Choi (2001) develops a variant of the test, still based on the p-values, that is asymptotically standard normally distributed. It should be evident that, like IPS, the Maddala–Wu–Choi approach does not require the same parameter, ρ, to apply to all of the series since the ADF test is run separately on each series in the panel.

11.8.4 Panel Stationarity Tests

The approaches described above are non-stationarity tests, and analogous to the Dickey–Fuller approach, they have non-stationarity under the null hypothesis. It is also possible, however, to construct a test where the null hypothesis is of stationarity for all series in the panel, analogous to the KPSS test of Kwaitkowski *et al.* (1992). In this case, the null hypothesis is that all of the series are stationary, which is rejected if at least one of them is non-stationary. This approach in the panel context was developed by

Hadri (2000), and leads to a test statistic that is asymptotically normally distributed. As in the univariate case, stationarity tests can be useful as a way to check for the robustness of the conclusions from unit root tests.

11.8.5 Allowing for Cross-Sectional Heterogeneity

The assumption of cross-sectional independence of the error terms in the panel regression is highly unrealistic and likely to be violated in practice. For example, in the context of testing for whether purchasing power parity holds, there are likely to be important unspecified factors that affect all exchange rates or groups of exchange rates in the sample, and will result in correlated residuals. O'Connell (1998) demonstrates the considerable size distortions that can arise when such cross-sectional dependencies are present but not accounted for – that is, the null hypothesis is rejected far too frequently when it is correct than should arise by chance alone if the distributional assumption holds for the test statistic. If the critical values employed in the tests are adjusted to remove the impacts of these size distortions, then the power of the tests will fall such that in extreme cases the benefit of using a panel structure could disappear completely. According to Maddala and Wu (1999), tests based on the Fisher statistic are more robust in the presence of unparameterised cross-sectional dependence than the IPS approach.

O'Connell proposes a feasible GLS estimator for ρ where an assumed non-zero form for the correlations between the disturbances is employed. To overcome the limitation that the correlation matrix must be specified (and this may be troublesome because it is not clear what form it should take), Bai and Ng (2004) propose an approach based on separating the data into a common factor component that is highly correlated across the series and a specific part that is idiosyncratic; a further approach is to proceed with OLS but to employ modified standard errors – so-called 'panel corrected standard errors' (PCSEs) – see, for example Breitung and Das (2005).

Overall, however, it is clear that satisfactorily dealing with cross-sectional dependence makes an already complex issue considerable harder still. In the presence of such dependencies, the test statistics are affected in a non-trivial way by the nuisance parameters. As a result, despite their inferiority in theory, the first generation approaches that ignore cross-sectional dependence are still widely employed in the empirical literature.

11.8.6 Panel Cointegration

It is often remarked in the literature that the development of the techniques for panel cointegration modelling is still in its infancy, while that for panel unit root testing is already quite mature. Testing for cointegration in panels is a rather complex issue, since one must consider the possibility of cointegration across groups of variables (what we might term ‘cross-sectional cointegration’) as well as within the groups. It is also possible that the parameters in the cointegrating series and even the number of cointegrating relationships could differ across the panel.

Most of the work so far has relied upon a generalisation of the single equation methods of the Engle–Granger type following the pioneering work by Pedroni (1999, 2004). His setup is very general and allows for separate intercepts for each group of potentially cointegrating variables and separate deterministic trends. For a set of variables y_{it} and $x_{m,i,t}$ that are individually integrated of order one and thought to be cointegrated

$$y_{it} = \alpha_i + \delta_i t + \beta_{1i}x_{1i,t} + \beta_{2i}x_{2i,t} + \ldots + \beta_{Mi}x_{Mi,t} + u_{i,t} \qquad \textbf{(11.22)}$$

where m = 1, …, M are the explanatory variables in the potentially cointegrating regression; t = 1, …, T and i = 1, …, N.

The residuals from this regression, $\hat{u}_{i,t}$ are then subjected to separate Dickey–Fuller or augmented Dickey–Fuller type regressions for each group of variables to determine whether they are I(1) – for example

$$\hat{u}_{i,t} = \rho_i \hat{u}_{i,t-1} + \sum_{j=1}^{p_i} \psi_{i,j}\Delta\hat{u}_{i,t-j} + v_{i,t} \qquad \textbf{(11.23)}$$

The null hypothesis is that the residuals from all of the test regressions are unit root processes ($H_0 : \rho_i = 1$), and therefore that there is no cointegration. Pedroni proposes two possible alternative hypotheses – first, that all of the autoregressive dynamics are the same stationary process ($H_1 : \rho_i = \rho < 1 \ \forall \ i$) and second, that the dynamics from each test equation follow a different stationary process ($H_1 : \rho_i < 1 \ \forall \ i$). Hence, in the first case no heterogeneity is permitted, while in the second it is – analogous to the difference between LLC and IPS as described above. Pedroni then constructs a raft of different test statistics based on standardised versions of the usual t-ratio from equation (11.23). The standardisation required is a function of whether an intercept or trend is included in equation (11.23), and the value of M. These standardised test statistics are each asymptotically standard normally distributed.

Kao (1999) essentially develops a restricted version of Pedroni's approach, where the slope parameters in equation (11.22) are assumed to be fixed across the groups, although the intercepts are still permitted to vary. Then the DF or ADF test regression is run on a pooled sample assuming homogeneity in the value of ρ. These restrictions allow some simplification in the testing approach.

As well as testing for cointegration using the residuals following these extensions of Engle and Granger, it is also possible, although in general more complicated, to use a generalisation of the Johansen technique. This approach is deployed by Larsson, Lyhagen and Lothgren (2001), but a simpler alternative is to apply the Johansen approach to each group of series separately, collect the p-values for the trace test and then take −2 times the sum of their logs following Maddala and Wu (1999) as in equation (11.21) above. A full systems approach based on a 'global VAR' is possible but with considerable additional complexity – see Breitung and Pesaran (2008) and the many references therein for further details.

11.8.7 An Illustration of the Use of Panel unit Root and Cointegration Tests: The Link Between Financial Development and GDP Growth

An important issue for developing countries from a policy perspective is the extent to which economic growth and the sophistication of the country's financial markets are linked. It has been argued in the relevant literature that excessive government regulations (such as limits on lending, restrictions on lending and borrowing interest rates, the barring of foreign banks, etc.) may impede the development of the financial markets and consequently economic growth will be slower than if the financial markets were more vibrant. On the other hand, if economic agents are able to borrow at reasonable rates of interest or raise funding easily on the capital markets, this can increase the viability of real investment opportunities and allow for a more efficient allocation of capital.

Both the theoretical and empirical research in this area has led to mixed conclusions; the theoretical models arrive at different findings dependent upon the framework employed and the assumptions made. And on the empirical side, many existing studies in this area are beset by two issues: first, the direction of causality between economic and financial development could go the other way: if an economy grows, then the demand for financial products will itself increase. Thus it is possible that economic growth leads to financial market development rather than the other line of causality. Second, given that long time series are typically unavailable for developing economies, traditional unit root and cointegration tests that examine the link between these two variables suffer from low power. In particular, while research has been able to identify a link between economic growth and stock market development, such an effect could not be identified for the sophistication of the banking sector. This provides a strong motivation for the use of panel techniques, which are more powerful, and which constitute the approach adopted by Christopoulos and Tsionas (2004). Some of the key methodologies and findings of their paper will now be discussed.

Defining real output for country i as y_{it}, financial 'depth' as F, the proportion of total output that is investment as S, and the rate of inflation as p, the core model they employ is

$$y_{it} = \beta_{0i} + \beta_{1i}F_{it} + \beta_{2i}S_{it} + \beta_{3i}\dot{p}_{it} + u_{it}. \quad \textbf{(11.24)}$$

Financial depth, F, is proxied using the ratio of total bank liabilities to GDP. Christopoulos and Tsionas obtain data from the IMF's *International Financial Statistics* for ten countries (Colombia, Paraguay, Peru, Mexico, Ecuador, Honduras, Kenya, Thailand, the Dominican Republic and Jamaica) over the period 1970–2000.

The regression in equation (11.24) has national output as the dependent variable, and financial development as one of the independent variables, but Christopoulos and Tsionas also investigate the reverse causality with F as the dependent variable and y as one of the independent variables. They first apply unit root tests to each of the individual series (output, financial depth, investment share in GDP, and inflation) separately for the ten countries. The findings are mixed, but show that most series are best characterised by unit root processes in levels but are stationary in first differences. They then employ the panel unit root tests of Im, Pesaran and Shin, and the Maddala–Wu chi-squared test separately for each variable, but now using a panel comprising all ten countries. The number of lags of Δy_{it} is determined using AIC. The null hypothesis in all cases is that the process is a unit root. Now the results, presented here in Table 11.4, are much stronger and show conclusively that all four series are non-stationary in levels but stationary in differences.

Table 11.4 Panel unit root test results for economic growth and financial development

Variables	Levels		First differences	
	IPS	Maddala–Wu	IPS	Maddala–Wu
Output (y)	−0.18	27.12	−4.52***	58.33***
Financial depth (F)	2.71	14.77	−6.63***	83.64***
Investment share (S)	−0.04	30.37	−5.81***	62.98***
Inflation ($\dot{p}$)	−0.47	26.37	−5.19***	74.29***

Notes: The critical value for the Maddala–Wu test is 37.57 at the 1% level. *** denotes rejection of the null hypothesis of a unit root at the 1% level.

Source: Christopoulos and Tsionas (2004). Reprinted with the permission of Elsevier.

The next stage is to test whether the series are cointegrated, and again this is first conducted separately for each country and then using a panel approach. Focusing on the latter, the LLC approach is used along with the Harris–Tzavalis (1999) technique, which is broadly the same as LLC but has slightly different correction factors in the limiting distribution owing to its assumption that T is fixed as N tends to infinity. As discussed in the previous sub-section, these techniques are based on a unit root test on the residuals from the potentially cointegrating regression, and Christopoulos and Tsionis investigate the use of panel cointegration tests with fixed effects, and with both fixed effects and a deterministic trend in the test regressions. These are applied to the regressions both with y, and separately F, as the dependent variables.

The results in Table 11.5 quite strongly demonstrate that when the dependent variable is output, the LLC approach rejects the null hypothesis of a unit root in the potentially cointegrating regression residuals when fixed effects only are included in the test regression, but not when a trend is also included. In the context of the Harris–Tzavalis variant of the residuals-based test, for both the fixed effects and the fixed effects + trend regressions, the null is rejected. When financial depth is instead used as the dependent variable, none of these tests reject the null hypothesis. Thus, the weight of evidence from the residuals-based tests is that cointegration exists when output is the dependent variable, but it does not when financial depth is. The authors interpret this result as implying that causality runs from output to financial depth but not the other way around.

Table 11.5 Panel cointegration test results for economic growth and financial development

	LLC		Harris–Tzavalis	
	Fixed effects	**Fixed effects + trend**	**Fixed effects**	**Fixed effects + trend**
Dep. var.: y	−8.36***	0.89	−77.13***	−5.57***
Dep. var.: F	−1.2 $r = 0$	0.5 $r \leq 1$	−0.85 $r \leq 2$	−1.65 $r \leq 3$
Fisher χ^2	76.09***	30.73	28.91	23.26

Notes: 'Dep. var.' denotes the dependent variable; *** denotes rejection of the null hypothesis of no cointegration at the 2% level. The critical values for the Fisher test are 37.57 and 31.41 at the 1% and 5% levels, respectively.

Source: Christopoulos and Tsionas (2004). Reprinted with the permission of Elsevier.

In the final row of Table 11.5, a systems approach to testing for cointegration, based on the sum of the logs of the *p*-values from the Johansen test, shows that the null hypothesis of no cointegrating vectors ($H_0 : r = 0$) is rejected, while ($H_0 : r \leq 1$) and above are all not rejected. Thus the conclusion is that one cointegrating relationship exists between the four variables across the panel. Note that in this case, since cointegration is tested within a VAR system, all variables are treated in parallel, and hence there are not separate results for different dependent variables.

11.9 Further Feading

Some readers may feel that further instruction in this area could be useful. If so, the classic specialist references to panel data techniques are Baltagi (2005) and Hsiao (2003) and further references are Arellano (2003) and Wooldridge (2010). All four are extremely detailed and have excellent referencing to recent developments in the theory of panel model specification, estimation and testing. However, all also require a high level of mathematical and econometric ability on the part of the reader. A more intuitive and accessible, but less detailed, treatment is given in Kennedy (2003, Chapter 17). Some examples of financial studies that employ panel techniques and outline the methodology sufficiently descriptively to be worth reading as aids to learning are given in the examples above. The book by Maddala and Kim (1999) provides a fairly accessible treatment of unit roots and cointegration generally, although the time of publication implies that the most recent developments are excluded. Breitung and Pesaran (2008) is a more recent survey and is comprehensive, but at a higher technical level.

KEY CONCEPTS

The key terms to be able to define and explain from this chapter are

- pooled data
- fixed effects
- random effects
- within transform
- between estimation
- panel cointegration test
- seemingly unrelated regression
- least squares dummy variable estimation
- Hausman test
- time-fixed effects
- panel unit root test

SELF-STUDY QUESTIONS

1. (a) What are the advantages of constructing a panel of data, if one is available, rather than using pooled data?

(b) What is meant by the term ‘seemingly unrelated regression’? Give examples from finance of where such an approach may be used.

(c) Distinguish between balanced and unbalanced panels, giving examples of each.

2. (a) Explain how fixed effects models are equivalent to an ordinary least squares regression with dummy variables.

(b) How does the random effects model capture cross-sectional heterogeneity in the intercept term?

(c) What are the relative advantages and disadvantages of the fixed versus random effects specifications and how would you choose between them for application to a particular problem?

3. Find a further example of where panel regression models have been used in the academic finance literature and do the following:

- Explain why the panel approach was used.
- Was a fixed effects or random effects model chosen and why?
- What were the main results of the study and is any indication given about whether the results would have been different had a pooled regression been employed instead in this or in previous studies?

4. (a) What are the advantages and disadvantages of conducting unit root tests within a panel framework rather than series by series?

(b) Explain the differences between panel unit root tests based on a common alternative hypothesis and those based on heterogeneous processes.

1 Hence, strictly, if the data are not on the same entities (for example, different firms or people) measured over time, then this would not be panel data.

2 Note that *k* is defined slightly differently in this chapter compared with others in the book. Here, *k* represents the number of slope parameters to be estimated (rather than the total number of parameters as it is elsewhere), which is equal to the number of explanatory variables in the regression model.

3 For example, the SUR framework has been used to test the impact of the introduction of the euro on the integration of European stock markets (Kim, Moshirian and Wu, 2005), in tests of the CAPM, and in tests of the forward rate unbiasedness hypothesis (Hodgson, Linton and Vorkink, 2004).

4 It is important to recognise this limitation of panel data techniques that the relationship between the explained and explanatory variables is assumed constant both cross-sectionally and over time, even if the varying intercepts allow the average values to differ. The use of panel techniques rather than estimating separate time series regressions for each object or estimating separate cross-sectional regressions for each time period thus implicitly assumes that the efficiency gains from doing so outweigh any biases that

may arise in the parameter estimation.

5 It is known as the *within transformation* because the subtraction is made within each cross-sectional object.

6 An advantage of running the regression on average values (the *between estimator*) over running it on the demeaned values (the *within estimator*) is that the process of averaging is likely to reduce the effect of measurement error in the variables on the estimation process.

7 Interestingly, while many casual observers believe that concentration in UK retail banking has grown considerably, it actually fell slightly between 1986 and 2002.

8 A Chow test for structural stability reveals a structural break between the two sub-samples. No other commentary on the results of the equilibrium regression is given by the authors.

9 The notation used here is a slightly modified version of Kennedy (2003, p. 315).

10 de Haas and van Lelyveld employ corrections to the standard errors for heteroscedasticity and autocorrelation. They additionally conduct regressions including interactive dummy variables, although these are not discussed here.

11 Both tests presume that there is a balanced panel – that is, the number of time series observations is the same for each cross-sectional entity.

12

Limited Dependent Variable Models

LEARNING OUTCOMES

In this chapter, you will learn how to

- Compare between different types of limited dependent variables and select the appropriate model
- Interpret and evaluate logit and probit models
- Distinguish between the binomial and multinomial cases
- Deal appropriately with censored and truncated dependent variables

12.1 Introduction and Motivation

Chapters 5 and 10 have shown various uses of dummy variables to numerically capture the information qualitative variables – for example, day-of-the-week effects, gender, credit ratings, etc. When a dummy is used as an explanatory variable in a regression model, this usually does not give rise to any particular problems (so long as one is careful to avoid the *dummy variable trap* – see Chapter 10). However, there are many situations in financial research where it is the explained variable, rather than one or more of the explanatory variables, that is qualitative. The qualitative information would then be coded as a dummy variable and the situation would be referred to as a *idiscrete choice* variable and needs to be treated differently. The term refers to any problem where the values that the dependent variables may take are limited to certain integers (e.g., 0, 1, 2, 3, 4) or even where it is a binary number (only 0 or 1, which would then be known as a *binary choice* variable).

Discrete choice variables are one set from among what are known more generally as *limited dependent variables*, since the values they can take are limited to only certain integers. Another class of limited dependent variables are where the data that we see are censored or truncated in some way – in other words, we can only observe the true values for part of the distribution while for the remainder above or below some fixed threshold, the true values remain latent. We will return to censored and truncated series – and the differences between them – later in the chapter.

There are numerous examples of instances where the dependent variable may arise from a binary choice, for example where we want to model

- Why firms choose to list their shares on the NASDAQ rather than the NYSE
- Why some stocks pay dividends while others do not
- What factors affect whether countries default on their sovereign debt
- Why some firms choose to issue new stock to finance an expansion while others issue bonds
- Why some firms choose to engage in stock splits while others do not.

It is fairly easy to see in all these cases that the appropriate form for the dependent variable would be a 0–1 dummy variable since there are only two possible outcomes. There are, of course, also situations where it would be more useful to allow the dependent variable to take on other values, but these will be considered later in Section 12.9. We will first examine a simple and obvious, but unfortunately flawed, method for dealing with binary dependent variables, known as the *linear probability model*.

12.2 The Linear Probability Model

The linear probability model (LPM) is by far the simplest way of dealing with binary dependent variables, and it is based on an assumption that the probability of an event occurring, P_i, is linearly related to a set of explanatory variables $x_{2i}, x_{3i}, \ldots, x_{ki}$

$$P_i = p(y_i = 1) = \beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + \cdots + \beta_k x_{ki} + u_i, \quad i = 1, \ldots, N \qquad \textbf{(12.1)}$$

The actual probabilities cannot be observed, so we would estimate a model where the outcomes, y_i (the series of zeros and ones), would be the dependent variable. This is then a linear regression model and would be estimated by OLS. The set of explanatory variables could include either quantitative variables or dummies or both. The fitted values from this regression are the estimated probabilities for $y_i = 1$ for each observation i. The slope estimates for the linear probability model can be interpreted as the change in the probability that the dependent variable will equal 1 for a one-unit change in a given explanatory variable, holding the effect of all other explanatory variables fixed. Suppose, for example, that we wanted to model the probability that a firm i will pay a dividend ($y_i = 1$) as a function of its market capitalisation (x_{2i}, measured in millions of US dollars), and we fit the following line:

$$\hat{P}_i = -0.3 + 0.012x_{2i} \tag{12.2}$$

where $\hat{P}_i$ denotes the fitted or estimated probability for firm i. This model suggests that for every \$1m increase in size, the probability that the firm will pay a dividend increases by 0.012 (or 1.2%). A firm whose stock is valued at \$50m will have a $-0.3 + 0.012 \times 50 = 0.3$ (or 30%) probability of making a dividend payment. Graphically, this situation may be represented as in Figure 12.1.

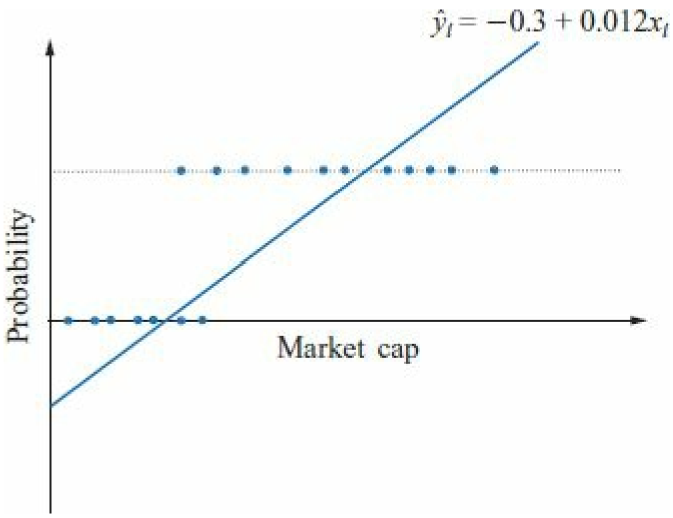

Figure 12.1 The fatal flaw of the linear probability model

While the linear probability model is simple to estimate and intuitive to interpret, the diagram should immediately signal a problem with this setup. For any firm whose value is less than \$25m, the model-predicted probability of dividend payment is negative, while for any firm worth more than \$88m, the probability is greater than one. Clearly, such predictions cannot be allowed to stand, since the probabilities should lie within the range (0,1). An obvious solution is to truncate the probabilities at 0 or 1, so that a probability of −0.3, say, would be set to zero, and a probability of, say, 1.2 would be set to 1. However, there are at least two reasons why this is still not adequate

(1) The process of truncation will result in too many observations for which the estimated probabilities are exactly zero or one.

(2) More importantly, it is simply not plausible to suggest that the firm's probability of paying a dividend is either exactly zero or exactly one. Are we really certain that very small firms will definitely never pay a dividend and that large firms will always make a payout? Probably not, so a different kind of model is usually used for binary dependent variables – either a *logit* or a *probit* specification. These approaches will be discussed in the following sections. But before moving on, it is worth noting that the LPM also suffers from a couple of more standard econometric problems that we have examined in previous chapters. First, since the dependent variable takes only one of two values, for given (fixed in repeated samples) values of the explanatory variables, the disturbance term will also take on only one of two values.[1] Consider again equation (12.1). If $y_i = 1$, then by definition

$$u_i = 1 - \beta_1 - \beta_2 x_{2i} - \beta_3 x_{3i} - \cdots - \beta_k x_{ki};$$

but if $y_i = 0$, then

$$u_i = -\beta_1 - \beta_2 x_{2i} - \beta_3 x_{3i} - \cdots - \beta_k x_{ki}.$$

Hence the error term cannot plausibly be assumed to be normally distributed. Since u_i changes systematically with the explanatory variables, the disturbances will also be heteroscedastic. It is therefore essential that heteroscedasticity-robust standard errors are always used in the context of limited dependent variable models.

12.3 The Logit Model

Both the logit and probit model approaches are able to overcome the limitation of the LPM that it can produce estimated probabilities that are negative or greater than one. They do this by using a function that effectively transforms the regression model so that the fitted values are bounded within the (0,1) interval. Visually, the fitted regression model will appear as an S-shape rather than a straight line, as was the case for the LPM. This is shown in Figure 12.2.

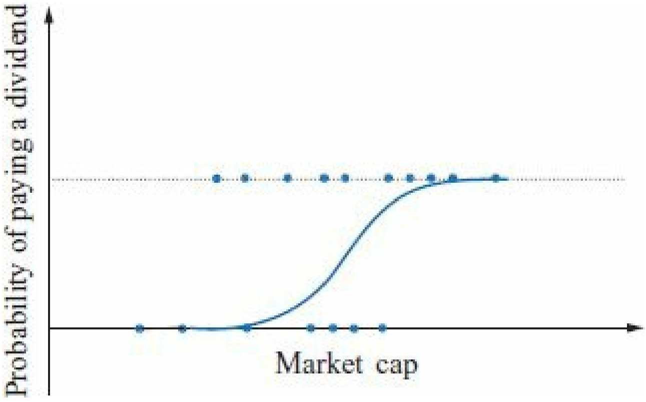

Figure 12.2 The logit model

The logistic function F, which is a function of any random variable, z, would be

$$F(z_i) = \frac{e^{z_i}}{1 + e^{z_i}} = \frac{1}{1 + e^{-z_i}} \tag{12.3}$$

where e is the exponential under the logit approach. The model is so called because the function F is in fact the cumulative logistic distribution. So the logistic model estimated would be

$$P_i = \frac{1}{1 + e^{-(\beta_1 + \beta_2 x_{2i} + \cdots + \beta_k x_{ki} + u_i)}} \tag{12.4}$$

where again P_i is the probability that $y_i = 1$.

With the logistic model, 0 and 1 are asymptotes to the function and thus the probabilities will never actually fall to exactly zero or rise to one, although they may come infinitesimally close. In equation (12.3), as z_i tends to infinity, e^{-z_i} tends to zero and $1/(1 + e^{-z_i})$ tends to 1; as z_i tends to minus infinity, e^{-z_i} tends to infinity and $1/(1 + e^{-z_i})$ tends to 0.

Clearly, this model is not linear (and cannot be made linear by a transformation) and thus is not estimable using OLS. Instead, maximum likelihood is usually used – this is discussed in Section 12.7 and in more detail in the appendix to this chapter.

12.4 Using a Logit to Test the Pecking Order

Hypothesis

This section examines a study of the pecking order hypothesis due to Helwege and Liang (1996). The theory of firm financing suggests that corporations should use the cheapest methods of financing their activities first (i.e. the sources of funds that require payment of the lowest rates of return to investors) and switch to more expensive methods only when the cheaper sources have been exhausted. This is known as the ‘pecking order hypothesis’, initially proposed by Myers (1984). Differences in the relative cost of the various sources of funds are argued to arise largely from information asymmetries since the firm’s senior managers will know the true riskiness of the business, whereas potential outside investors will not.[2] Hence, all else equal, firms will prefer internal finance and then, if further (external) funding is necessary, the firm’s riskiness will determine the type of funding sought. The more risky the firm is perceived to be, the less accurate will be the pricing of its securities.

Helwege and Liang (1996) examine the pecking order hypothesis in the context of a set of US firms that had been newly listed on the stock market in 1983, with their additional funding decisions being tracked over the 1984–92 period. Such newly listed firms are argued to experience higher rates of growth, and are more likely to require additional external funding than firms which have been stock market listed for many years. They are also more likely to exhibit information asymmetries due to their lack of a track record. The list of initial public offerings (IPOs) came from the Securities Data Corporation and the Securities and Exchange Commission with data obtained from Compustat.

A core objective of the paper is to determine the factors that affect the probability of raising external financing. As such, the dependent variable will be binary – that is, a column of 1s (firm raises funds externally) and 0s (firm does not raise any external funds). Thus OLS would not be appropriate and hence a logit model is used. The explanatory variables are a set that aims to capture the relative degree of information asymmetry and degree of riskiness of the firm. If the pecking order hypothesis is supported by the data, then firms should be more likely to raise external funding the less internal cash they hold. Hence variable ‘deficit’ measures (capital expenditures + acquisitions + dividends − earnings). ‘Positive deficit’ is a variable identical to deficit but with any negative deficits (i.e. surpluses) set to zero; ‘surplus’ is equal to the negative of deficit for firms where deficit is negative; ‘positive deficit × operating income’ is an interaction term where the two variables are multiplied together to capture cases where firms have strong investment opportunities but limited access to internal funds; 'assets' is used as a measure of firm size; 'industry asset growth' is the average rate of growth of assets in that firm's industry over the 1983–92 period; 'previous financing' is a dummy variable equal to 1 for firms that obtained external financing in the previous year. The results from the logit regression are presented in Table 12.1.

Table 12.1 Logit estimation of the probability of external financing

Variable	(1)	(2)	(3)
Intercept	−0.29 (−3.42)	−0.72 (−7.05)	−0.15 (−1.58)
Deficit	0.04 (0.34)	0.02 (0.18)	
Positive deficit			−0.24 (−1.19)
Surplus			−2.06 (−3.23)
Positive deficit × operating income			−0.03 (−0.59)
Assets	0.0004 (1.99)	0.0003 (1.36)	0.0004 (1.99)
Industry asset growth	−0.002 (−1.70)	−0.002 (−1.35)	−0.002 (−1.69)
Previous financing		0.79 (8.48)	

Note: a blank cell implies that the particular variable was not included in that regression; *t*-ratios in parentheses; only figures for all years in the sample are presented.

Source: Helwege and Liang (1996). Reprinted with the permission of Elsevier.

The key variable, 'deficit,' has a parameter that is not statistically significant and hence the probability of obtaining external financing does not depend on the size of a firm's cash deficit.[3] The parameter on the 'surplus' variable has the correct negative sign, indicating that the larger a firm's surplus, the less likely it is to seek external financing, which provides some limited support for the pecking order hypothesis. Larger firms (with larger total assets) are more likely to use the capital markets, as are firms that have already obtained external financing during the previous year.

12.5 The Probit Model

Instead of using the cumulative logistic function to transform the model, the cumulative normal distribution is sometimes used instead. This gives rise to the probit model. The function F in equation (12.3) is replaced by

$$F(z_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_i} e^{-\frac{z_i^2}{2}} dz \tag{12.5}$$

This function is the cumulative distribution function for a standard normally distributed random variable. As for the logistic approach, this function provides a transformation to ensure that the fitted probabilities will lie between zero and one. Also as for the logit model, the marginal impact of a unit change in an explanatory variable, x_{4i} say, will be given by $\beta_4 F(z_i)$, where β_4 is the parameter attached to x_{4i} and $z_i = \beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + \cdots + u_i$.

12.6 Choosing Between the Logit and Probit Models

For the majority of the applications, the logit and probit models will give very similar characterisations of the data because the densities are very similar. That is, the fitted regression plots (such as figure 12.2) will be virtually indistinguishable and the implied relationships between the explanatory variables and the probability that $y_i = 1$ will also be very similar. Both approaches are much preferred to the linear probability model. The only instance where the models may give non-negligibility different results occurs when the split of the y_i between 0 and 1 is very unbalanced – for example, when $y_i = 1$ occurs only 10% of the time.

Stock and Watson (2011) suggest that the logistic approach was traditionally preferred since the function does not require the evaluation of an integral and thus the model parameters could be estimated faster. However, this argument is no longer relevant given the computational speeds now achievable and the choice of one specification rather than the other is now usually arbitrary.

12.7 Estimation of Limited Dependent Variable Models

Given that both logit and probit are non-linear models, they cannot be estimated by OLS. While the parameters could, in principle, be estimated using non-linear least squares (NLS), maximum likelihood (ML) is simpler and is invariably used in practice. As discussed in Chapter 9, the principle is that the parameters are chosen to jointly maximise a log-likelihood function (LLF). The form of this LLF will depend upon whether the logit or probit model is used, but the general principles for parameter estimation described in Chapter 9 will still apply. That is, we form the appropriate log-likelihood function and then the software package will find the values of the parameters that jointly maximise it using an iterative search procedure. A derivation of the ML estimator for logit and probit models is given in the appendix to this chapter. Box 12.1 shows how to interpret the estimated parameters from probit and logit models.

BOX 12.1 Parameter interpretation for probit and logit models

Standard errors and *t*-ratios will automatically be calculated by the econometric software package used, and hypothesis tests can be conducted in the usual fashion. However, interpretation of the coefficients needs slight care. It is tempting, but incorrect, to state that a 1-unit increase in x_{2i}, for example, causes a $100 \times \beta_2$% increase in the probability that the outcome corresponding to $y_i = 1$ will be realised. This would have been the correct interpretation for the linear probability model.

However, for logit models, this interpretation would be incorrect because the form of the function is not $P_i = \beta_i + \beta_2 x_i + u_i$, for example, but rather $P_i = F(\beta_i + \beta_2 x_i + u_i)$, where F represents the (non-linear) logistic function. To obtain the required relationship between changes in x_{2i} and P_i, we would need to differentiate F with respect to x_{2i} and it turns out that this derivative is $F(x_{2i})(1 - F(x_{2i}))$. So in fact, a 1-unit increase in x_{2i} will cause a $\beta_2 F(x_{2i})(1 - F(x_{2i}))$ increase in probability. Usually, these impacts of incremental changes in an explanatory variable are evaluated by setting each of them to their mean values. For example, suppose we have estimated the following logit model with three explanatory variables using maximum likelihood

$$\hat{P}_i = \frac{1}{1 + e^{-(0.1 + 0.3x_{2i} - 0.6x_{3i} + 0.9x_{4i})}} \quad \textbf{(12.7)}$$

Thus we have $\hat{\beta}_1 = 0.1, \hat{\beta}_2 = 0.3, \hat{\beta}_3 = -0.6, \hat{\beta}_4 = 0.9.$ We now need to calculate $F(z_i)$, for which we need the means of the explanatory variables, where z_i is defined as before. Suppose that these are $\bar{x}_2 = 1.6$, $\bar{x}_3 = 0.2$, $\bar{x}_4 = 0.1$, then the estimate of $F(z_i)$ will be given by

$$\hat{P}_i = \frac{1}{1+e^{-(0.1+0.3\times1.6-0.6\times0.2+0.9\times0.1)}} = \frac{1}{1+e^{-0.55}} = 0.63 \quad \textbf{(12.8)}$$

Thus a 1-unit increase in x_2 will cause an increase in the probability that the outcome corresponding to $y_i = 1$ will occur by 0.3 × 0.63 × 0.37 = 0.07. The corresponding changes in probability for variables x_3 and x_4 are −0.6 × 0.63 × 0.37 = −0.14 and 0.9 × 0.63 × 0.37 = 0.21, respectively. These estimates are sometimes known as the *marginal effects*.

There is also another way of interpreting discrete choice models, known as the random utility model. The idea is that we can view the value of y that is chosen by individual i (either 0 or 1) as giving that person a particular level of utility, and the choice that is made will obviously be the one that generates the highest level of utility. This interpretation is particularly useful in the situation where the person faces a choice between more than two possibilities as in Section 12.9 below.

Once the model parameters have been estimated, standard errors can be calculated and hypothesis tests conducted. While t-test statistics are constructed in the usual way, the standard error formulae used following the ML estimation are valid asymptotically only. Consequently, it is common to use the critical values from a normal distribution rather than a t distribution with the implicit assumption that the sample size is sufficiently large.

12.8 Goodness of Fit Measures for Linear Dependent Variable Models

While it would be possible to calculate the values of the standard goodness of fit measures such as RSS, R^2 or $\bar{R}^2$ for linear dependent variable models, these cease to have any real meaning. The objective of ML is to maximise the value of the LLF, not to minimise the RSS. Moreover, R^2 and adjusted R^2, if calculated in the usual fashion, will be misleading because the fitted values from the model can take on any value but the actual values will be only either 0 and 1. To illustrate, suppose that we are considering a situation where a bank either grants a loan (y_i = 1) or it refuses (y_i = 0). Does, say, $\hat{P}_i = 0.8$ mean the loan is offered or not? In order to answer this question, sometimes, any value of $\hat{P}_i > 0.5$ would be rounded up to one and any value <0.5 rounded down to zero. However, this approach is unlikely to work well when most of the observations on the dependent variable are one or when most are zero. In such cases, it makes more sense to use the unconditional probability that y = 1 (call this $\bar{y}$) as the threshold rather than 0.5. So if, for example, only 20% of the observations have y = 1 (so $\bar{y} = 0.2$), then we would deem the model to have correctly predicted the outcome concerning whether the bank would grant the loan to the customer where $\hat{P}_i > 0.2$ and y_i = 1 and where $\hat{P}_i < 0.2$ and y_i = 0.

Thus if y_i = 1 and $\hat{P}_i = 0.8$, the model has effectively made the correct prediction (either the loan is granted or refused – we cannot have any outcome in between), whereas R^2 and $\bar{R}^2$ will not give it full credit for this. Two goodness of fit measures that are commonly reported for limited dependent variable models are as follows

(1) The percentage of y_i values correctly predicted, defined as 100 × the number of observations predicted correctly divided by the total number of observations:

$$\text{Percent correct predictions} = \frac{100}{N} \sum_{i=1}^{N} y_i I(\hat{P}_i) + (1 - y_i)(1 - I(\hat{P}_i)) \qquad \textbf{(12.6)}$$

where $I(\hat{y}_i) = 1$ if $\hat{y}_i > \bar{y}$ and 0 otherwise.

Obviously, the higher this number, the better the fit of the model. Although this measure is intuitive and easy to calculate, Kennedy (2003) suggests that it is not ideal, since it is possible that a 'nave predictor' could do better than any model if the sample is unbalanced between 0 and 1. For example, suppose that y_i = 1 for 80% of the observations. A simple rule that the prediction is always 1 is likely to outperform any more complex model on this measure but is unlikely to be very useful. Kennedy (2003, p. 267) suggests measuring goodness of fit as the percentage of y_i = 1 correctly predicted plus the percentage of y_i = 0 correctly predicted. Algebraically, this can be calculated as

$$\text{Percent correct predictions} = 100 \times \left[\frac{\sum y_i I(\hat{P}_i)}{\sum y_i} + \frac{\sum (1 - y_i)(1 - I(\hat{P}_i))}{N - \sum y_i} \right] \quad \textbf{(12.9)}$$

Again, the higher the value of the measure, the better the fit of the model.

(2) A measure known as 'pseudo-R^2', defined as

$$\text{pseudo} - R^2 = 1 - \frac{LLF}{LLF_0} \quad \textbf{(12.10)}$$

where LLF is the maximised value of the log-likelihood function for the logit and probit model and LLF_0 is the value of the log-likelihood function for a restricted model where all of the slope parameters are set to zero (i.e. the model contains only an intercept). Pseudo-R^2 will have a value of zero for the restricted model, as with the traditional R^2, but this is where the similarity ends. Since the likelihood is essentially a joint probability, its value must be between zero and one, and therefore taking its logarithm to form the LLF must result in a negative number. Thus, as the model fit improves, LLF will become less negative and therefore pseudo-R^2 will rise. The maximum value of one could be reached only if the model fitted perfectly (i.e., all the $\hat{P}_i$ were either exactly zero or one corresponding to the actual values). This could never occur in reality and therefore pseudo-R^2 has a maximum value less than one. We also lose the simple interpretation of the standard R^2 that it measures the proportion of variation in the dependent variable that is explained by the model. Indeed, pseudo-R^2 does not have any intuitive interpretation.

This definition of pseudo-R^2 is also known as McFadden's R^2, but it is also possible to specify the metric in other ways. For example, we could define pseudo-R^2 as $[1 - (RSS/TSS)]$ where RSS is the residual sum of squares from the fitted model and TSS is the total sum of squares of y_i.

12.9 Multinomial Linear Dependent Variables

All of the examples that have been considered so far in this chapter have concerned situations where the dependent variable is modelled as a binary (0,1) choice. But there are also many instances where investors or financial agents are faced with more alternatives. For example, a company may be considering listing on the NYSE, the NASDAQ or the AMEX markets; a firm that is intending to take over another may choose to pay by cash, with shares, or with a mixture of both; a retail investor may be choosing between five different mutual funds; a credit ratings agency could assign one of sixteen (AAA to B3/B−) different ratings classifications to a firm's debt.

Notice that the first three of these examples are different from the last one. In the first three cases, there is no natural ordering of the alternatives: the choice is simply made between them. In the final case, there is an obvious ordering, because a score of 1, denoting a AAA-rated bond, is better than a score of 2, denoting a AA1/AA+-rated bond, and so on (see Section 5.15 in Chapter 5). These two situations need to be distinguished and a different approach used in each case. In the first (when there is no natural ordering), a multinomial logit or probit would be used, while in the second (where there is an ordering), an ordered logit or probit would be used. This latter situation will be discussed in the next section, while multinomial models will be considered now.

When the alternatives are unordered, this is sometimes called a *discrete choice* or *multiple choice* problem. The models used are derived from the principles of utility maximisation – that is, the agent chooses the alternative that maximises his utility relative to the others. Econometrically, this is captured using a simple generalisation of the binary setup discussed earlier. When there were only two choices (0,1), we required just one equation to capture the probability that one or the other would be chosen. If there are now three alternatives, we would need two equations; for four alternatives, we would need three equations. In general, if there are m possible alternative choices, we need $m - 1$ equations.

The situation is best illustrated by first examining a multinomial linear probability model. This still, of course, suffers from the same limitations as it did in the binary case (i.e., the same problems as the LPM), but it nonetheless serves as a simple example by way of introduction.[4] The multiple choice example most commonly used is that of the selection of the mode of transport for travel to work.[5] Suppose that the journey may be made by car, bus, or bicycle (three alternatives), and suppose that the explanatory variables are the person's income (I), total hours worked (H), their gender (G) and the distance travelled (D).[6] We could set up two equations

$$\begin{aligned} \mathrm{BUS}_i &= \alpha_1 + \alpha_2 I_i + \alpha_3 H_i + \alpha_4 G_i + \alpha_5 D_i + u_i \\ \mathrm{CAR}_i &= \beta_1 + \beta_2 I_i + \beta_3 H_i + \beta_4 G_i + \beta_5 D_i + v_i \end{aligned} \qquad \textbf{(12.11)}$$

(12.12)

where BUS_i = 1 if person i travels by bus and 0 otherwise; CAR_i = 1 if person i travels by car and 0 otherwise.

There is no equation for travel by bicycle and this becomes a sort of reference point, since if the dependent variables in the two equations are both zero, the person must be travelling by bicycle.[7] In fact, we do not need to estimate the third equation (for travel by bicycle) since any quantity of interest can be inferred from the other two. The fitted values from the equations can be interpreted as probabilities and so, together with the third possibility, they must sum to unity. Thus, if, for a particular individual i, the probability of travelling by car is 0.4 and by bus is 0.3, then the possibility that she will travel by bicycle must be 0.3 (1−0.4−0.3). Also, the intercepts for the three equations (the two estimated equations plus the missing one) must sum to zero across the three modes of transport.

While the fitted probabilities will always sum to unity by construction, as with the binomial case, there is no guarantee that they will all lie between 0 and 1 – it is possible that one or more will be greater than 1 and one or more will be negative. In order to make a prediction about which mode of transport a particular individual will use, given that the parameters in equations (12.11) and (12.12) have been estimated and given the values of the explanatory variables for that individual, the largest fitted probability would be set to 1 and the others set to 0. So, for example, if the estimated probabilities of a particular individual travelling by car, bus and bicycle are 1.1, 0.2 and −0.3, these probabilities would be rounded to 1, 0, and 0. So the model would predict that this person would travel to work by car.

Exactly as the LPM has some important limitations that make logit and probit the preferred models, in the multiple choice context multinomial logit and probit models should be used. These are direct generalisations of the binary cases, and as with the multinomial LPM, $m - 1$ equations must be estimated where there are m possible outcomes or choices. The outcome for which an equation is not estimated then becomes the reference choice, and thus the parameter estimates must be interpreted slightly differently. Suppose that travel by bus (B) or by car (C) have utilities for person i that depend on the characteristics described above (I_i, H_i, G_i, D_i), then the car will be chosen if

(12.13)

$$(\beta_1 + \beta_2 I_i + \beta_3 H_i + \beta_4 G_i + \beta_5 D_i + v_i)$$
$$> (\alpha_1 + \alpha_2 I_i + \alpha_3 H_i + \alpha_4 G_i + \alpha_5 D_i + u_i)$$

That is, the probability that the car will be chosen will be greater than that of the bus being chosen if the utility from going by car is greater. Equation (12.13) can be rewritten as

$$(\beta_1 - \alpha_1) + (\beta_2 - \alpha_2) I_i + (\beta_3 - \alpha_3) H_i + (\beta_4 - \alpha_4) G_i + (\beta_5 - \alpha_5) D_i > (u_i - v_i) \quad \textbf{(12.14)}$$

If it is assumed that u_i and v_i independently follow a particular distribution, then the difference between them will follow a logistic distribution.[8] Thus we can write

$$P(C_i/B_i) = \frac{1}{1 + e^{-z_i}} \quad \textbf{(12.15)}$$

where z_i is the function on the LHS side of (12.14), i.e., $(\beta_1 - \alpha_1) + (\beta_2 - \alpha_2) I_i + \cdots$ and travel by bus becomes the reference category. $P(C_i/B_i)$ denotes the probability that individual i would choose to travel by car rather than by bus.

Equation (12.15) implies that the probability of the car being chosen in preference to the bus depends upon the logistic function of the differences in the parameters describing the relationship between the utilities from travelling by each mode of transport. Of course, we cannot recover both β_2 and α_2 for example, but only the difference between them (call this $\gamma_2 = \beta_2 - \alpha_2$). These parameters measure the impact of marginal changes in the explanatory variables on the probability of travelling by car relative to the probability of travelling by bus. Note that a unit increase in I_i will lead to a $\gamma_2 F(I_i)$ increase in the probability and not a γ_2 increase – see equations (12.5) and (12.7) above. For this trinomial problem, there would need to be another equation – for example, based on the difference in utilities between travelling by bike and by bus. These two equations would be estimated simultaneously using maximum likelihood.

For the multinomial logit model, the error terms in the equations (u_i and v_i in the example above) must be assumed to be independent. However, this creates a problem whenever two or more of the choices are very similar to one another. This problem is known as the 'independence of irrelevant alternatives'. To illustrate how this works, Kennedy (2003, p.

270) uses an example where another choice to travel by bus is introduced and the only thing that differs is the colour of the bus. Suppose that the original probabilities for the car, bus and bicycle were 0.4, 0.3 and 0.3. If a new green bus were introduced in addition to the existing red bus, we would expect that the overall probability of travelling by bus should stay at 0.3 and that bus passengers should split between the two (say, with half using each coloured bus). This result arises since the new colour of the bus is irrelevant to those who have already chosen to travel by car or bicycle. Unfortunately, the logit model will not be able to capture this and will seek to preserve the relative probabilities of the old choices (which could be expressed as $\frac{4}{10}, \frac{3}{10}$ and $\frac{3}{10}$ respectively). These will become $\frac{4}{13}, \frac{3}{13}, \frac{3}{13}$ and $\frac{3}{13}$ for car, green bus, red bus and bicycle, respectively – a long way from what intuition would lead us to expect.

Fortunately, the multinomial probit model, which is the multiple choice generalisation of the probit model discussed in Section 12.5 above, can handle this. The multinomial probit model would be set up in exactly the same fashion as the multinomial logit model, except that the cumulative normal distribution is used for $(u_i - v_i)$ instead of a cumulative logistic distribution. This is based on an assumption that u_i and v_i are multivariate normally distributed but unlike the logit model, they can be correlated. A positive correlation between the error terms can be employed to reflect a similarity in the characteristics of two or more choices. However, such a correlation between the error terms makes estimation of the multinomial probit model using maximum likelihood difficult because multiple integrals must be evaluated. Kennedy (2003, p. 271) suggests that this has resulted in continued use of the multinomial logit approach despite the independence of irrelevant alternatives problem.

12.10 The Pecking Order Hypothesis Revisited: The Choice Between Financing Methods

In Section 12.4, a logit model was used to evaluate whether there was empirical support for the pecking order hypothesis where the hypothesis boiled down to a consideration of the probability that a firm would seek external financing or not. But suppose that we wish to examine not only whether a firm decides to issue external funds but also which method of funding it chooses when there are a number of alternatives available. As discussed above, the pecking order hypothesis suggests that the least costly methods, which, everything else equal, will arise where there is least information asymmetry, will be used first, and the method used will also depend on the riskiness of the firm. Returning to Helwege and Liang's study, they argue that if the pecking order is followed, low-risk firms will issue public debt first, while moderately risky firms will issue private debt and the most risky companies will issue equity. Since there is more than one possible choice, this is a multiple choice problem and consequently, a binary logit model is inappropriate and instead, a multinomial logit is used. There are three possible choices here: bond issue, equity issue and private debt issue. As is always the case for multinomial models, we estimate equations for one fewer than the number of possibilities, and so equations are estimated for equities and bonds, but not for private debt. This choice then becomes the reference point, so that the coefficients measure the probability of issuing equity or bonds rather than private debt, and a positive parameter estimate in, say, the equities equation implies that an increase in the value of the variable leads to an increase in the probability that the firm will choose to issue equity rather than private debt.

The set of explanatory variables is slightly different now given the different nature of the problem at hand. The key variable measuring risk is now the 'unlevered *Z* score', which is Altman's *Z* score constructed as a weighted average of operating earnings before interest and taxes, sales, retained earnings and working capital. All other variable names are largely self-explanatory and so are not discussed in detail, but they are divided into two categories – those measuring the firm's level of risk (unlevered *Z*-score, debt, interest expense and variance of earnings) and those measuring the degree of information asymmetry (R&D expenditure, venture-backed, age, age over fifty, plant property and equipment, industry growth, non-financial equity issuance, and assets). Firms with heavy R&D expenditure, those receiving venture capital financing, younger firms, firms with less property, plant and equipment, and smaller firms are argued to suffer from greater information asymmetry. The parameter estimates for the multinomial logit are presented in Table 12.2, with equity issuance as a (0,1) dependent variable in the second column and bond issuance as a (0,1) dependent variable in the third column.

Table 12.2 Multinomial logit estimation of the type of external financing

Variable	Equity equation	Bonds equation
Intercept	−4.67 (−6.17)	−4.68 (−5.48)
Unlevered *Z*-score	0.14 (1.84)	0.26 (2.86)
Debt	1.72 (1.60)	3.28 (2.88)
Interest expense	−9.41 (−0.93)	−4.54 (−0.42)
Variance of earnings	−0.04 (−0.55)	−0.14 (−1.56)
R&D	0.61 (1.28)	0.89 (1.59)
Venture-backed	0.70 (2.32)	0.86 (2.50)
Age	−0.01 (−1.10)	−0.03 (−1.85)
Age over fifty	1.58 (1.44)	1.93 (1.70)
Plant, property and equipment	(0.62) (0.94)	0.34 (0.50)
Industry growth	0.005 (1.14)	0.003 (0.70)
Non-financial equity issuance	0.008 (3.89)	0.005 (2.65)
Assets	−0.001 (−0.59)	0.002 (4.11)

Notes: t-ratios in parentheses; only figures for all years in the sample are presented.
Source: Helwege and Liang (1996). Reprinted with the permission of Elsevier.

Overall, the results paint a very mixed picture about whether the pecking order hypothesis is validated or not. The positive (significant) and negative (insignificant) estimates on the unlevered *Z*-score and interest expense variables, respectively, suggest that firms in good financial health (i.e. less risky firms) are more likely to issue equities or bonds rather than private debt. Yet the positive sign of the parameter on the debt variable is suggestive that riskier firms are more likely to issue equities or bonds; the variance of earnings variable has the wrong sign but is not statistically significant. Almost all of the asymmetric information variables have statistically insignificant parameters. The only exceptions are that firms having venture backing are more likely to seek capital market financing of either type, as are non-financial firms. Finally, larger firms are more likely to issue bonds (but not equity). Thus the authors conclude that the results ‘do not indicate that firms strongly avoid external financing as the pecking order predicts’ and ‘equity is not the least desirable source of financing since it appears to dominate bank loans’ (Helwege and Liang, 1996, p. 458).

12.11 Ordered Response Linear Dependent Variables Models

Some limited dependent variables can be assigned numerical values that have a natural ordering. The most common example in finance is that of credit ratings, as discussed previously, but a further application is to modelling a security’s bid-ask spread (see, for example, ap Gwilym, Clare and Thomas, 1998). In such cases, it would not be appropriate to use multinomial logit or probit since these techniques cannot take into account any ordering in the dependent variables. Notice that ordinal variables are still distinct from the usual type of data that were employed in the early chapters in this book, such as stock returns, GDP, interest rates, etc. These are examples of cardinal numbers, since additional information can be inferred from their actual values relative to one another. To illustrate, an increase in house prices of 20% represents twice as much growth as a 10% rise. The same is not true of ordinal numbers, where (returning to the credit ratings example) a rating of AAA, assigned a numerical score of 16, is not ‘twice as good’ as a rating of Baa2/BBB, assigned a numerical score of 8. Similarly, for ordinal data, the difference between a score of, say, 15 and of 16 cannot be assumed to be equivalent to the difference between the scores of 8 and 9. All we can say is that as the score increases, there is a monotonic increase in the credit quality. Since only the ordering can be interpreted with such data and not the actual numerical values, OLS cannot be employed and a technique based on ML is used instead. The models used are generalisations of logit and probit, known as *ordered logit* and *ordered probit*.

Using the credit rating example again, the model is set up so that a particular bond falls in the AA+ category (using Standard and Poor’s terminology) if its unobserved (latent) creditworthiness falls within a certain range that is too low to classify it as AAA and too high to classify it as AA. The boundary values between each rating are then estimated along with the model parameters.

12.12 Are Unsolicited Credit Ratings Biased Downwards? An Ordered Probit Analysis

Modelling the determinants of credit ratings is one of the most important uses of ordered probit and logit models in finance. The main credit ratings agencies construct what may be termed *solicited* ratings, which are those where the issuer of the debt contacts the agency and pays them a fee for producing the rating. Many firms globally do not seek a rating (because, for example, the firm believes that the ratings agencies are not well placed to evaluate the riskiness of debt in their country or because they do not plan to issue any debt or because they believe that they would be awarded a low rating), but the agency may produce a rating anyway. Such 'unwarranted and unwelcome' ratings are known as *unsolicited* ratings. All of the major ratings agencies produce unsolicited ratings as well as solicited ones, and they argue that there is a market demand for this information even if the issuer would prefer not to be rated.

Companies in receipt of unsolicited ratings argue that these are biased downwards relative to solicited ratings and that they cannot be justified without the level of detail of information that can be provided only by the rated company itself. A study by Poon (2003) seeks to test the conjecture that unsolicited ratings are biased after controlling for the rated company's characteristics that pertain to its risk.

The data employed comprise a pooled sample of all companies that appeared on the annual 'issuer list' of S&P during the years 1998–2000. This list contains both solicited and unsolicited ratings covering 295 firms over fifteen countries and totalling 595 observations. In a preliminary exploratory analysis of the data, Poon finds that around half of the sample ratings were unsolicited, and indeed the unsolicited ratings in the sample are on average significantly lower than the solicited ratings.[9] As expected, the financial characteristics of the firms with unsolicited ratings are significantly weaker than those for firms that requested ratings. The core methodology employs an ordered probit model with explanatory variables comprising firm characteristics and a dummy variable for whether the firm's credit rating was solicited or not

$$R_i^* = X_i\beta + \epsilon_i \tag{12.16}$$

with

$$R_i = \begin{cases} 1 & \text{if } R_i^* \le \mu_0 \\ 2 & \text{if } \mu_0 < R_i^* \le \mu_1 \\ 3 & \text{if } \mu_1 < R_i^* \le \mu_2 \\ 4 & \text{if } \mu_2 < R_i^* \le \mu_3 \\ 5 & \text{if } R_i^* > \mu_3 \end{cases}$$

where R_i are the observed ratings scores that are given numerical values as follows: AA or above = 6, A = 5, BBB = 4, BB = 3, B = 2 and CCC or below = 1; R_i^* is the unobservable 'true rating' (or 'an unobserved continuous variable representing S&P's assessment of the creditworthiness of issuer *i*'), X_i is a vector of variables that explains the variation in ratings; β is a vector of coefficients; μ_i are the threshold parameters to be estimated along with β; and ϵ_i is a disturbance term that is assumed normally distributed.

The explanatory variables attempt to capture the creditworthiness using publicly available information. Two specifications are estimated: the first includes the variables listed below, while the second additionally incorporates an interaction of the main financial variables with a dummy variable for whether the firm's rating was solicited (SOL) and separately with a dummy for whether the firm is based in Japan.[10] The financial variables are ICOV – interest coverage (i.e. earnings interest), ROA – return on assets, DTC – total debt to capital, and SDTD – short-term debt to total debt. Three variables – SOVAA, SOVA and SOVBBB – are dummy variables that capture the debt issuer's sovereign credit rating.[11] Table 12.3 presents the results from the ordered probit estimation.

Table 12.3 Ordered probit model results for the determinants of credit ratings

Explanatory variables	Model 1		Model 2	
	Coefficient	Test statistic	Coefficient	Test statistic
Intercept	2.324	8.960***	1.492	3.155***
SOL	0.359	2.105**	0.391	0.647
JP	−0.548	−2.949***	1.296	2.441**
JP∗SOL	1.614	7.027***	1.487	5.183***
SOVAA	2.135	8.768***	2.470	8.975***
SOVA	0.554	2.552**	0.925	3.968***
SOVBBB	−0.416	−1.480	−0.181	−0.601
ICOV	0.023	3.466***	−0.005	−0.172
ROA	0.104	10.306***	0.194	2.503**
DTC	−1.393	−5.736***	−0.522	−1.130
SDTD	−1.212	−5.228***	0.111	0.171
SOL∗ICOV	–	–	0.005	0.163
SOL∗ROA	–	–	−0.116	−1.476
SOL∗DTC	–	–	0.756	1.136
SOL∗SDTD	–	–	−0.887	−1.290
JP∗ICOV	–	–	0.009	0.275
JP∗ROA	–	–	0.183	2.200**
JP∗DTC	–	–	−1.865	−3.214***
JP∗SDTD	–	–	−2.443	−3.437***
AA or above	>5.095		>5.578	
A	>3.788 and ≤5.095	25.278***	>4.147 and ≤5.578	23.294***
BBB	>2.550 and ≤3.788	19.671***	>2.803 and ≤4.147	19.204***
BB	>1.287 and ≤2.550	14.342***	>1.432 and ≤2.803	14.324***
B	>0 and ≤1.287	7.927***	>0 and ≤1.432	7.910***
CCC or below	≤0		≤0	

Note: *, ** and *** denote significance at the 10%, 5% and 1% levels, respectively.
Source: Poon (2003). Reprinted with the permission of Elsevier.

The key finding is that the SOL variable is positive and statistically significant in Model 1 (and it is positive but insignificant in Model 2), indicating that even after accounting for the financial characteristics of the firms, unsolicited firms receive ratings on average 0.359 units lower than an otherwise identical firm that had requested a rating. The parameter estimate for the interaction term between the solicitation and Japanese dummies (SOL*JP) is positive and significant in both specifications, indicating strong evidence that Japanese firms soliciting ratings receive higher scores. On average, firms with stronger financial characteristics (higher interest coverage, higher return on assets, lower debt to total capital, or a lower ratio of short-term debt to long-term debt) have higher ratings.

A major flaw that potentially exists within the above analysis is the *self-selection bias* or *sample selection bias* that may have arisen if firms that would have received lower credit ratings (because they have weak financials) elect not to solicit a rating. If the probit equation for the determinants of ratings is estimated ignoring this potential problem and it exists, the coefficients will be inconsistent. To get around this problem and to control for the sample selection bias, Heckman (1979) proposed a two-step procedure that in this case would involve first estimating a 0–1 probit model for whether the firm chooses to solicit a rating and second estimating the ordered probit model for the determinants of the rating. The first-stage probit model is

$$Y_i^* = Z_i\gamma + \xi_i \qquad \textbf{(12.17)}$$

where $Y_i = 1$ if the firm has solicited a rating and 0 otherwise, and Y_i^* denotes the latent propensity of issuer i to solicit a rating, Z_i are the variables that explain the choice to be rated or not, and γ are the parameters to be estimated. When this equation has been estimated, the rating R_i as defined above in equation (12.16) will be observed only if $Y_i = 1$. The error terms from the two equations, ϵ_i and ξ_i, follow a bivariate standard normal distribution with correlation $\rho_{\epsilon\xi}$. Table 12.4 shows the results from the two-step estimation procedure, with the estimates from the binary probit model for the decision concerning whether to solicit a rating in panel A and the determinants of ratings for rated firms in panel B.

Table 12.4 Two-step ordered probit model allowing for selectivity bias in the determinants of credit ratings

Explanatory variable Panel A: Decision to be rated	Coefficient	Test statistic
Intercept	1.624	3.935***
JP	−0.776	−4.951***
SOVAA	−0.959	−2.706***
SOVA	−0.614	−1.794*
SOVBBB	−1.130	−2.899***
ICOV	−0.005	−0.922
ROA	0.051	6.537***
DTC	0.272	1.019
SDTD	−1.651	−5.320***
Panel B: Rating determinant equation		
Intercept	1.368	2.890***
JP	2.456	3.141***
SOVAA	2.315	6.121***
SOVA	0.875	2.755***
SOVBBB	0.306	0.768
ICOV	0.002	0.118
ROA	0.038	2.408**
DTC	−0.330	−0.512
SDTD	0.105	0.303
JP∗ICOV	0.038	1.129
JP∗ROA	0.188	2.104**
JP∗DTC	−0.808	−0.924
JP∗SDTD	−2.823	−2.430**
Estimated correlation	−0.836	−5.723***
AA or above	>4.275	
A	>2.841 and ≤4.275	8.235***
BBB	>1.748 and ≤2.841	9.164***
BB	>0.704 and ≤1.748	6.788***
B	>0 and ≤0.704	3.316***
CCC or below	≤0	

Note: *, ** and *** denote significance at the 10%, 5% and 1% levels, respectively.
Source: Poon (2003). Reprinted with the permission of Elsevier.

A positive parameter value in panel A indicates that higher values of the associated variable increases the probability that a firm will elect to be rated. Of the four financial variables, only the return on assets and the short-term debt as a proportion of total debt have correctly signed and significant (positive and negative, respectively) impacts on the decision to be rated. The parameters on the sovereign credit rating dummy variables (SOVAA, SOVA and SOVB) are all significant and negative in sign, indicating that any debt issuer in a country with a high sovereign rating is less likely to solicit its own rating from S&P, other things equal.

These sovereign rating dummy variables have the opposite sign in the ratings determinant equation (panel B) as expected, so that firms in countries where government debt is highly rated are themselves more likely to receive a higher rating. Of the four financial variables, only ROA has a significant (and positive) effect on the rating awarded. The dummy for Japanese firms is also positive and significant, and so are three of the four financial variables when interacted with the Japan dummy, indicating that S&P appears to attach different weights to the financial variables when assigning ratings to Japanese firms compared with comparable firms in other countries.

Finally, the estimated correlation between the error terms in the decision to be rated equation and the ratings determinant equation, $\rho_{\epsilon\xi}$, is significant and negative (−0.836), indicating that the results in Table 12.3 above would have been subject to self-selection bias and hence the results of the two-stage model are to be preferred. The only disadvantage of this approach, however, is that by construction it cannot answer the core question of whether unsolicited ratings are on average lower after allowing for the debt issuer's financial characteristics, because only firms with solicited ratings are included in the sample at the second stage!

12.13 Censored and Truncated Dependent Variables

Censored or truncated variables occur when the range of values observable for the dependent variables is limited for some reason. Unlike the types of limited dependent variables examined so far in this chapter, censored or truncated variables may not necessarily be dummies. A standard example is that of charitable donations by individuals. It is likely that some people would actually prefer to make negative donations (that is, to receive from the charity rather than to donate to it), but since this is not possible, there will be many observations at exactly zero. So suppose, for example, that we wished to model the relationship between donations to charity and people's annual income, in pounds. The situation we might face is illustrated in Figure 12.3.

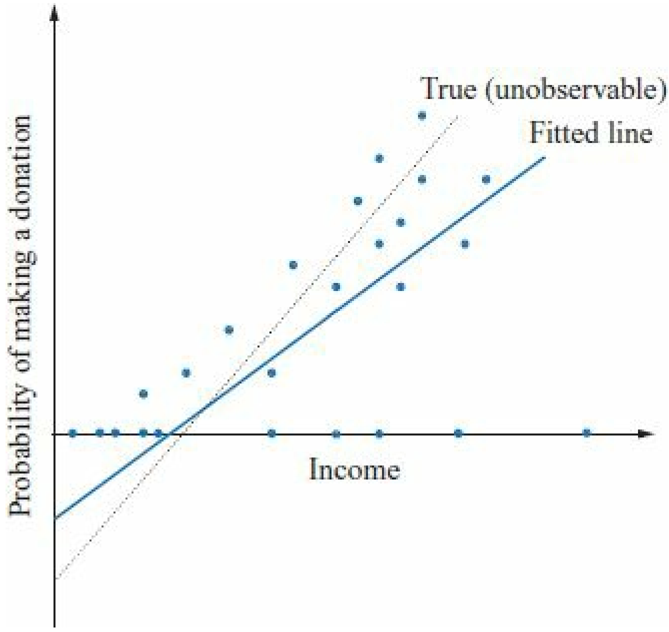

Figure 12.3 Modelling charitable donations as a function of income

Given the observed data, with many observations on the dependent variable stuck at zero, OLS would yield biased and inconsistent parameter estimates. An obvious but flawed way to get around this would be just to remove all of the zero observations altogether, since we do not know whether they should be truly zero or negative. However, as well as being inefficient (since information would be discarded), this would still yield biased and inconsistent estimates. This arises because the error term, u_i, in such a regression would not have an expected value of zero, and it would also be correlated with the explanatory variable(s), violating the assumption that $Cov(u_i, x_{ki}) = 0 \; \forall \; k$.

The key differences between censored and truncated data are highlighted in Box 12.2. For both censored and truncated data, OLS will not be appropriate, and an approach based on maximum likelihood must be used, although the model in each case would be slightly different. In both cases, we can work out the marginal effects given the estimated parameters, but these are now more complex than in the logit or probit cases.

BOX 12.2 The differences between censored and truncated dependent variables

Although at first sight the two words might appear interchangeable, when the terms are used in econometrics, censored and truncated data are different.

- Censored data occur when the dependent variable has been '*censored*' at a certain point so that values above (or below) this cannot be observed. Even though the dependent variable is censored, the corresponding values of the independent variables are still observable.
- As an example, suppose that a privatisation IPO is heavily oversubscribed, and you were trying to model the demand for the shares using household income, age, education and region of residence as explanatory variables. The number of shares allocated to each investor may have been capped at, say, 250, resulting in a truncated distribution.
- In this example, even though we are likely to have many share allocations at 250 and none above this figure, all of the observations on the independent variables are present and hence the dependent variable is censored, not truncated.
- A truncated dependent variable, meanwhile, occurs when the observations for both the dependent and the independent variables are missing when the dependent variable is above (or below) a certain threshold. Thus the key difference from censored data is that we cannot observe the x_is either, and so some observations are completely cut out or *truncated* from the sample. For example, suppose that a bank were interested in determining the factors (such as age, occupation and income) that affected a customer's decision as to whether to undertake a transaction in a branch or online. Suppose also that the bank tried to achieve this by encouraging clients to fill in an online questionnaire when they log on. There would be no data at all for those who opted to transact in person since they probably would not have even logged on to the bank's web-based system and so would not have the opportunity to complete the questionnaire. This is a common problem, which will result whenever data for buyers or users only

can be observed while data for non-buyers or non-users cannot. Of course, it is possible, although unlikely, that the population of interest is focused only on those who use the internet for banking transactions, in which case there would be no problem.

12.13.1 Censored Dependent Variable Models

The approach usually used to estimate models with censored dependent variables is known as tobit analysis, named after Tobin (1958). To illustrate, suppose that we wanted to model the demand for privatisation IPO shares, as discussed above, as a function of income (x_{2i}), age (x_{3i}), education (x_{4i}) and region of residence (x_{5i}). The model would be

$$
\begin{aligned}
y_i^* &= \beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \beta_5 x_{5i} + u_i \\
y_i &= \begin{cases} y_i^* & \text{for } y_i^* < 250 \\ 250 & \text{for } y_i^* \geq 250 \end{cases}
\end{aligned}
\qquad \textbf{(12.18)}
$$

y_i^* represents the true demand for shares (i.e. the number of shares requested) and this will be observable only for demand less than 250. Thus 250 is effectively like a threshold. In Tobin's original model, the threshold was assumed to be zero, which simplifies matters slightly.

It is important to note in this model that β_2, β_3, etc. represent the impact on the number of shares demanded (of a unit change in x_{2i}, x_{3i}, etc.) and not the impact on the actual number of shares that will be bought (allocated).

More generally, a dependent variable can be either right-censored (upper censored) as in the example above where observations above a certain threshold (call it *b*) are not observable and are equal to the threshold, or it could be left-censored (lower censored) where observations below a certain threshold (call this *a*) are not observable and so are equal to *a*.

A commonly employed illustration of the latter (left-censored) variable is to return to the example above relating to charitable donations made by individuals. To see how this would work, the explanatory variables could be exactly as in the above example with IPOs, and y_i would be the actual amount donated while y_i^* is the unobservable amount that a person *i* would actually like to give to charity (this may be negative, which would be interpreted as suggesting that the person would prefer to take money from the charity rather than donate to it if that were possible). Algebraically, we would write

$$y_i^* = \beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \beta_5 x_{5i} + u_i$$
$$y_i = \begin{cases} y_i^* & \text{for } y_i^* > 0 \\ 0 & \text{for } y_i^* \leq 0 \end{cases} \tag{12.19}$$

A final possibility is that the dependent variable is double-censored so that neither observations at or below a certain threshold *a* nor observations at or above a certain other threshold *b* can be observed. We could write this as

$$y_i^* = \beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \beta_5 x_{5i} + u_i$$
$$y_i = \begin{cases} a & \text{for } y_i^* \leq a \\ y_i^* & \text{for } a < y_i^* < b \\ b & \text{for } y_i^* \geq b \end{cases} \tag{12.20}$$

Tobit models can be estimated in a fairly straightforward fashion using maximum likelihood under the assumption that the threshold(s) (*a* or/and *b*) is/are known and that the disturbances, u_i, follow a normal distribution with mean 0 and constant variance σ^2. The log-likelihood function for a double-censored Tobit model is

$$LLF = \sum_{i=1}^{N} \left[I_i^a \ln F\left(\frac{a - (XB)}{\sigma}\right) + I_i^b \ln F\left(\frac{(XB) - b}{\sigma}\right) + (1 - I_i^a - I_i^b)\left(\ln f\left(\frac{y - (XB)}{\sigma}\right) - \ln \sigma\right)\right] \tag{12.21}$$

where *XB* is a shorthand for all of the parameters multiplied by their corresponding explanatory variables ($\beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \beta_5 x_{5i}$), $F(.)$ and $f(.)$ are the standard normal cdf and pdf respectively, and I_i^a and I_i^b are indicator functions that, respectively, pull out the observations below the lower and above the upper thresholds respectively. The latter can be defined, respectively, as

$$I_i^a = \begin{cases} 1 & \text{if } y_i < a \\ 0 & \text{if } y_i \geq a \end{cases}$$

and

$$I_i^b = \begin{cases} 1 & \text{if } y_i > b \\ 0 & \text{if } y_i \le b \end{cases}$$

Effectively, there is a pdf (in essence, a linear part) for the observed portion of the distribution and a cdf (or two cdfs) for the truncated part(s). Equation (12.21) above is the most general form of log-likelihood function in this class with both left– and right-censoring. A more restricted version would be where the dependent variable was censored on only one side, in which case one of the first two terms in the equation would drop out: the first term drops out if there is no left-censoring and $a = -\infty$, while the second term drops out if there is no right-censoring and $b = \infty$.

An interesting financial application of the tobit approach is due to Haushalter (2000), who employs it to model the determinants of the extent of hedging by oil and gas producers using futures or options over the 1992–4 period. The dependent variable used in the regression models, the proportion of production hedged, is clearly censored because around half of all of the observations are exactly zero (i.e., the firm does not hedge at all).[12] The censoring of the proportion of production hedged may arise because of high fixed costs that prevent many firms from being able to hedge even if they wished to. Moreover, if companies expect the price of oil or gas to rise in the future, they may wish to increase rather than reduce their exposure to price changes (i.e., 'negative hedging'), but this would not be observable given the way that the data are constructed in the study.

The main results from the study are that the proportion of exposure hedged is negatively related to creditworthiness, positively related to indebtedness, to the firm's marginal tax rate, and to the location of the firm's production facility. The extent of hedging is not, however, affected by the size of the firm as measured by its total assets.

Before moving on, two important limitations of tobit modelling should be noted. First, such models are much more seriously affected by non-normality and heteroscedasticity than are standard regression models (see Amemiya, 1984), and biased and inconsistent estimation will result. Second, as Kennedy (2003, p. 283) argues, the tobit model requires it to be plausible that the dependent variable can have values close to the limit. There is no problem with the privatisation IPO example discussed above since the demand could be for 249 shares. However, it would not be appropriate to use the tobit model in situations where this is not the case, such as the number of shares issued by each firm in a particular month. For most companies, this figure will be exactly zero, but for those where it is not, the number will be much higher and thus it would not be feasible to issue, say, one or three or fifteen shares. In this case, an alternative approach should be used.

12.13.2 Truncated Dependent Variable Models

As stated above, a truncated dependent variable occurs when both the dependent and the independent variables for a particular section of the population are missing or unobservable. Thus, dealing with truncated data is really a sample selection problem because the sample of data that can be observed is not representative of the population of interest – the sample is biased, very likely resulting in biased (towards zero) and inconsistent parameter estimates. Thus we cannot use OLS to estimate the model parameters and again we use maximum likelihood with a slight modification in the likelihood function to transform the data so that the cumulative probabilities still sum to one but over the observed part of the distribution only. The appropriate log-likelihood function (although there are many different ways to write it) would be

$$LLF = \sum_{i=1}^{N} \left[\ln F\left(\frac{a-(XB)}{\sigma}\right) + \ln F\left(\frac{(XB)-b}{\sigma}\right) - \left(\left(\frac{y-(XB)}{2\sigma^2}\right) - \ln \sigma\right)\right] \quad \textbf{(12.22)}$$

where as above XB is a shorthand for all of the parameters multiplied by their corresponding explanatory variables ($\beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \beta_5 x_{5i}$), $F(.)$ is the standard normal cdf.

Usually, however, for truncated data a more general model is employed that contains two equations – one for whether a particular data point will fall into the observed or constrained categories and another for modelling the resulting variable. The second equation is equivalent to the tobit approach. This two-equation methodology allows for a different set of factors to affect the sample selection (for example, the decision to set up internet access to a bank account) from the equation to be estimated (for example, to model the factors that affect whether a particular transaction will be conducted online or in a branch). If it is thought that the two sets of factors will be the same, then a single equation can be used and the tobit approach is sufficient. In many cases, however, the researcher may believe that the variables in the sample selection and estimation equations should be different. Thus the equations could be

$$a_i^* = \alpha_1 + \alpha_2 z_{2i} + \alpha_3 z_{3i} + \cdots + \alpha_m z_{mi} + \varepsilon_i \quad \textbf{(12.23)}$$

$$y_i^* = \beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + \cdots + \beta_k x_{ki} + u_i \quad \textbf{(12.24)}$$

where $y_i = y_i^*$ for $a_i^* > 0$ and, y_i is unobserved for $a_i^* \leq 0$. a_i^*denotes the relative 'advantage' of being in the observed sample relative to the unobserved sample.

The first equation determines whether the particular data point i will be observed or not, by regressing a proxy for the latent (unobserved) variable a_i^* on a set of factors, z_i. The second equation is similar to the tobit model. Ideally, the two equations (12.23) and (12.24) will be fitted jointly by maximum likelihood. This is usually based on the assumption that the error terms, ε_i and u_i, are multivariate normally distributed and allowing for any possible correlations between them. However, while joint estimation of the equations is more efficient, it is computationally more complex and hence a two-stage procedure popularised by Heckman (1976) is often used. The Heckman procedure allows for possible correlations between ε_i and u_i while estimating the equations separately in a clever way – see Maddala (1983).

It is useful to note that for both censored and truncated data, the parameter estimates arising from maximum likelihood estimation are the marginal effects for the whole population – that is, we can interpret them in the usual way rather than having to calculate them separately in a second step as we would have to for probit or logit models. The reason is that the latter types of models effectively involve a nonlinear transformation of the data through the normal or logistic functions, which is not the case for censored or truncated data.

KEY CONCEPTS

The key terms to be able to define and explain from this chapter are

- limited dependent variable
- probit
- truncated variables
- multinomial logit
- pseudo-R^2
- logit
- censored variables
- ordered response
- marginal effects

Appendix 12.1 The Maximum Likelihood Estimator for Logit and Probit Models

Recall that under the logit formulation, the estimate of the probability that $y_i = 1$ will be given from equation (12.4), which was

$$P_i = \frac{1}{1 + e^{-(\beta_1 + \beta_2 x_{2i} + \ldots + \beta_k x_{ki} + u_i)}} \quad \textbf{(12A.1)}$$

Set the error term, u_i, to its expected value for simplicity and again, let $z_i = \beta_1 + \beta_2 x_{2i} + \cdots + \beta_k x_{ki}$, so that we have

$$P_i = \frac{1}{1 + e^{-z_i}} \quad \textbf{(12A.2)}$$

We will also need the probability that $y_i \neq 1$ or equivalently the probability that $y_i = 0$. This will be given by 1 minus the probability in equation (12A.2).[13] Given that we can have actual zeros and ones only for y_i rather than probabilities, the likelihood function for each observation y_i will be

$$L_i = \left(\frac{1}{1 + e^{-z_i}}\right)^{y_i} \times \left(\frac{1}{1 + e^{z_i}}\right)^{(1-y_i)} \quad \textbf{(12A.3)}$$

The likelihood function that we need will be based on the joint probability for all N observations rather than an individual observation i. Assuming that each observation on y_i is independent, the joint likelihood will be the product of all N marginal likelihoods. Let $L(\theta \mid x_{2i}, x_{3i}, \ldots, x_{ki}; i = 1, N)$ denote the likelihood function of the set of parameters $(\beta_1, \beta_2, \ldots, \beta_k)$ given the data. Then the likelihood function will be given by

$$L(\theta) = \prod_{i=1}^{N} \left(\frac{1}{1 + e^{-z_i}}\right)^{y_i} \times \left(\frac{1}{1 + e^{z_i}}\right)^{(1-y_i)} \quad \textbf{(12A.4)}$$

As for maximum likelihood estimator of GARCH models, it is computationally much simpler to maximise an additive function of a set of variables than a multiplicative function, so long as we can ensure that the parameters required to achieve this will be the same. We thus take the natural logarithm of equation (12A.4) and this loglikelihood function is maximised

$$LLF = -\sum_{i=1}^{N}[y_i \ln(1+e^{-z_i}) + (1-y_i)\ln(1+e^{z_i})] \quad \textbf{(12A.5)}$$

Estimation for the probit model will proceed in exactly the same way, except that the form for the likelihood function in equation (12A.4) will be slightly different. It will instead be based on the familiar normal distribution function described in Appendix 9.1 to Chapter 9.

SELF-STUDY QUESTIONS

1. Explain why the linear probability model is inadequate as a specification for limited dependent variable estimation.
2. Compare and contrast the probit and logit specifications for binary choice variables.
3. (a) Describe the intuition behind the maximum likelihood estimation technique used for limited dependent variable models.
   (b) Why do we need to exercise caution when interpreting the coefficients of a probit or logit model?
   (c) How can we measure whether a logit model that we have estimated fits the data well or not?
   (d) What is the difference, in terms of the model setup, in binary choice versus multiple choice problems?
4. (a) Explain the difference between a censored variable and a truncated variable as the terms are used in econometrics.
   (b) Give examples from finance (other than those already described in this book) of situations where you might meet each of the types of variable described in part (a) of this question.
   (c) With reference to your examples in part (b), how would you go about specifying such models and estimating them?

[1] N.B. The discussion refers to the disturbance, u_i, rather than the residual, $\hat{u}_i$.

[2] 'Managers have private information regarding the value of assets in place and investment opportunities that cannot credibly be conveyed to the market. Consequently, any risky security offered by the firm will not be

priced fairly from the manager's point of view' (Helwege and Liang, 1996, p. 438).

3 Or an alternative explanation, as with a similar result in the context of a standard regression model, is that the probability varies widely across firms with the size of the cash deficit so that the standard errors are large relative to the point estimate.

4 Multinomial models are clearly explained with intuitive examples in Halcoussis (2005, Chapter 12).

5 This illustration is used in Greene (2002) and Kennedy (2003), for example.

6 Note that the same variables must be used for all equations for this approach to be valid.

7 We are assuming that the choices are exhaustive and mutually exclusive – that is, one and only one method of transport can be chosen!

8 In fact, they must follow independent log Weibull distributions.

9 We are assuming here that the broader credit rating categories, of which there are six (AAA, AA, A, BBB, BB, B), are being used rather than the finer categories used by Cantor and Packer (1996).

10 The Japanese dummy is used since a disproportionate number of firms in the sample are from this country.

11 So SOVAA = 1 if the sovereign (i.e., the government of that country) has debt with a rating of AA or above and 0 otherwise; SOVA has a value 1 if the sovereign has a rating of A; and SOVBBB has a value 1 if the sovereign has a rating of BBB; any firm in a country with a sovereign whose rating is below BBB is assigned a zero value for all three sovereign rating dummies.

12 Note that this is an example of a *censored* rather than a *truncated* dependent variable because the values of all of the explanatory variables are still available from the annual accounts even if a firm does not hedge at all.

13 We can use the rule that

$$1-\frac{1}{1+e^{-z_i}}=\frac{1+e^{-z_i}-1}{1+e^{-z_i}}=\frac{e^{-z_i}}{1+e^{-z_i}}=\frac{e^{-z_i}}{1+\frac{1}{e^{z_i}}}=\frac{e^{-z_i}\times e^{z_i}}{1+e^{z_i}}=\frac{1}{1+e^{z_i}}.$$

13

Simulation Methods

LEARNING OUTCOMES

In this chapter, you will learn how to

- Design simulation frameworks to solve a variety of problems in finance
- Explain the difference between pure simulation and bootstrapping
- Describe the various techniques available for reducing Monte Carlo sampling variability

13.1 Motivations

There are numerous situations, in finance and in econometrics, where the researcher has essentially no idea what is going to happen! To offer one illustration, in the context of complex financial risk measurement models for portfolios containing large numbers of assets whose movements are dependent on one another, it is not always clear what will be the effect of changing circumstances. For example, following full European Monetary Union (EMU) and the replacement of member currencies with the euro, it is widely believed that European financial markets have become more integrated, leading the correlation between movements in their equity markets to rise. What would be the effect on the properties of a portfolio containing equities of several European countries if correlations between the markets rose to 99%? Clearly, it is probably not possible to be able to answer such a question using actual historical data alone, since the event (a correlation of 99%) has not yet happened.

The practice of econometrics is made difficult by the behaviour of series and interrelationships between them that render model assumptions at best questionable. For example, the existence of fat tails, structural breaks and bi-directional causality between dependent and independent variables, etc. will make the process of parameter estimation and inference less reliable. Real data is messy, and no one really knows all of the features that lurk inside it. Clearly, it is important for researchers to have an idea of what the effects of such phenomena will be for model estimation and inference.

By contrast, simulation is the econometrician's chance to behave like a 'real scientist', conducting experiments under controlled conditions. A simulations experiment enables the econometrician to determine what the effect of changing one factor or aspect of a problem will be, while leaving all other aspects unchanged. Thus, simulations offer the possibility of complete flexibility. Simulation may be defined as an approach to modelling that seeks to mimic a functioning system as it evolves. The simulations model will express in mathematical equations the assumed form of operation of the system. In econometrics, simulation is particularly useful when models are very complex or sample sizes are small.

13.2 Monte Carlo Simulations

Simulations studies are usually used to investigate the properties and behaviour of various statistics of interest. The technique is often used in econometrics when the properties of a particular estimation method are not known. For example, it may be known from asymptotic theory how a particular test behaves with an infinite sample size, but how will the test behave if only fifty observations are available? Will the test still have the desirable properties of being correctly sized and having high power? In other words, if the null hypothesis is correct, will the test lead to rejection of the null 5% of the time if a 5% rejection region is used? And if the null is incorrect, will it be rejected a high proportion of the time?

Examples from econometrics of where simulation may be useful include

- Quantifying the simultaneous equations bias induced by treating an endogenous variable as exogenous
- Determining the appropriate critical values for a Dickey–Fuller test
- Determining what effect heteroscedasticity has upon the size and power of a test for autocorrelation.

Simulations are also often extremely useful tools in finance, in situations such as:

- The pricing of exotic options, where an analytical pricing formula is unavailable
- Determining the effect on financial markets of substantial changes in the macroeconomic environment
- 'Stress-testing' risk management models to determine whether they generate capital requirements sufficient to cover losses in all situations.

In all of these instances, the basic way that such a study would be conducted (with additional steps and modifications where necessary) is shown in Box 13.1.

BOX 13.1 Conducting a Monte Carlo simulation

(1) Generate the data according to the desired data generating process (DGP), with the errors being drawn from some given distribution

(2) Do the regression and calculate the test statistic

(3) Save the test statistic or whatever parameter is of interest

(4) Go back to stage 1 and repeat N times.

A brief explanation of each of these steps is in order. The first stage involves *specifying the model* that will be used to generate the data. This may be a pure time series model or a structural model. Pure time series models are usually simpler to implement, as a full structural model would also require the researcher to specify a data generating process for the explanatory variables as well. Assuming that a time series model is deemed appropriate, the next choice to be made is of the *probability distribution* specified for the errors. Usually, standard normal draws are used, although any other empirically plausible distribution (such as a Student's t) could also be used.

The second stage involves estimation of the parameter of interest in the study. The parameter of interest might be, for example, the value of a coefficient in a regression, or the value of an option at its expiry date. It could instead be the value of a portfolio under a particular set of scenarios governing the way that the prices of the component assets move over time.

The quantity N is known as the number of replications, and this should be as large as is feasible. The central idea behind Monte Carlo is that of random sampling from a given distribution. Therefore, if the number of replications is set too small, the results will be sensitive to ‘odd’ combinations of random number draws. It is also worth noting that asymptotic arguments apply in Monte Carlo studies as well as in other areas of econometrics. That is, the results of a simulation study will be equal to their analytical counterparts (assuming that the latter exist) asymptotically.

13.3 Variance Reduction Techniques

Suppose that the value of the parameter of interest for replication i is denoted x_i. If the average value of this parameter is calculated for a set of, say, N = 1,000 replications, and another researcher conducts an otherwise identical study with different sets of random draws, a different average value of x is almost certain to result. This situation is akin to the problem of selecting only a sample of observations from a given population in standard regression analysis. The sampling variation in a Monte Carlo study is measured by the standard error estimate, denoted S_x

$$S_x = \sqrt{\frac{\mathrm{var}(x)}{N}} \qquad \textbf{(13.1)}$$

where var(x) is the variance of the estimates of the quantity of interest over the N replications. It can be seen from this equation that to reduce the Monte Carlo standard error by a factor of 10, the number of replications must be increased by a factor of 100. Consequently, in order to achieve acceptable accuracy, the number of replications may have to be set at an infeasibly high level. An alternative way to reduce Monte Carlo sampling error is to use a variance reduction technique. There are many variance reduction techniques available. Two of the intuitively simplest and most widely used methods are the method of *antithetic variates* and the method of *control variates*. Both of these techniques will now be described.

13.3.1 Antithetic Variates

One reason that a lot of replications are typically required of a Monte Carlo study is that it may take many, many repeated sets of sampling before the entire probability space is adequately covered. By their very nature, the values of the random draws are random, and so after a given number of replications, it may be the case that not the whole range of possible outcomes has actually occurred.[1] What is really required is for successive replications to cover different parts of the probability space – that is, for the random draws from different replications to generate outcomes that span the entire spectrum of possibilities. This may take a long time to achieve naturally.

The antithetic variate technique involves taking the complement of a set of random numbers and running a parallel simulation on those. For example, if the driving stochastic force is a set of T $N(0, 1)$ draws, denoted u_t, for each replication, an additional replication with errors given by $-u_t$ is also used. It can be shown that the Monte Carlo standard error is reduced when antithetic variates are used. For a simple illustration of this, suppose that the average value of the parameter of interest across two sets of Monte Carlo replications is given by

$$\bar{x} = (x_1 + x_2)/2 \tag{13.2}$$

where x_1 and x_2 are the average parameter values for replications sets 1 and 2, respectively. The variance of $\bar{x}$ will be given by

$$\text{var}(\bar{x}) = \frac{1}{4}\left(\text{var}(x_1) + \text{var}(x_2) + 2\text{cov}(x_1, x_2)\right) \tag{13.3}$$

If no antithetic variates are used, the two sets of Monte Carlo replications will be independent, so that their covariance will be zero, i.e.

$$\text{var}(\bar{x}) = \frac{1}{4}\left(\text{var}(x_1) + \text{var}(x_2)\right) \tag{13.4}$$

However, the use of antithetic variates would lead the covariance in (13.3) to be negative, and therefore the Monte Carlo sampling error to be reduced.

It may at first appear that the reduction in Monte Carlo sampling variation from using antithetic variates will be huge since, by definition, $\text{corr}(u_t, -u_t) = \text{cov}(u_t, -u_t) = -1$. However, it is important to remember that the relevant covariance is between the simulated quantity of interest for the standard replications and those using the antithetic variates. But the perfect negative covariance is between the random draws (i.e. the error terms) and their antithetic variates. For example, in the context of option pricing (discussed below), the production of a price for the underlying security (and therefore for the option) constitutes a non-linear transformation of u_t. Therefore the covariances between the terminal prices of the underlying assets based on the draws and based on the antithetic variates will be negative, but not −1.

Several other variance reduction techniques that operate using similar principles are available, including stratified sampling, moment-matching and low-discrepancy sequencing. The latter are also known as *quasi-random sequences* of draws. These involve the selection of a specific sequence of representative samples from a given probability distribution. Successive samples are selected so that the unselected gaps left in the probability distribution are filled by subsequent replications. The result is a set of random draws that are appropriately distributed across all of the outcomes of interest. The use of low-discrepancy sequences leads the Monte Carlo standard errors to be reduced in direct proportion to the number of replications rather than in proportion to the square root of the number of replications. Thus, for example, to reduce the Monte Carlo standard error by a factor of 10, the number of replications would have to be increased by a factor of 100 for standard Monte Carlo random sampling, but only 10 for low-discrepancy sequencing. Further details of low-discrepancy techniques are beyond the scope of this text, but can be seen in Boyle (1977) or Press *et al.* (1992). The former offers a detailed and relevant example in the context of options pricing.

13.3.2 Control Variates

The application of control variates involves employing a variable similar to that used in the simulation, but whose properties are known prior to the simulation. Denote the variable whose properties are known by *y*, and that whose properties are under simulation by *x*. The simulation is conducted on *x* and also on *y*, with the same sets of random number draws being employed in both cases. Denoting the simulation estimates of *x* and *y* by $\hat{x}$ and $\hat{y}$, respectively, a new estimate of *x* can be derived from

$$x^* = y + (\hat{x} - \hat{y}) \tag{13.5}$$

Again, it can be shown that the Monte Carlo sampling error of this quantity, x^*, will be lower than that of *x* provided that a certain condition holds. The control variates help to reduce the Monte Carlo variation owing to particular sets of random draws by using the same draws on a related problem whose solution is known. It is expected that the effects of sampling error for the problem under study and the known problem will be similar, and hence can be reduced by calibrating the Monte Carlo results using the analytic ones.

It is worth noting that control variates succeed in reducing the Monte Carlo sampling error only if the control and simulation problems are very closely related. As the correlation between the values of the control statistic and the statistic of interest is reduced, the variance reduction is weakened. Consider again equation (13.5), and take the variance of both sides

$$\text{var}(x^*) = \text{var}(y + (\hat{x} - \hat{y})) \qquad \textbf{(13.6)}$$

var(y) = 0 since y is a quantity which is known analytically and is therefore not subject to sampling variation, so equation (13.6) can be written

$$\text{var}(x^*) = \text{var}(\hat{x}) + \text{var}(\hat{y}) - 2\text{cov}(\hat{x}, \hat{y}) \qquad \textbf{(13.7)}$$

The condition that must hold for the Monte Carlo sampling variance to be lower with control variates than without is that var(x^*) is less than $\text{var}(\hat{x})$. Taken from equation (13.7), this condition can also be expressed as

$$\text{var}(\hat{y}) - 2\text{cov}(\hat{x}, \hat{y}) < 0$$

or

$$\text{cov}(\hat{x}, \hat{y}) > \frac{1}{2}\text{var}(\hat{y})$$

Divide both sides of this inequality by the products of the standard deviations, i.e. by $(\text{var}(\hat{x}), \text{var}(\hat{y}))^{1/2}$, to obtain the correlation on the LHS

$$\text{corr}(\hat{x}, \hat{y}) > \frac{1}{2}\sqrt{\frac{\text{var}(\hat{y})}{\text{var}(\hat{x})}}$$

To offer an illustration of the use of control variates, a researcher may be interested in pricing an arithmetic Asian option using simulation. Recall that an arithmetic Asian option is one whose payoff depends on the arithmetic average value of the underlying asset over the lifetime of the averaging; at the time of writing, an analytical (closed-form) model is not yet available for pricing such options. In this context, a control variate price could be obtained by finding the price via simulation of a similar derivative whose value is known analytically – e.g. a vanilla European option. Thus, the Asian and vanilla options would be priced using simulation, as shown below, with the simulated price given by P_A and P^*_{BS}, respectively. The price of the vanilla option, P_{BS} is also calculated using an analytical formula, such as Black–Scholes. The new estimate of the Asian option price, P^*_A, would then be given by

$$P^*_A = (P_A - P_{BS}) + P^*_{BS} \tag{13.8}$$

13.3.3 Random Number Re-Usage Across Experiments

Although of course it would not be sensible to re-use sets of random number draws within a Monte Carlo experiment, using the same sets of draws across experiments can greatly reduce the variability of the difference in the estimates across those experiments. For example, it may be of interest to examine the power of the Dickey–Fuller test for samples of size 100 observations and for different values of ϕ (to use the notation of Chapter 8). Thus, for each experiment involving a different value of ϕ, the same set of standard normal random numbers could be used to reduce the sampling variation across experiments. However, the accuracy of the actual estimates in each case will not be increased, of course.

Another possibility involves taking long series of draws and then slicing them up into several smaller sets to be used in different experiments. For example, Monte Carlo simulation may be used to price several options of different times to maturity, but which are identical in all other respects. Thus, if six-month, three-month and one-month horizons were of interest, sufficient random draws to cover six months would be made. Then the six-months' worth of draws could be used to construct two replications of a three-month horizon, and six replications for the one-month horizon. Again, the variability of the simulated option prices across maturities would be reduced, although the accuracies of the prices themselves would not be increased for a given number of replications.

Random number re-usage is unlikely to save computational time, for making the random draws usually takes a very small proportion of the overall time taken to conduct the whole experiment.

13.4 Bootstrapping

Bootstrapping is related to simulation, but with one crucial difference. With simulation, the data are constructed completely artificially. Bootstrapping, on the other hand, is used to obtain a description of the properties of empirical estimators by using the sample data points themselves, and it involves sampling repeatedly with replacement from the actual data. Many econometricians were initially highly sceptical of the usefulness of the technique, which appears at first sight to be some kind of magic trick – creating useful additional information from a given sample. Indeed, Davison and Hinkley (1997, p. 3), state that the term 'bootstrap' in this context comes from an analogy with the fictional character Baron Munchhausen, who got out from the bottom of a lake by pulling himself up by his bootstraps.

Suppose a sample of data, $\mathbf{y} = y_1, y_2, \ldots, y_T$ are available and it is desired to estimate some parameter θ. An approximation to the statistical properties of $\hat{\theta}_T$ can be obtained by studying a sample of bootstrap estimators. This is done by taking N samples of size T with replacement from $\mathbf{y}$ and re-calculating $\hat{\theta}$ with each new sample. A series of $\hat{\theta}$ estimates is then obtained, and their distribution can be considered.

The advantage of bootstrapping over the use of analytical results is that it allows the researcher to make inferences without making strong distributional assumptions, since the distribution employed will be that of the actual data. Instead of imposing a shape on the sampling distribution of the $\hat{\theta}$ value, bootstrapping involves empirically estimating the sampling distribution by looking at the variation of the statistic within-sample.

A set of new samples is drawn with replacement from the sample and the test statistic of interest calculated from each of these. Effectively, this involves sampling from the sample, i.e. treating the sample as a population from which samples can be drawn. Call the test statistics calculated from the new samples $\hat{\theta}^*$. The samples are likely to be quite different from each other and from the original $\hat{\theta}$ value, since some observations may be sampled several times and others not at all. Thus a distribution of values of $\hat{\theta}^*$ is obtained, from which standard errors or some other statistics of interest can be calculated.

Along with advances in computational speed and power, the number of bootstrap applications in finance and in econometrics have increased rapidly in previous years. For example, in econometrics, the bootstrap has been used in the context of unit root testing. Scheinkman and LeBaron (1989) also suggest that the bootstrap can be used as a 'shuffle diagnostic', where as usual the original data are sampled with replacement to form new data series. Successive applications of this procedure should generate a collection of data sets with the same distributional properties, on average, as the original data. But any kind of dependence in the original series (e.g., linear or non-linear autocorrelation) will, by definition, have been removed. Applications of econometric tests to the shuffled series can then be used as a benchmark with which to compare the results on the actual data or to construct standard error estimates or confidence intervals.

In finance, an application of bootstrapping in the context of risk management is discussed below. Another important recent proposed use of the bootstrap is as a method for detecting data snooping (data mining) in the context of tests of the profitability of technical trading rules. Data snooping occurs when the same set of data is used to construct trading rules and also to test them. In such cases, if a sufficient number of trading rules are examined, some of them are bound, purely by chance alone, to generate statistically significant positive returns. Intra-generational data snooping is said to occur when, over a long period of time, technical trading rules that 'worked' in the past continue to be examined, while the ones that did not fade away. Researchers are then made aware of only the rules that worked, and not the other, perhaps thousands, of rules that failed.

Data snooping biases are apparent in other aspects of estimation and testing in finance. Lo and MacKinlay (1990) find that tests of financial asset pricing models (CAPM) may yield misleading inferences when properties of the data are used to construct the test statistics. These properties relate to the construction of portfolios based on some empirically motivated characteristic of the stock, such as market capitalisation, rather than a theoretically motivated characteristic, such as dividend yield.

Sullivan, Timmermann and White (1999) and White (2000) propose the use of a bootstrap to test for data snooping. The technique works by placing the rule under study in the context of a 'universe' of broadly similar trading rules. This gives some empirical content to the notion that a variety of rules may have been examined before the final rule is selected. The bootstrap is applied to each trading rule, by sampling with replacement from the time series of observed returns for that rule. The null hypothesis is that there does not exist a superior technical trading rule. Sullivan, Timmermann and White show how a *p*-value of the 'reality check' bootstrap-based test can be constructed, which evaluates the significance of the returns (or excess returns) to the rule after allowing for the fact that the whole universe of rules may have been examined.

13.4.1 An Example of Bootstrapping in a Regression Context

Consider a standard regression model

$$y = X\beta + u \tag{13.9}$$

The regression model can be bootstrapped in two ways.

Re-sample the Data

This procedure involves taking the data, and sampling the entire rows corresponding to observation *i* together. The steps would then be as shown in Box 13.2.

BOX 13.2 Re-sampling the data

(1) Generate a sample of size T from the original data by sampling with replacement from the whole rows taken together (that is, if observation 32 is selected, take y_{32} and all values of the explanatory variables for observation 32).

(2) Calculate $\hat{\beta}*$, the coefficient matrix for this bootstrap sample.

(3) Go back to stage 1 and generate another sample of size T. Repeat these stages a total of N times. A set of N coefficient vectors, $\hat{\beta}*$, will thus be obtained and in general they will all be different, so that a distribution of estimates for each coefficient will result.

A methodological problem with this approach is that it entails sampling from the regressors, and yet under the CLRM, these are supposed to be fixed in repeated samples, which would imply that they do not have a sampling distribution. Thus, resampling from the data corresponding to the explanatory variables is not in the spirit of the CLRM.

As an alternative, the only random influence in the regression is the errors, u, so why not just bootstrap from those?

Re-sampling from the Residuals

This procedure is ‘theoretically pure’ although harder to understand and to implement. The steps are shown in Box 13.3.

BOX 13.3 Re-sampling from the residuals

(1) Estimate the model on the actual data, obtain the fitted values $\hat{y}$, and calculate the residuals, $\hat{u}$

(2) Take a sample of size T with replacement from these residuals (and call these $\hat{u}^*$), and generate a bootstrapped-dependent variable by adding the fitted values to the bootstrapped residuals

$$y^* = \hat{y} + \hat{u}^* \qquad \textbf{(13.10)}$$

(3) Then regress this new dependent variable on the original X data to get a bootstrapped coefficient vector, $\hat{\beta}^*$

(4) Go back to stage 2, and repeat a total of N times.

13.4.2 Situations where the Bootstrap will be Ineffective

There are at least two situations where the bootstrap, as described above, will not work well.

Outliers in the Data

If there are *outliers* in the data, the conclusions of the bootstrap may be affected. In particular, the results for a given replication may depend critically on whether the outliers appear (and how often) in the bootstrapped sample.

Non-Independent Data

Use of the bootstrap implicitly assumes that the data are *independent of one another*. This would obviously not hold if, for example, there were autocorrelation in the data. A potential solution to this problem is to use a ‘moving block bootstrap’. Such a method allows for the dependence in the series by sampling whole blocks of observations at a time. These, and many other issues relating to the theory and practical usage of the bootstrap are given in Davison and Hinkley (1997); see also Efron (1979, 1982).

It is also worth noting that variance reduction techniques are also available under the bootstrap, and these work in a very similar way to those described above in the context of pure simulation.

13.5 Random Number Generation

Most econometrics computer packages include a random number generator. The simplest class of numbers to generate are from a uniform (0,1) distribution. A uniform (0,1) distribution is one where only values between zero and one are drawn, and each value within the interval has an equal chance of being selected. Uniform draws can be either discrete or continuous. An example of a discrete uniform number generator would be a die or a roulette wheel. Computers generate continuous uniform random number draws.

Numbers that are a continuous uniform (0,1) can be generated according to the following recursion

$$y_{i+1} = (ay_i + c) \text{ modulo } m, i = 0, 1, \ldots, T \quad \textbf{(13.11)}$$

then

$$R_{i+1} = y_{i+1}/m \text{ for } i = 0, 1, \ldots, T \quad \textbf{(13.12)}$$

for T random draws, where y_0 is the seed (the initial value of y), a is a multiplier and c is an increment. All three of these are simply constants. The 'modulo operator' simply functions as a clock, returning to one after reaching m.

Any simulation study involving a recursion, such as that described by equation (13.11) to generate the random draws, will require the user to specify an initial value, y_0, to get the process started. The choice of this value will, undesirably, affect the properties of the generated series. This effect will be strongest for y_1, y_2, …, but will gradually die away. For example, if a set of random draws is used to construct a time series that follows a GARCH process, early observations on this series will behave less like the GARCH process required than subsequent data points. Consequently, a good simulation design will allow for this phenomenon by generating more data than are required and then dropping the first few observations. For example, if 1,000 observations are required, 1,200 observations might be generated, with observations 1 to 200 subsequently deleted and 201 to 1,200 used to conduct the analysis.

These computer-generated random number draws are known as *pseudo-random numbers*, since they are in fact not random at all, but entirely deterministic, since they have been derived from an exact formula! By carefully choosing the values of the user-adjustable parameters, it is possible to get the pseudo-random number generator to meet all the statistical properties of true random numbers. Eventually, the random number sequences will start to repeat, but this should take a long time to happen. See Press *et al.* (1992) for more details and Fortran code, or Greene (2002) for an example.

The U(0,1) draws can be transformed into draws from any desired distribution – for example a normal or a Student's *t*. Usually, econometric software packages with simulations facilities would do this automatically.

13.6 Disadvantages of the Simulation Approach to Econometric or Financial Problem Solving

- *It might be computationally expensive*
  That is, the number of replications required to generate precise solutions may be very large, depending upon the nature of the task at hand. If each replication is relatively complex in terms of estimation issues, the problem might be computationally infeasible, such that it could take days, weeks or even years to run the experiment. Although CPU time is becoming ever cheaper as faster computers are brought to market, the technicality of the problems studied seems to accelerate just as quickly!
- *The results might not be precise*
  Even if the number of replications is made very large, the simulation experiments will not give a precise answer to the problem if some unrealistic assumptions have been made of the data generating process. For example, in the context of option pricing, the option valuations obtained from a simulation will not be accurate if the data generating process assumed normally distributed errors while the actual underlying returns series is fat-tailed.
- *The results are often hard to replicate*
  Unless the experiment has been set up so that the sequence of random draws is known and can be reconstructed, which is rarely done in practice, the results of a Monte Carlo study will be somewhat specific to the given investigation. In that case, a repeat of the experiment would involve different sets of random draws and therefore would be likely to yield different results, particularly if the number of replications is small.
- *Simulation results are experiment-specific*

The need to specify the data generating process using a single set of equations or a single equation implies that the results could apply to only that exact type of data. Any conclusions reached may or may not hold for other data generating processes. To give one illustration, examining the power of a statistical test would, by definition, involve determining how frequently a wrong null hypothesis is rejected. In the context of DF tests, for example, the power of the test as determined by a Monte Carlo study would be given by the percentage of times that the null of a unit root is rejected. Suppose that the following data generating process is used for such a simulation experiment

$$y_t = 0.99y_{t-1} + u_t, \qquad u_t \sim \mathrm{N}(0,1) \tag{13.13}$$

Clearly, the null of a unit root would be wrong in this case, as is necessary to examine the power of the test. However, for modest sample sizes, the null is likely to be rejected quite infrequently. It would not be appropriate to conclude from such an experiment that the DF test is generally not powerful, since in this case the null (ϕ = 1) is not very wrong! This is a general problem with many Monte Carlo studies. The solution is to run simulations using as many different and relevant data generating processes as feasible. Finally, it should be obvious that the Monte Carlo data generating process should match the real-world problem of interest as far as possible.

To conclude, simulation is an extremely useful tool that can be applied to an enormous variety of problems. The technique has grown in popularity over the past decade, and continues to do so. However, like all tools, it is dangerous in the wrong hands. It is very easy to jump into a simulation experiment without thinking about whether such an approach is valid or not.

13.7 An example of Monte Carlo Simulation in Econometrics: Deriving a Set of Critical Values for a Dickey–Fuller Test

Recall, that the equation for a Dickey–Fuller (DF) test applied to some series y_t is the regression

$$y_t = \phi y_{t-1} + u_t$$

(13.14)

so that the test is one of H_0: $\phi = 1$ against H_1: $\phi < 1$. The relevant test statistic is given by

$$\tau = \frac{\hat{\phi} - 1}{SE(\hat{\phi})} \quad \textbf{(13.15)}$$

Under the null hypothesis of a unit root, the test statistic does not follow a standard distribution, and therefore a simulation would be required to obtain the relevant critical values. Obviously, these critical values are well documented, but it is of interest to see how one could generate them. A very similar approach could then potentially be adopted for situations where there has been less research and where the results are relatively less well known. The simulation would be conducted in the four steps shown in Box 13.4.

BOX 13.4 Setting up a Monte Carlo simulation

(1) Construct the data generating process under the null hypothesis – that is, obtain a series for y that follows a unit root process. This would be done by:

- Drawing a series of length T, the required number of observations, from a normal distribution. This will be the error series, so that $u_t \sim N(0,1)$.
- Assuming a first value for y, i.e., a value for y at time $t = 1$.
- Constructing the series for y recursively, starting with y_2, y_3, and so on

$$\begin{aligned} y_2 &= y_1 + u_2 \\ y_3 &= y_2 + u_3 \\ &\cdots \\ y_T &= y_{T-1} + u_T \end{aligned} \quad \textbf{(13.16)}$$

(2) Calculating the test statistic, τ.

(3) Repeating steps 1 and 2 N times to obtain N replications of the experiment. A distribution of values for τ will be obtained across the replications.

(4) Ordering the set of N values of τ from the lowest to the highest. The relevant 5% critical value will be the 5th percentile of this distribution.

13.8 An Example of how to Simulate the Price of a Financial Option

A simple example of how to use a Monte Carlo study for obtaining a price for a financial option is shown below. Although the option used for illustration here is just a plain vanilla European call option which could be valued analytically using the standard Black–Scholes (1973) formula, again, the method is sufficiently general that only relatively minor modifications would be required to value more complex options. Boyle (1977) gives an excellent and highly readable introduction to the pricing of financial options using Monte Carlo. The steps involved are shown in Box 13.5.

BOX 13.5 Simulating the price of an option

(1) *Specify a data generating process for the underlying asset*. A random walk with drift model is usually assumed. Specify also the assumed size of the drift component and the assumed size of the volatility parameter. Specify also a strike price K, and a time to maturity, T.

(2) Draw a series of length T, the required number of observations for the life of the option, from a normal distribution. This will be the *error series*, so that ε_t ~N(0,1).

(3) Form a series of observations of length T on the *underlying asset*.

(4) *Observe the price of the underlying asset at maturity observation T*. For a call option, if the value of the underlying asset on maturity date, $P_T \leq K$, the option expires worthless for this replication. If the value of the underlying asset on maturity date, $P_T > K$, the option expires in the money, and has value on that date equal to $P_T - K$, which should be discounted back to the present day using the risk-free rate. Use of the risk-free rate relies upon risk-neutrality arguments (see Duffie, 1996).

(5) Repeat steps 1 to 4 a total of N times, and take the average value of the option over the N replications. This average will be the *price of the option*.

13.8.1 Simulating the Price of a Financial Option Using a Fat-Tailed Underlying Process

A fairly limiting and unrealistic assumption in the above methodology for pricing options is that the underlying asset returns are normally distributed, whereas in practice, it is well know that asset returns are fat-tailed. There are several ways to remove this assumption. First, one could employ draws from a fat-tailed distribution, such as a Student's t, in step (2) above. Another method, which would generate a distribution of returns with fat tails, would be to assume that the errors and therefore the returns follow a GARCH process. To generate draws from a GARCH process, do the steps shown in Box 13.6.

BOX 13.6 Generating draws from a GARCH process

(1) Draw a series of length T, the required number of observations for the life of the option, from a normal distribution. This will be the error series, so that $\varepsilon_t \sim N(0, 1)$.

(2) Recall that one way of expressing a GARCH model is

$$r_t = \mu + u_t \qquad u_t = \varepsilon_t \sigma_t \qquad \varepsilon_t \sim N(0, 1) \tag{13.17}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta \sigma_{t-1}^2 \tag{13.18}$$

A series of ε_t, have been constructed and it is necessary to specify initialising values y_1 and σ_1^2 and plausible parameter values for α_0, α_1, β. Assume that y_1 and σ_1^2 are set to μ and one, respectively, and the parameters are given by $\alpha_0 = 0.01$, $\alpha_1 = 0.15$, $\beta = 0.80$. The equations above can then be used to generate the model for r_t as described above.

13.8.2 Simulating the Price of an Asian Option

An Asian option is one whose payoff depends upon the average value of the underlying asset over the averaging horizon specified in the contract. Most Asian options contracts specify that arithmetic rather than geometric averaging should be employed. Unfortunately, the arithmetic average of a unit root process with a drift is not well defined. Additionally, even if the asset prices are assumed to be log-normally distributed, the arithmetic average of them will not be. Consequently, a closed-form analytical expression for the value of an Asian option has yet to be developed. Thus, the pricing of Asian options represents a natural application for simulations methods. Determining the value of an Asian option is achieved in almost exactly the same way as for a vanilla call or put. The simulation is conducted identically, and the only difference occurs in the very last step where the value of the payoff at the date of expiry is determined.

13.9 An Example of Bootstrapping to Calculate Capital Risk Requirements

13.9.1 Financial Motivation

Risk management modelling has, in this author's opinion, been one of the most rapidly developing areas of application of econometric techniques over the past decade or so. One of the most popular approaches to risk measurement is by calculating what is known as an institution's 'value-at-risk', denoted VaR. Broadly speaking, value-at-risk is an estimation of the *probability of likely losses which could arise from changes in market prices*. More precisely, it is defined as the money-loss of a portfolio that is expected to occur over a pre-determined horizon and with a pre-determined degree of confidence. The roots of VaR's popularity stem from the simplicity of its calculation, its ease of interpretation and from the fact that VaR can be suitably aggregated across an entire firm to produce a single number which broadly encompasses the risk of the positions of the firm as a whole. The value-at-risk estimate is also often known as the position risk requirement or minimum capital risk requirement (MCRR); the three terms will be used interchangeably in the exposition below. There are various methods available for calculating value-at-risk, including the 'delta-normal' method; historical simulation, involving the estimation of the quantile of returns of the portfolio; and structured Monte Carlo simulation; see Dowd (1998) or Jorion (2006) for thorough introductions to value-at-risk.

The *Monte Carlo* approach involves two steps. First, a data generating process is specified for the underlying assets in the portfolio. Second, possible future paths are simulated for those assets over given horizons, and the value of the portfolio at the end of the period is examined. Thus the returns for each simulated path are obtained, and from this distribution across the Monte Carlo replications, the VaR as a percentage of the initial value of the portfolio can be measured as the first or fifth percentile.

The Monte Carlo method is clearly a very powerful and flexible method for generating VaR estimates, since any stochastic process for the underlying assets can be specified. The effect of increasing variances or correlations, etc. can easily be incorporated into the simulation design. However, there are at least two drawbacks with the use of Monte Carlo simulation for estimating VaR. First, for a large portfolio, the computational time required to compute the VaR may be excessively great. Second, and more fundamentally, the calculated VaR may be inaccurate if the stochastic process that has been assumed for the underlying asset is inappropriate. In particular, asset prices are often assumed to follow a random walk or a random walk with drift, where the driving disturbances are random draws from a normal distribution. Since it is well known that asset returns are fat-tailed, the use of Gaussian draws in the simulation is likely to lead to a systematic underestimate of the VaR, as extremely large positive or negative returns are more likely in practice than would arise under a normal distribution. Of course, the normal random draws could be replaced by draws from a *t*-distribution, or the returns could be assumed to follow a GARCH process, both of which would generate an unconditional distribution of returns with fat tails. However, there is still some concern as to whether the distribution assumed in designing the simulations framework is really appropriate.

An alternative approach, that could potentially overcome this criticism, would be to use bootstrapping rather than Monte Carlo simulation. In this context, the future simulated prices are generated using random draws with replacement from the actual returns themselves, rather than artificially generating the disturbances from an assumed distribution. Such an approach is used in calculating MCRRs by Hsieh (1993) and by Brooks, Clare and Persand (2000). The methodology proposed by Hsieh will now be examined.

Hsieh (1993) employs daily log returns on foreign currency (against the US dollar) futures series from 22 February 1985 until 9 March 1990 (1,275 observations) for the British pound (denoted BP), the German mark (DM), the Japanese yen (JY) and the Swiss franc (SF). The first stage in setting up the bootstrapping framework is to form a model that fits the data and adequately describes its features. Hsieh employs the BDS test (discussed briefly in Chapter 9) to determine an appropriate class of models. An application of the test to the raw returns data shows that the data are not random, and that there is some structure in the data. The dependence in the series, shown in the rejection of randomness by the test implies that there is either

- a linear relationship between y_t and $y_{t-1}, y_{t-2}, \ldots$or
- a non-linear relationship between y_t and $y_{t-1}, y_{t-2}, \ldots$

The Box–Pierce Q test is applied to test for both, on the returns for the former, and on the squared or absolute values of the returns for the latter. The results of this test are not shown but effectively rule out the possibility of linear dependence (so that, for example, an ARMA model would not be appropriate for the returns), but there appears to be evidence of non-linear dependence in the series. Therefore, a second question, is whether the non-linearity is in-mean or in-variance (see Chapter 8 for elucidation). Hsieh uses a bicorrelation test to show that there is no evidence for non-linearity in-mean. Therefore, the most appropriate class of models for the returns series is a model which has time-varying (conditional) variances. Hsieh employs two types of model: EGARCH and autoregressive volatility (ARV) models. The coefficient estimates for the EGARCH model are reported in Table 13.1.

Table 13.1 EGARCH estimates for currency futures returns

$x_t = \mu + \sigma_t \eta_t$ $\eta_t \sim N(0,1)$ $\ln \sigma_t^2 = \alpha + \beta \ln \sigma_{t-1}^2 + \phi(\lvert\eta_{t-1}\rvert - (2/\pi)^{1/2}) + \gamma \eta_{t-1}$				
Coefficient	BP	DM	JY	SF
μ	0.000319 (0.000208)	0.000377 (0.000214)	0.000232 (0.000189)	0.000239 (0.000235)
α	−0.688127 (0.030088)	−1.072229 (0.041828)	−4.438289 (0.756704)	−0.993241 (0.032479)
β	0.928780 (0.002995)	0.889511 (0.004386)	0.550707 (0.075851)	0.895527 (0.003508)
ϕ	0.135854 (0.019961)	0.187005 (0.028388)	0.282167 (0.093357)	0.157669 (0.024013)
γ	−0.110718 (0.177458)	0.084173 (0.147279)	0.313274 (0.201531)	0.129035 (0.166507)

Note: Standard errors in parentheses.

Source: Hsieh (1993). Reprinted with the permission of School of Business Administration, University of Washington.

Several features of the EGARCH estimates are worth noting. First, as one may anticipate for a set of currency futures returns, the asymmetry terms (i.e., the estimated values of γ) are not significant for any of the four series. The high estimated values of β suggest a high degree of persistence in volatility in all cases except the Japanese yen. Brooks, Clare and Persand (2000) suggest that such persistence may be excessive in the sense that the volatility implied by the estimated conditional variance is too persistent to reproduce the profile of the volatility of the actual returns series. Such excessive volatility persistence could lead to an overestimate of the VaR. Leaving this issue aside, Hsieh continues to evaluate the effectiveness of the EGARCH models in capturing all of the non-linear dependence in the data. This is achieved by reapplying the BDS test to the standardised residuals, constructed by taking the residuals from the estimated models, and dividing them by their respective conditional standard deviations. If the model has captured all of the important features of the data, the standardised residual series should be completely random. It is observed that the EGARCH model cannot capture all of the non-linear dependence in the mark or franc series.

A second approach to modelling volatility is derived from a high/low volatility estimator. A daily volatility series is thus constructed using a re-scaled estimate of the range over the trading day

$$\sigma_{P,t} = (0.361 \times 1440/M)^{1/2} ln(High_t/Low_t) \quad \textbf{(13.19)}$$

where $High_t$ and Low_t are the highest and lowest transacted prices on day t and M is the number of trading minutes during the day. The volatility series, $\sigma_{P,t}$ can now be modelled as any other series. A natural model to propose, given the dependence (or persistence) in volatility over time, is an autoregressive model in the volatility. The formulation used for the price series is known as an autoregressive volatility (ARV) model

$$x_t = \sigma_{P,t} u_t \quad \textbf{(13.20)}$$

$$\ln \sigma_{P,t} = \alpha + \sum_i \beta_i \ln \sigma_{P,t-i} + v_t \quad \textbf{(13.21)}$$

where v_t is an error term. The appropriate lag length for the ARV model is determined using Schwarz's information criterion, which suggests that 8, 8, 5 and 8 lags should be used for the pound, mark, yen and franc series, respectively. The coefficient estimates for the ARV models are given in Table 13.2.

Table 13.2 Autoregressive volatility estimates for currency futures returns

$$x_t = \sigma_{P,t} u_t$$
$$\ln \sigma_{P,t} = \alpha + \sum_i \beta_i \ln \sigma_{P,t-i} + \nu_t$$

Coefficient	BP	DM	JY	SF
α	−1.037 (0.171)	−1.139 (0.187)	−1.874 (0.199)	−1.219 (0.193)
β_1	0.192 (0.028)	0.153 (0.028)	0.208 (0.028)	0.115 (0.028)
β_2	0.134 (0.029)	0.111 (0.028)	0.137 (0.028)	0.106 (0.028)
β_3	0.062 (0.029)	0.052 (0.028)	0.058 (0.029)	0.068 (0.028)
β_4	0.069 (0.029)	0.092 (0.028)	0.109 (0.028)	0.091 (0.028)
β_5	0.137 (0.028)	0.091 (0.028)	0.112 (0.028)	0.118 (0.028)
β_6	0.027 (0.029)	0.072 (0.028)		0.074 (0.028)
β_7	0.073 (0.028)	0.110 (0.028)		0.086 (0.028)
β_8	0.088 (0.028)	0.079 (0.028)		0.078 (0.028)
$\bar{R}^2$	0.274	0.227	0.170	0.193

Note: Standard errors in parentheses.

Source: Hsieh (1993). Reprinted with the permission of School of Business Administration, University of Washington.

The degrees of persistence for each exchange rate series implied by the ARV estimates is given by the sums of the β coefficients, which are 0.78, 0.76, 0.62, 0.74, respectively. These figures are high, although less so than under the EGARCH formulation. The standardised residuals from this model are given by $x_t/\hat{\sigma}_{P,t}$, where $\hat{\sigma}_{P,t}$ are the fitted values of volatility. An application of the BDS test to these standardised residuals shows no evidence of further structure apart from in the Swiss franc case, where the test statistics are marginally significant. Thus, since these standardised residuals are iid, it is valid to sample from them using the bootstrap technique.

To summarise, it is concluded that both the EGARCH and ARV models present reasonable descriptions of the futures returns series, which are then employed in conjunction with the bootstrap to estimate the value at risk estimates. This is achieved by simulating the future values of the futures price series, using the parameter estimates from the two models, and using disturbances obtained by sampling with replacement from the standardised residuals $(\hat{\eta}_t/\hat{h}_t^{1/2})$ for the EGARCH model and from u_t and v_t for ARV models. In this way, 10,000 possible future paths of the series are simulated (i.e. 10,000 replications are used), and in each case, the maximum drawdown (loss) can be calculated over a given holding period by

$$Q = (P_0 - P_1) \times number\ of\ contracts \tag{13.22}$$

where P_0 is the initial value of the position, and P_1 is the lowest simulated price (for a long position) or highest simulated price (for a short position) over the holding period. The maximum loss is calculated assuming holding periods of 1, 5, 10, 15, 20, 25, 30, 60, 90 and 180 days. It is assumed that the futures position is opened on the final day of the sample used to estimate the models, 9 March 1990.

The ninetieth percentile of these 10,000 maximum losses can be taken to obtain a figure for the amount of capital required to cover losses on 90% of days. It is important for firms to consider the maximum daily losses arising from their futures positions, since firms will be required to post additional funds to their margin accounts to cover such losses. If funds are not made available to the margin account, the firm is likely to have to liquidate its futures position, thus destroying any hedging effects that the firm required from the futures contracts in the first place.

However, Hsieh (1993) uses a slightly different approach to the final stage, which is as follows. Assuming (without loss of generality) that the number of contracts held is 1, the following can be written for a long position

$$\frac{Q}{x_0} = \left(1 - \frac{x_1}{x_0}\right) \tag{13.23}$$

or

$$\frac{Q}{x_0} = \left(\frac{x_1}{x_0} - 1\right) \tag{13.24}$$

for a short position. x_1 is defined as the minimum price for a long position (or the maximum price for a short position) over the horizon that the position is held. In either case, since x_0 is a constant, the distribution of Q will depend on the distribution of x_1. Hsieh (1993) assumes that prices are lognormally distributed, i.e., that the logs of the ratios of the prices,

$$\ln\left(\frac{x_1}{x_0}\right)$$

are normally distributed. This being the case, an alternative estimate of the fifth percentile of the distribution of returns can be obtained by taking the relevant critical value from the normal statistical tables, multiplying it by the standard deviation and adding it to the mean of the distribution.

The MCRRs estimated using the ARV and EGARCH models are compared with those estimated by bootstrapping from the price changes themselves, termed the 'unconditional density model'. The estimated MCRRs are given in Table 13.3.

Table 13.3 Minimum capital risk requirements for currency futures as a percentage of the initial value of the position

	No. of days	Long position			Short position		
		AR	Unconditional density	EGARCH	AR	Unconditional density	EGARCH
BP	1	0.73	0.91	0.93	0.80	0.98	1.05
	5	1.90	2.30	2.61	2.18	2.76	3.00
	10	2.83	3.27	4.19	3.38	4.22	4.88
	15	3.54	3.94	5.72	4.45	5.48	6.67
	20	4.10	4.61	6.96	5.24	6.33	8.43
	25	4.59	5.15	8.25	6.20	7.36	10.46
	30	5.02	5.58	9.08	7.11	8.33	12.06
	60	7.24	7.44	14.50	11.64	12.87	20.71
	90	8.74	8.70	17.91	15.45	16.90	28.03
	180	11.38	10.67	24.25	25.81	27.36	48.02
DM	1	0.72	0.87	0.83	0.89	1.00	0.95
	5	1.89	2.18	2.34	2.23	2.70	2.91
	10	2.77	3.14	3.93	3.40	4.12	5.03
	15	3.52	3.86	5.37	4.36	5.30	6.92
	20	4.05	4.45	6.54	5.19	6.14	8.91
	25	4.55	4.90	7.86	6.14	7.21	10.69
	30	4.93	5.37	8.75	7.02	7.88	12.36
	60	7.16	7.24	13.14	11.36	12.38	20.86
	90	8.87	8.39	16.06	14.68	16.16	27.75
	180	11.38	10.35	21.69	24.25	26.25	45.68
JY	1	0.56	0.74	0.72	0.68	0.87	0.86
	5	1.61	1.99	2.22	1.92	2.36	2.73
	10	2.59	2.82	3.46	3.06	3.53	4.41
	15	3.30	3.46	4.37	4.11	4.60	5.79
	20	3.95	4.10	5.09	5.13	5.45	6.77
	25	4.42	4.58	5.78	5.91	6.30	7.98
	30	4.95	4.92	6.34	6.58	6.85	8.81
	60	6.99	6.84	8.72	10.53	10.74	13.58
	90	8.43	8.00	10.51	13.61	14.00	17.63
	180	10.97	10.27	13.99	21.86	22.21	27.39

SF	1	0.82	0.97	0.89	0.93	1.12	0.98
	5	1.99	2.51	2.48	2.23	2.93	2.98
	10	2.87	3.60	4.12	3.37	4.53	5.09
	15	3.67	4.35	5.60	4.22	5.67	7.03
	20	4.24	5.10	6.82	5.09	6.69	8.86
	25	4.81	5.65	8.12	5.90	7.77	10.93
	30	5.23	6.20	9.12	6.70	8.47	12.50
	60	7.69	8.41	13.73	10.55	13.10	21.27
	90	9.23	9.93	16.89	13.60	17.06	27.80
	180	12.18	12.57	22.92	21.72	27.45	45.47

Source: Hsieh (1993). Reprinted with the permission of School of Business Administration, University of Washington.

The entries in Table 13.3 refer to the amount of capital required to cover 90% of expected losses, as percentages of the initial values of the positions. For example, according to the EGARCH model, approximately 14% of the initial value of a long position should be held in the case of the yen to cover 90% of expected losses for a 180-day horizon. The results contain several interesting features. First, the MCRRs derived from bootstrapping the price changes themselves (the 'unconditional approach') are in most cases higher than those generated from the other two methods, especially at short investment horizons. This is argued to have occurred owing to the fact that the level of volatility at the start of the MCRR calculation period was low relative to its historical level. Therefore, the conditional estimation methods (EGARCH and ARV) will initially forecast volatility to be lower than the historical average. As the holding period increases from 1 towards 180 days, the MCRR estimates from the ARV model converge upon those of the unconditional densities. On the other hand, those of the EGARCH model do not converge, even after 180 days (in fact, in some cases, the EGARCH MCRR seems oddly to diverge from the unconditionally estimated MCRR as the horizon increases). It is thus argued that the EGARCH model may be inappropriate for MCRR estimation in this application.

It can also be observed that the MCRRs for short positions are larger than those of comparative long positions. This could be attributed to an upward drift in the futures returns over the sample period, suggesting that on average an upwards move in the futures price was slightly more likely than a fall.

A further step in the analysis, which Hsieh did not conduct, but which is shown in Brooks, Clare and Persand (2000), is to evaluate the performance of the MCRR estimates in an out-of-sample period. Such an exercise would evaluate the models by assuming that the MCRR estimated from the model had been employed, and by tracking the change in the value of the position over time. If the MCRR is adequate, the 90% nominal estimate should be sufficient to cover losses on 90% of out-of-sample testing days. Any day where the MCRR is insufficient to cover losses is termed an 'exceedence' or an 'exception'. A model that leads to more than 10% exceptions for a nominal 90% coverage is deemed unacceptable on the grounds that on average, the MCRR was insufficient. Equally, a model that leads to considerably less than the expected 10% exceptions would also be deemed unacceptable on the grounds that the MCRR has been set at an inappropriately high level, leading capital to be unnecessarily tied up in a liquid and unprofitable form. Brooks, Clare and Persand (2000) observe, as Hsieh's results forewarn, that the MCRR estimates from GARCH-type models are too high, leading to considerably fewer exceedences than the nominal proportion.

KEY CONCEPTS

The key terms to be able to define and explain from this chapter are

- simulation
- Monte Carlo sampling variability
- antithetic variates
- bootstrapping
- pseudo-random number
- control variates

SELF-STUDY QUESTIONS

1. (a) Present two examples in finance and two in econometrics (ideally other than those listed in this chapter!) of situations where a simulation approach would be desirable. Explain in each case why simulations are useful.

(b) Distinguish between pure simulation methods and bootstrapping. What are the relative merits of each technique? Therefore, which situations would benefit more from one technique than the other?

(c) What are variance reduction techniques? Describe two such techniques and explain how they are used.

(d) Why is it desirable to conduct simulations using as many replications of the experiment as possible?

(e) How are random numbers generated by a computer?

(f) What are the drawbacks of simulation methods relative to analytical approaches, assuming that the latter are available?

2. A researcher tells you that she thinks the properties of the Ljung–Box test (i.e., the size and power) will be adversely affected by ARCH in the data. Design a simulations experiment to test this proposition.

3. (a) Consider the following AR(1) model

$$y_t = \phi y_{t-1} + u_t$$

Design a simulation experiment to determine the effect of increasing the value of ϕ from 0 to 1 on the distribution of the *t*-ratios.

(b) Consider again the AR(1) model from part (a) of this question. As stated in Chapter 4, the explanatory variables in a regression model are assumed to be non-stochastic, and yet y_{t-1} is stochastic. The result is that the estimator for ϕ will be biased in small samples. Design a simulation experiment to investigate the effect of the value of ϕ and the sample size on the extent of the bias.

4. A barrier option is a path-dependent option whose payoff depends on whether the underlying asset price traverses a barrier. A knock-out call is a call option that ceases to exist when the underlying price falls below a given barrier level H. Thus the payoff is given by

$$\begin{aligned} \max[0, S_T - K] \quad & \text{if } S_t > H \; \forall \, t \leq T \\ 0 \quad & \text{if } S_t \leq H \text{ for any } t \leq T. \end{aligned}$$

where S_T is the underlying price at expiry date T, and K is the exercise price. Suppose that a knock-out call is written on the FTSE 100 Index. The current index value, S_0 = 5000, K = 5100, time to maturity = 1 year, H = 4900, IV = 25%, risk-free rate = 5%, dividend yield = 2%.

Design a Monte Carlo simulation to determine the fair price to pay for this option. Using the same set of random draws, what is the value of an otherwise identical call without a barrier?

[1] Obviously, for a continuous random variable, there will be an infinite number of possible values. In this context, the problem is simply that if the probability space is split into arbitrarily small intervals, some of those intervals will not have been adequately covered by the random draws that were actually selected.

14

Additional Econometric Techniques for Financial Research

LEARNING OUTCOMES

In this chapter, you will learn how to

- Set up and conduct a valid event study
- Employ the Fama–MacBeth and Fama–French approaches to testing asset pricing models and explaining the variation in asset returns
- Work with extreme value distributions
- Implement the generalised method of moments for parameter estimation

This chapter collects together several additional econometric tools that are often employed in contemporary research in finance. The techniques are very different from one another in some ways, and there are few thematic linkages except that they might all be used in the context of financial research.

The chapter begins by discussing two of the most important approaches to conducting research in finance that have emerged over the past two or three decades: the event study methodology and the Fama–French approach. Although neither of these requires any new econometric tools that were not covered in previous chapters, the terminology employed is quite specific to this part of the literature and thus a focused discussion of how to implement these techniques may prove useful. The chapter then discusses how to estimate and interpret extreme value distributions for modelling tail behaviour. Finally, the generalised method of moments

(GMM) – an important and increasingly widely used estimation process – which is particularly relevant for asset pricing models, is presented and discussed in detail.

14.1 Event Studies

Event studies are very useful in finance research and as a result they are extremely commonly employed in the literature. In essence, they represent an attempt to gauge the effect of an identifiable *event* on a financial variable, usually stock returns. So, for example, research has investigated the impact of various types of announcements (e.g., dividends, stock splits, entry into or deletion from a stock index) on the returns of the stocks concerned. Event studies are often considered to be tests for market efficiency: if the financial markets are informationally efficient, there should be an immediate reaction to the event on the announcement date and no further reaction on subsequent trading days.

MacKinlay (1997) argues that conducting event studies initially appears difficult but is in fact easy; my view is that exactly the reverse is true: in principle, event studies are simple to understand and to conduct, but to do so in a rigorous manner requires a great deal of thought. There is a bewildering array of approaches that can be deployed, and at first blush it is not at all clear which of them is appropriate or optimal. The main groundwork for conducting modern event studies was established by Ball and Brown (1968) and by Fama *et al.* (1969), but as MacKinlay notes, something like them was conducted more than three decades earlier.

While there are now many useful survey papers that describe the various aspects of event studies in considerable detail, unfortunately each has its own notation and approach which can be confusing. Corrado (2011) is a recent example, although Armitage (1995) and MacKinlay (1997) are particularly clearly explained and more closely resemble the treatment given here. A similar discussion is offered by Campbell *et al.* (1997) but using matrix notation.

14.1.1 Some Notation and a Description of the Basic Approach

We of course need to be able to define precisely the dates on which the events occur, and the sample data are usually aligned with respect to this date. If we have N events in the sample, we usually specify an ‘event window’, which is the period of time over which we investigate the impact of the event. The length of this window will be set depending on whether we wish to investigate the short- or long-run effects of the event. It is common to examine a period comprising, say, ten trading days before the event up to ten trading days after as a short-run event window, while long-run windows can cover a month, a year, or even several years after the event.

A first question to ask once the event has been identified is what frequency of data should be employed in the analysis. MacKinlay (1997) shows that the power of event studies to detect abnormal performance is much greater when daily data are employed rather than weekly or monthly observations, so that the same power can be achieved with a much smaller *N*, or for given *N*, the power will be much larger. Although it would in some cases be possible to use intra-daily data, these are harder to collect and may bring additional problems including nuisance microstructure effects; this is perhaps why daily observations are the frequency of choice for most studies in the literature.[1]

Define the return for each firm *i* on each day *t* during the event window as R_{it}. We can conduct the following approach separately for each day within the event window – for example, we might investigate it for all of ten days before the event up to ten days after (where $t = 0$ represents the date of the event and $t = -10, -9, -8, \ldots, -1, 0, 1, 2, \ldots, 8, 9, 10$). Note that we will need to exercise care in the definition of the reference day $t = 0$ if the announcement is made after the close of the market.

In most cases, we need to be able to separate the impact of the event from other, unrelated movements in prices. For example, if it is announced that a firm will become a member of a widely followed stock index and its share price that day rises by 4%, but on average the prices of all other stocks also rise by 4%, it would be unwise to conclude that all of the increase in the price of the stock under study is attributable to the announcement. This motivates the idea of constructing abnormal returns, denoted AR_{it}, which are calculated by subtracting an expected return from the actual return

$$AR_{it} = R_{it} - E(R_{it}) \tag{14.1}$$

There are numerous ways that the expected returns can be calculated, but usually this is achieved using a sample of data before the event window so that the nature of the event is not allowed to 'contaminate' estimation of the expected returns. Armitage (1995) suggests that estimation periods can comprise anything from 100 to 300 days for daily observations and 24 to 60 months when the analysis is conducted on a monthly basis. Longer estimation windows will in general increase the precision of parameter estimation, although with it the likelihood of a structural break and so there is a trade-off.

If the event window is very short (e.g., a day or a few days), then we need be far less concerned about constructing an expected return since it is likely to be very close to zero over such a short horizon. In such circumstances, it will probably be acceptable to simply use the actual returns in place of abnormal returns.

It is further often the case that a gap is left between the estimation period and the event window, to be completely sure that anticipation (i.e., 'leakage') of the event does not affect estimation of the expected return equation. However, it might well be the case that in practice we do not have the luxury of doing this since the sample period available is insufficient. Clearly, what we would like to do is to calculate the return that would have been expected for that stock if the event did not happen at all so that we can isolate the impact of the event from any unrelated incidents that may be occurring at the same time.

The simplest method for constructing expected returns (apart from setting them to zero) is to assume a constant mean return, so that the expected return is simply the average return for each stock *i* calculated at the same frequency over the estimation window, which we might term $\bar{R}_i$. Brown and Warner (1980, 1985) conduct a simulation experiment to compare methods of estimating expected returns for event studies. They find that an approach simply using historical return averages outperforms many more complicated approaches because of the estimation error that comes with the latter.

A second, slightly more sophisticated approach, is to subtract the return on a proxy for the market portfolio that day *t* from the individual return. This will certainly overcome the impact of general market movements in a rudimentary way, and is equivalent to the assumption that the stock's beta in the market model or the CAPM is unity.

Probably the most common approach to constructing expected returns, however, is to use the market model. This in essence works by constructing the expected return using a regression of the return to stock *i* on a constant and the return to the market portfolio

$$R_{it} = \alpha_i + \beta_i R_{mt} + u_{it} \quad \textbf{(14.2)}$$

The expected return for firm *i* on any day *t* during the event window would then be calculated as the beta estimate from this regression multiplied by the actual market return on day *t*.

An interesting question is whether the expected return should incorporate the α from the estimation period in addition to β multiplied by the market return. Most applications of event studies include this, and indeed the original study by Fama *et al.* (1969) includes an alpha. However, we need to exercise caution when doing so since if – either because of some unrelated incident affecting the price of the stock or in anticipation of the event – the alpha is particularly high (particularly low) during the estimation period, it will push up (down) the expected return. Thus it may be preferable to assume an expected value of zero for the alpha and to exclude it from the event period abnormal return calculation.

In most applications, a broad stock index such as the FTSE All-Share or the S&P500 would be employed to proxy for the market portfolio. This equation can be made as complicated as desired – for example, by allowing for firm size or other characteristics – these would be included as additional factors in the regression with the expected return during the event window being calculated in a similar fashion. An approach based on the arbitrage pricing models of Chen, Roll and Ross (1986) or of Fama and French (1993) could also be employed – more discussion is made of this issue in the following section.

A final further approach would be to set up a 'control portfolio' of firms that have characteristics as close as possible to those of the event firm – for example, matching on firm size, beta, industry, book-to-market ratio, etc. – and then using the returns on this portfolio as the expected returns. Armitage (1995) reports the results of several Monte Carlo simulations that compare the results of various model frameworks that can be used for event studies.

The hypothesis testing framework is usually set up so that the null to be examined is of the event having no effect on the stock price (i.e. an abnormal return of zero). Under the null of no abnormal performance for firm *i* on day *t* during the event window, we can construct test statistics based on the standardised abnormal performance. These test statistics will be asymptotically normally distributed (as the length of the estimation window, *T*, increases)

$$AR_{it} \sim N(0, \sigma^2(AR_{it}))$$

where $\sigma^2(AR_{it})$ is the variance of the abnormal returns, which can be estimated in various ways. A simple method, used by Brown and Warner (1980) among others, is to use the time series of data from the estimation of the expected returns separately for each stock. So we could define $\hat{\sigma}^2(AR_{it})$ as being the equal to the variance of the residuals from the market model, which could be calculated for example using

$$\hat{\sigma}^2(AR_{it}) = \frac{1}{T-2}\sum_{t=2}^{T}\hat{u}_{it}^2 \quad \textbf{(14.3)}$$

where T is the number of observations in the estimation period. If instead the expected returns had been estimated using historical average returns, we would simply use the variance of those.

Sometimes, an adjustment is made to $\hat{\sigma}^2(AR_{it})$ that reflects the errors arising from estimation of α and β in the market model. Including the adjustment, the variance in the previous equation becomes

$$\hat{\sigma}^2(AR_{it}) = \frac{1}{T-2}\sum_{t=2}^{T}\left(\hat{u}_{it}^2 + \frac{1}{T}\left[1 + \frac{R_{mt} - \bar{R}_m}{\hat{\sigma}_m^2}\right]\right) \quad \textbf{(14.4)}$$

where $\bar{R}_m$ and $\hat{\sigma}_m^2$ are the average and variance of the returns on the market portfolio, respectively, during the estimation window. It should be clear that as the length of the estimation period, T, increases, this adjustment will gradually shrink to zero.

We can then construct a test statistic by taking the abnormal return and dividing it by its corresponding standard error, which will asymptotically follow a standard normal distribution[2]

$$S\hat{A}R_{it} = \frac{\hat{A}R_{it}}{[\hat{\sigma}^2(AR_{it})]^{1/2}} \sim N(0,1) \quad \textbf{(14.5)}$$

where $S\hat{A}R_{it}$ stands for the standardised abnormal return, which is the test statistic for each firm i and for each event day t.

It is likely that there will be quite a bit of variation of the returns across the days within the event window, with the price rising on some days and then falling on others. As such, it would be hard to identify the overall patterns. We may therefore consider computing the time series cumulative abnormal return over a multi-period event window (for example, over ten trading days) by summing the average returns over several periods, say from time T_1 to T_2

$$\hat{CAR}_i(T_1, T_2) = \sum_{t=T_1}^{T_2} \hat{AR}_{it} \quad \textbf{(14.6)}$$

Note that the time from T_1 to T_2 may constitute the entire event window or it might just be a sub-set of it. The variance of this $\hat{CAR}$ will be given by the number of observations in the event window plus one multiplied by the daily abnormal return variance calculated in equation (14.4) above

$$\hat{\sigma}^2(CAR_i(T_1, T_2)) = (T_2 - T_1 + 1)\hat{\sigma}^2(\hat{AR}_{it}) \quad \textbf{(14.7)}$$

This expression is essentially the sum of the individual daily variances over the days in T_1 to T_2 inclusive.[3]

We can now construct a test statistic for the cumulative abnormal return in the same way as we did for the individual dates, which will again be standard normally distributed

$$S\hat{CAR}_i(T_1, T_2) = \frac{\hat{CAR}_i(T_1, T_2)}{[\hat{\sigma}^2(CAR_i(T_1, T_2))]^{1/2}} \sim N(0, 1) \quad \textbf{(14.8)}$$

It is common to examine a pre-event window (to consider whether there is any anticipation of the event) and a post-event window – in other words, we sum the daily returns for a given firm i for days $t - 10$ to $t - 1$, say, and separately from $t + 1$ to $t + 10$, with the actual day of the event, t, being considered on its own.

Typically, some of the firms will show a negative abnormal return around the event when a positive figure was expected, and this is probably not very useful. But if we have N firms or N events, it is usually of more interest whether the return averaged across all firms is statistically different from zero than whether this is the case for any specific individual firm. We could define this average across firms for each separate day t during the event window as

$$\hat{AR}_t = \frac{1}{N}\sum_{i=1}^{N} \hat{AR}_{it} \quad \textbf{(14.9)}$$

This firm-average abnormal return, $\hat{AR}_t$ will have variance given by $1/N$ multiplied by the average of the variances of the individual firm returns

$$\hat{\sigma}^2(AR_t) = \frac{1}{N^2}\sum_{i=1}^{N} \hat{\sigma}^2(AR_{it}) \quad \textbf{(14.10)}$$

Thus the test statistic (the standardised return) for testing the null hypothesis that the average (across the N firms) return on day t is zero will be given by

$$S\hat{A}R_t = \frac{\hat{A}R_t}{[\hat{\sigma}^2(AR_t)]^{1/2}} = \frac{\frac{1}{N}\sum_{i=1}^{N}\hat{A}R_{it}}{[\frac{1}{N^2}\sum_{i=1}^{N}\hat{\sigma}^2(AR_{it})]^{1/2}} \sim N(0,1) \quad \textbf{(14.11)}$$

Finally, we can aggregate both across firms and over time to form a single test statistic for examining the null hypothesis that the average multi-horizon (i.e., cumulative) return across all firms is zero. We would get an equivalent statistic whether we first aggregated over time and then across firms or the other way around. The *CAR* calculated by averaging across firms first and then cumulating over time could be written

$$C\hat{A}R(T_1, T_2) = \sum_{t=T_1}^{T_2} \hat{A}R_t \quad \textbf{(14.12)}$$

Or equivalently, if we started with the $CAR_i(T_1, T_2)$ separately for each firm, we would take the average of these over the N firms

$$C\hat{A}R(T_1, T_2) = \frac{1}{N}\sum_{i=1}^{N} C\hat{A}R_i(T_1, T_2) \quad \textbf{(14.13)}$$

To obtain the variance of this $C\hat{A}R(T_1, T_2)$ we could take $1/N$ multiplied by the average of the variances of the individual $C\hat{A}R_i$.

$$\hat{\sigma}^2(CAR(T_1, T_2)) = \frac{1}{N^2}\sum_{i=1}^{N} \hat{\sigma}^2(CAR_i(T_1, T_2)) \quad \textbf{(14.14)}$$

And again we can construct a standard normally distributed test statistic as

$$S\hat{CAR}(T_1, T_2) = \frac{C\hat{A}R(T_1, T_2)}{[\hat{\sigma}^2(CAR(T_1, T_2))]^{1/2}} \sim N(0,1) \quad \textbf{(14.15)}$$

14.1.2 Cross-Sectional Regressions

The methodologies and formulae presented above provide various tools for examining whether abnormal returns are statistically significant or not. However, it will often be the case that we are interested in allowing for differences in the characteristics of a sub-section of the events and also examining the link between the characteristics and the magnitude of the abnormal returns. For example, does the event have a bigger impact on small firms? Or on firms which are heavily traded, etc.? The simplest way to achieve this would be to calculate the abnormal returns as desired using something like equations (14.1) and (14.2) above and then to use these as the dependent variable in a cross-sectional regression of the form

$$AR_i = \gamma_0 + \gamma_1 x_{1i} + \gamma_2 x_{2i} + \ldots + \gamma_M x_{Mi} + w_i \tag{14.16}$$

where AR_i is the abnormal return for firm i measured over some horizon, and x_{ji}, ($j = 1, \ldots, M$) are a set of M characteristics that are thought to influence the abnormal returns, γ_j measures the impact of the corresponding variable j on the abnormal return, and w_i is an error term. We can examine the sign, size and statistical significance of γ_0 as a test for whether the average abnormal return is significantly different from zero after allowing for the impacts of the M characteristics. MacKinlay (1997) advocates the use of heteroscedasticity-robust standard errors in the regression.

The abnormal return used in this equation would typically be measured over several days (or perhaps even the whole event window), but it could also be based on a single day.

14.1.3 Complications When Conducting Event Studies and Their Resolution

The above discussion presents the standard methodology that is commonly employed when conducting event studies, and most of the time it will provide appropriate inferences. However, as always in econometrics, the use of test statistics requires a number of assumptions about the nature of the data and models employed. Some of these assumptions will now be highlighted and their implications explored.

Cross-Sectional Dependence

A key assumption when the returns are aggregated across firms is that the events are independent of one another. Often, this will not be the case, particularly when the events are clustered through time. For example, if we were investigating the impact of index recompositions on the prices of the stocks concerned, these index constituents generally only change at specific times of the year. So, typically, a bunch of stocks will enter into an index on the same day, and then there may be no further such events for three or six months.

The impact of this clustering is that we cannot assume the returns to be independent across firms, and as a result the variances in the aggregates across firms (equations (14.10) and (14.14)) will not apply since these derivations have effectively assumed the returns to be independent across firms so that all of the covariances between returns across firms can be set to zero. An obvious solution to this would be not to aggregate the returns across firms, but simply to construct the test statistics on an event-by-event basis and then to undertake a summary analysis of them (e.g., reporting their means, variances, percentage of significant events, etc.).

A second solution would be to construct portfolios of firms having the event at the same time and then the analysis would be done on each of the portfolios. The standard deviation would be calculated using the cross-section of those portfolios' returns on day t (or on days T_1 to T_2, as desired). This approach will allow for cross-correlations since they will automatically be taken into account in constructing the portfolio returns and the standard deviations of those returns. But a disadvantage of this technique is that it cannot allow for different variances for each firm as all are equally weighted within the portfolio; the standard method described above would do so, however.

Changing Variances of Returns

It has been argued in the literature that often the variance of returns will increase over the event window, but the variance figure used in the testing procedure will have been calculated based on the estimation window, which is usually some time before the event. Either the event itself or the factors that led to it are likely to increase uncertainty and with it the volatility of returns. As a result, the measured variance will be too low and the null hypothesis of no abnormal return during the event will be rejected too often. To deal with this, Boehmer, Musumeci, and Poulsen (1991), among others, suggest estimating the variance of abnormal returns by employing the cross-sectional variance of returns across firms during the *event* window. Clearly, if we adopt this procedure we cannot estimate separate test statistics for each firm (although arguably these are usually of little interest anyway). The variance estimator in equation (14.10) would be replaced by

$$\hat{\sigma}^2(AR_t) = \frac{1}{N^2}\sum_{i=1}^{N}(\hat{AR}_{it} - \hat{AR}_t)^2 \tag{14.17}$$

with the test statistic following as before. A similar adjustment could be made for the variance of the cumulative abnormal return

$$\sigma^2(CAR(T_1, T_2)) = \frac{1}{N^2}\sum_{i=1}^{N}\left(C\hat{A}R_i(T_1, T_2) - C\hat{A}R(T_1, T_2)\right) \tag{14.18}$$

While this test statistic will allow for the variance to change over time, a drawback is that it does not allow for differences in return variances across firms and nor does it allow for cross-correlations in returns caused by event clustering.

Weighting the Stocks

Another issue is that the approach as stated above will not give equal weight to each stock's return in the calculation. The steps outlined above construct the cross-firm aggregate return (in equation (14.9)) and then standardise this using the aggregate standard deviation (as in equation (14.11)). An alternative method would be to first standardise each firm's abnormal return (dividing by its appropriate standard deviation) and then aggregating these standardised abnormal returns. If we take the standardised abnormal return for each firm, $S\hat{A}R_{i,t}$, from equation (14.5), we can calculate the average of these across the N firms

$$S\hat{A}R_t = \frac{1}{N}\sum_{i=1}^{N} S\hat{A}R_{it} \tag{14.19}$$

These *SAR*s have already been standardised so there is no need to divide them by the square root of the variance. If we take this SAR_t and multiply it by $\sqrt{N}$, we will get a test statistic that is asymptotically normally distributed and which, by construction, will give equal weight to each *SAR* (because we have taken an unweighted average of them)

$$\sqrt{N}SAR_t \sim N(0, 1)$$

We could similarly take an unweighted average of the standardised cumulative abnormal returns (*SCAR*)

$$\hat{SCAR}(T_1, T_2) = \frac{1}{N}\sum_{i=1}^{N} \hat{SCAR}_i(T_1, T_2) \tag{14.20}$$

and

$$\sqrt{N}SCAR(T_1, T_2) \sim N(0, 1)$$

If the true abnormal return is similar across securities, we would be better to equally weight the abnormal returns in calculating the test statistics (as in equations (14.19) and (14.20)), but if the abnormal return varies positively with its variance measure, then it would be better to give more weight to stocks with lower return variances (as in equation (14.15) for example).

Long Event Windows

Event studies are joint tests of whether the event-induced abnormal return is zero and whether the model employed to construct expected returns is correct. If we wish to examine the impact of an event over a long period (say, more than a few months), we need to be more careful about the design of the model for expected returns and also to ensure that this model appropriately allows for risk. Over short windows, discrepancies between models are usually small and any errors in model specification are almost negligible. But over the longer run, small errors in setting up the asset pricing model can lead to large errors in the calculation of abnormal returns and therefore the impact of the event.

A key question in conducting event studies to measure long-run impacts is whether to use cumulative abnormal returns (CARs), as described above, or buy-and-hold abnormal returns (BHARs). There are several important differences between the two. First, BHARs employ geometric returns rather than arithmetic returns (used in computing CARs) in calculating the overall return over the event period of interest. Thus the BHAR can allow for compounding whereas the CAR does not. The formula for calculating the BHAR is usually given by

$$\hat{BHAR}_i = [\Pi_{t=T_1}^{T_2}(1 + R_{it}) - 1] - [\Pi_{t=T_1}^{T_2}(1 + E(R_{it})) - 1] \tag{14.21}$$

where $E(R_{it})$ is the expected return. Usually, when constructing BHARs the expected return is based on a non-event firm or portfolio of firms that is matched in some way to the event firm (e.g., based on size, industry, etc.). Alternatively, although less desirably, it could be obtained from a benchmark such as a stock market index.

If desired, we can then sum the $BHAR_i$ across the N firms to construct an aggregate measure. BHARs have been advocated, among others, by Barber and Lyon (1997) and Lyon, Barber and Tsai (1999) because they better match the 'investor experience' than CARs given the former's use of geometric rather than arithmetic averaging. CARs represent biased estimates of the actual returns received by investors. However, by contrast, Fama (1998) in particular argues in favour of the use of CARs rather than BHARs. The latter seem to be more adversely affected by skewness in the sample of abnormal returns than the former because of the impact of compounding in BHARs.[4] In addition, Fama indicates that the average CAR increases at a rate of $(T_2 - T_1)$ with the number of months included in the sum, whereas its standard error increases only at a rate $\sqrt{(T_2 - T_1)}$. This is not true for BHARs where the standard errors grow at the faster rate $(T_2 - T_1)$ rather than its square root. Hence any inaccuracies in measuring expected returns will be more serious for BHARs as another consequence of compounding.

Event Time versus Calendar Time Analysis

All of the procedures discussed above have involved conducting analysis in *event time*. There is, however, an alternative approach that involves using *calendar time*, advocated by Fama (1998) and Mitchell and Stafford (2000) among others. In essence, using a calendar time methodology involves running a time series regression and examining the intercept from that regression. The dependent variable is a series of portfolio returns, which measure the average returns at each point in time of the set of firms that have undergone the event of interest within a pre-defined measurement period before that time. So, for example, we might choose to examine the returns of firms for a year after the event that they announce cessation of their dividend payments. Then, for each observation t, the dependent variable will be the average return on all firms that stopped paying dividends at any point during the past year. One year after the event, by construction the firm will drop out of the portfolio. Hence the number of firms within the portfolio will vary over time (as the number of firms ceasing dividend payment varies) and the portfolio will effectively be rebalanced each month. The explanatory variables may be risk measures from the Carhart (1997) four-factor model for example – this will be discussed in detail below.

The calendar time approach will weight each time period equally and thus the weight on each individual firm in the sample will vary inversely with the number of other firms that have undergone the event during the observation period. This may be problematic and will result in a loss of power to detect an effect if managers time their events to take advantage of misvaluations.

Small Samples and Non-Normality

The test statistics presented in the previous section are all asymptotic, and problems may arise either if the estimation window (T) is too short, or if the number of firms (N) is too small when the firm-aggregated statistic is used. As we discussed earlier in the book, it is well known that stock returns are leptokurtic and tend to have longer lower tails than upper tails. And particularly with small samples, the presence of outliers – for example, very large returns during the estimation window affecting the market model parameter estimation or the residual variance estimates – may also be problematic. One possible remedy would be to use a bootstrap approach to computing the test statistics.

A second strategy for dealing with non-normality would be to use a non-parametric test. Such tests are robust in the presence of non-normal distributions, although they are usually less powerful than their parametric counterparts. In the present context, we could test the null hypothesis that the proportion of positive abnormal returns is not affected by the event. In other words, the proportion of positive abnormal returns across firms remains at the expected level. We could then use the test statistic, Z_p

$$Z_p = \frac{[p - p^*]}{[p^*(1 - p^*)/N]^{1/2}} \qquad \textbf{(14.22)}$$

where p is the actual proportion of negative abnormal returns during the event window and p^* is the expected proportion of negative abnormal returns. Under the null hypothesis, the test statistic follows a binomial distribution, which can be approximated by the standard normal distribution. Sometimes p^* is set to 0.5, but this may not be appropriate if the return distribution is skewed, which is typically the case. Instead, it is better to calculate p^* based on the proportion of negative abnormal returns during the estimation window. The Wilcoxon signed-rank test can also be used.

Event Studies: Some Further Issues

A further implicit assumption in the standard event test methodology is that the events themselves occur involuntarily. In practice, however, firms often have discretion about the extent, timing and presentational forms of the announcements that they make. Thus they are likely to use any discretion they have to make announcements when market reactions are going to be the most favourable. For example, where the local regulatory rules allow discretion, firms may release bad news when the markets are closed or when the media and investors are preoccupied with other significant news items. Prabhala (1997) discusses the implications of and solutions to the endogeneity of the firm's decision about when (and perhaps even whether) to make an announcement. When a firm chooses not to announce at a particular time, we have a sort of truncated sample since we can observe events only for firms who choose to make an announcement.

A way of simultaneously dealing with a number of the issues highlighted above (i.e., differing return variances across firms, changing return variances over time, and clustering of events across firms) is to use what has been termed generalised least squares (GLS) in constructing the test statistics. In essence this works by constructing a variance–covariance matrix from the abnormal returns and using this to weight the returns in computing the aggregate test statistic – see Armitage (1995) for further details.

We can see from the above that a range of procedures exists for conducting event studies. The core of the approach is the same in each case, but they differ according to how the aggregation is performed over time and across firms and this affects the method of calculation of the standard deviations. So how do we choose which approach to use? Hopefully, given the context and the nature of the events under consideration, we can gain a reasonable idea of which approach is likely to be the most justifiable. For example, is clustering an issue? Is it expectable that the return variances will have changed over time? Is it important to allow for the variances of returns to vary between firms? By answering these questions, we can usually select the appropriate procedure. But if in doubt, it is always advisable to examine a range of methods and to compare the results as a robustness check. With luck, the various calculation techniques will lead to the same conclusion.

14.1.4 Conducting an Event Study Using Excel

This section will now use the core of the approaches described above in order to conduct an event study. While this ought to be enough to get started and to obtain some indicative results, it is important to note that there is far more that can be done with event studies to make them more rigorous than the approach presented here and readers are encouraged to consult the papers cited above for further details.

The first step would be to decide which event to consider the impact of, and there is certainly no shortage of possibilities (dividend announcements; stock spit announcements; index composition changes; merger announcements; CEO turnover; new contract announcements; macroeconomic announcements, etc.). Once this is done and the data are collected, the time-consuming part is to then organise them in a way to make them easy to work with. It would be possible to conduct the analysis in any software package for data analysis, including EViews. However, since the bulk of the task involves data arrangement and the econometric part is usually not sophisticated (in most cases, a regression will not even be conducted), it probably makes sense to revert back to Microsoft Excel or a similar spreadsheet package.[5]

The starting point for the analysis conducted here are the abnormal returns for $N = 20$ firms, which are given in the Excel file 'Event.xls', and have already been calculated using the market model using equations (14.1) and (14.2). The returns are given for days −259 to +263. The raw data are on the sheet 'abnormal returns'. The spreadsheet has been set up with the data aligned on the event day, so while the firms underwent the event on different days, the spreadsheet is constructed so that day '0' is the event day in the same row for all firms. The estimation period is from day −259 to day −10 inclusive (249 days), while the event periods examined are $(T - 10, T - 1)$, day T itself, $(T + 1, T + 10)$ and $(T + 1, T + 250)$. The first of these windows allows us to examine whether there was any leakage of information that affected stock returns prior to the event. Whether there is an immediate effect on the day that the event occurs will depend on whether the announcement is made in advance or it is a 'surprise' to the markets. If the event was known in advance to be happening on day T then the impact on the markets that day may be muted since it could have already been reflected in prices. Note that in this case the adjustment in equation (14.4) is not employed since the estimation period ($T = 249$) is quite long and would render the adjustment term negligible.

We first calculate the average return across all twenty firms for each day during the estimation and event windows in column V of the 'abnormal returns' sheet using the Excel AVERAGE formula in the usual way. All of the calculations of the key statistics are done on a separate sheet which I have called 'summary stats'. The sheet first calculates the AR for day T and the CARs for the date ranges using equations (14.1) and (14.6), respectively, for each individual firm and also for the average across all firms.

The next step is to calculate the variances of the abnormal returns or cumulative abnormal returns. For day T, this is done using equation (14.3), which is simply the time series variance of returns during the estimation window and placed in row 2 (and copied directly into row 11). For the multi-day event windows, the one-day variance from equation (14.3) is scaled up by the number of days in the event window (10 or 250) using equation (14.7). Then the test statistics are calculated by dividing the AR by its respective standard deviation (i.e. the square root of the variance) using equation (14.5) or its CAR equivalent in equation (14.8). Finally, the easiest way to obtain p-values for the tests is to use the TDIST function in Excel for a two-sided test and with a large number of degrees of freedom (say, 1,000), so that it approximates the normal distribution.

As discussed in the previous section, there are several potential issues with the fairly simple event study methodology just described. So, for robustness, it is a good idea to examine different ways of tackling the problem, and two possible checks are given in columns X and Y of the 'summary stats' sheet. Both of these procedures can only be undertaken based on the average across firms and not at the individual firm level. The first tweak is to calculate the standard deviation used in the test statistics cross-sectionally in order to allow for the possibility that the return variances may have changed (typically, risen) around the time of the event. Thus we simply take the variance across firms for the abnormal return or cumulative abnormal return of interest, divide this by N (i.e. 20) and then proceed in the usual way.

A further possibility examined in column Y is to equally weight firms by calculating the average of the standardised abnormal returns as in equations (14.19) or (14.20). Then the test statistic is simply this average multiplied by the square root of the N.

If we now consider the results on this sheet, it is clear that there is little evidence of a short-term reaction to the event. During the two trading weeks before the event, ($T - 10$ to $T - 1$), only one firm has a significant abnormal return at the 5% level (firm 20 has a CAR of 15.43% with a test statistic of 2.02). None of the individual firms have significant returns on the event date (T), and neither do any of them show significance in the short post-event window ($T + 1$ to $T + 10$). It is over the longer term – the next trading year – where there is some action. Now five firms have statistically significant returns together with economically quite large cumulative abnormal returns of 20% to 55%.

Examining the aggregate-level results, it is reassuring that the three slightly different approaches in columns W to Y yield very similar conclusions. Here the null hypothesis is that the average abnormal return (or average cumulative abnormal return) is zero. There is again no discernible market reaction before, on, or in the short-run after, the event. However, the long-run abnormal return is positive and highly statistically significant whichever of the three approaches is considered. Interestingly, the variance estimates before the event (at times $t - 10$ to $T - 1$) are higher for the cross-sectional approach in equation (14.18), although they are lower for cross-sectional approach during and after the event.

Finally, in the third sheet of the Event.xls workbook, labelled 'non-parametric test', the non-parametric statistic Z_p of equation (14.22) is calculated and then the p-value is obtained using the TDIST function as above. This examines the null hypothesis that the proportion of abnormal returns around the event is the same as it was during the estimation window. So the first calculation row of the sheet (row 2) calculates p^*, the expected proportion of negative returns based on data from the estimation window. Then for each event period range, we calculate p, the actual proportion of negative returns.[6]

The expected proportion of negative returns varies from 0.43 for firm 18 to 0.55 for firm 8, but the actual proportions for the short pre- and post-event windows are often much lower than that. For example, for firm 1, p was 0.3 (i.e., negative returns on only three days from ten) before the event. Pre-event, six of the twenty firms have significant differences between p and p^*, while for the two weeks immediately after the event, only three of them show significant differences. Over the long-run, however, there are no significant differences between the expected and actual proportions of negative return days – either for any of the individual firms or for the average.

14.2 Tests of the CAPM and the Fama–French Methodology

14.2.1 Testing the CAPM

The Basics

Before moving on to the more sophisticated multi-factor models, it may be useful to review the standard approach that was developed for testing the CAPM. This is not the place for a detailed discussion of the motivation for the CAPM or its derivation – such a discussion can be found at an accessible level in Bodie *et al.* (2014) or most other finance textbooks; alternatively, see Campbell, Lo and Mackinlay (1997) for a more technical treatment. A good introduction to the general area of asset pricing tests is given in the book by Cuthbertson and Nitzsche (2004).

The most commonly quoted equation for the CAPM is

$$E(R_i) = R_f + \beta_i[E(R_m) - R_f] \tag{14.23}$$

So the CAPM states that the expected return on any stock *i* is equal to the risk-free rate of interest, R_f, plus a risk premium. This risk premium is equal to the risk premium per unit of risk, also known as the market risk premium, $[E(R_m) - R_f]$, multiplied by the measure of how risky the stock is, known as 'beta', β_i. Beta is not observable from the market and must be calculated, and hence tests of the CAPM are usually done in two steps – first, estimating the stock betas and second, actually testing the model. It is important to note that the CAPM is an equilibrium model, or a model in terms of expectations. Thus, we would not expect the CAPM to hold in every time period for every stock. But if it is a good model, then it should hold 'on average'. Usually, we will use a broad stock market index as a proxy for the market portfolio and the yield on short-term Treasury bills as the risk-free rate.

A stock's beta can be calculated in two ways – one approach is to calculate it directly as the covariance between the stock's *excess* return and the excess return on the market portfolio, divided by the variance of the excess returns on the market portfolio

$$\beta_i = \frac{Cov(R_i^e, R_m^e)}{Var(R_m^e)} \tag{14.24}$$

where the ^{e} superscript denotes excess returns (i.e., the return with the risk-free rate subtracted from it). Alternatively, and equivalently, we can run a simple *time-series* regression of the excess stock returns on the excess returns to the market portfolio separately for each stock, and the slope estimate will be the beta

$$R^e_{i,t} = \alpha_i + \beta_i R^e_{m,t} + u_{i,t}, \qquad i = 1, \ldots, N; \quad t = 1, \ldots, T \tag{14.25}$$

where N is the total number of stocks in the sample and T is the number of time series observations on each stock.

The intercept estimate $(\hat{\alpha}_i)$ from this regression would be ‘Jensen’s alpha’ for the stock, which would measure how much the stock underperformed or outperformed what would have been expected given its level of market risk. It is probably not very interesting to examine the alpha for an individual stock, but we could use exactly the same regression to test the performance of portfolios, trading strategies and so on – all we would do would be to replace the excess returns that comprise the dependent variable with those from the portfolio or trading rule.

Returning to testing the CAPM, suppose that we had a sample of 100 stocks (N = 100) and their returns using five years of monthly data (T = 60). The first step would be to run 100 time series regressions (one for each individual stock), the regressions being run with the sixty monthly data points. Then the second stage would involve a single cross-sectional regression of the average (over time) of the stock returns on a constant and the betas

$$\bar{R}_i = \lambda_0 + \lambda_1 \beta_i + v_i, \qquad i = 1, \ldots, N \tag{14.26}$$

where $\bar{R}_i$ is the return for stock i averaged over the sixty months. Notice that, unlike the first stage, this second stage regression now involves the actual returns and not excess returns. Essentially, the CAPM says that stocks with higher betas are more risky and therefore should command higher average returns to compensate investors for that risk.

If the CAPM is a valid model, two key predictions arise which can be tested using this second stage regression: $\lambda_0 = R_f$ and $\lambda_1 = [R_m - R_f]$. So, to find support for the CAPM, we would expect to see the intercept estimate being close to the risk-free rate of interest and the slope being close to the market risk premium.

Two further implications of the CAPM being valid are first, that there is a linear relationship between a stock’s return and its beta and second, that no other variables should help to explain the cross-sectional variation in returns. So, in other words, any additional variable we add to the second stage regression in equation (14.26) should not have a statistically significant parameter estimate attached to it. We could thus for example run the augmented regression

$$\bar{R}_i = \lambda_0 + \lambda_1 \beta_i + \lambda_2 \beta_i^2 + \lambda_3 \sigma_i^2 + v_i \tag{14.27}$$

where β_i^2 is the squared beta for stock i and σ_i^2 is the variance of the residuals from the first stage regression, which is a measure of idiosyncratic risk for stock i. The squared beta term can capture whether there is any non-linearity in the relationship between returns and beta. If the CAPM is a valid and complete model, then we should see that $\lambda_2 = 0$ and $\lambda_3 = 0$.

However, research has indicated that the CAPM is not a complete model of stock returns. In particular, it has been found that returns are systematically higher for small capitalisation stocks than the CAPM would predict, and similarly, returns are systematically higher for ‘value’ stocks (those with low market-to-book or priceto-earnings ratios) than the CAPM would predict. We can test this directly using a different augmented second stage regression

$$\bar{R}_i = \lambda_0 + \lambda_1 \beta_i + \lambda_2 MV_i + \lambda_3 BTM_i + v_i \tag{14.28}$$

where MV_i is the market capitalisation for stock i and BTM_i is the ratio of its book value to its market value of equity.[7] This is the kind of model employed by Fama and French (1992), as discussed below. As for equation (14.27), the test for the CAPM to be supported by the data would be $\lambda_2 = 0$ and $\lambda_3 = 0$.

Unfortunately, returns data are beset by problems that can render the results from tests of the CAPM dubious or possibly even invalid. First, the familiar non-normality of returns can lead to problems with tests in finite samples – while normality is not a specific theoretical requirement of the CAPM, it is required for valid hypothesis testing. Second, there is also likely to be heteroscedasticity in the returns. More recent research testing the CAPM has used the generalised method of moments (GMM), where estimators can be constructed that are robust to these issues – see for, example, Cochrane (2005). A final important problem is the measurement error in beta discussed extensively in Section 5.13 of this book. In order to minimise such measurement errors, the beta estimates can be based on portfolios rather than individual securities. Alternatively, the Shanken

(1992) correction can be applied, where the standard deviation in the test statistic is multiplied by a factor to adjust for the measurement error.

The Fama–MacBeth Approach

Fama and MacBeth (1973) used the two stage approach to testing the CAPM outlined above, but using a *time series of cross-sections*. The basics are exactly as described above, but instead of running a single time-series regression for each stock and then a single cross-sectional regression, the estimation is conducted with a rolling window.

Fama and MacBeth employ five years of observations to estimate the CAPM betas and the other risk measures (i.e., the standard deviation and squared beta) and these are used as the explanatory variables in a set of cross-sectional regressions each month for the following four years. The estimation period is then rolled forward four years and the process continues until the end of the sample period is reached.[8] To illustrate, their initial time series estimation period for the betas is January 1930 to December 1934. The cross-sectional regressions are run with monthly returns on each stock as the dependent variable for January 1935, and then separately for February 1935, …, to December 1938. The sample is then rolled forward with the beta estimation from January 1934 to December 1938 and the cross-sectional regressions now beginning January 1939. In this way, they end up with a cross-sectional regression for every month in the sample (except for the first five years used for the initial beta estimations).

Since we will have one estimate of the lambdas, $\hat{\lambda}_{j,t}$ $(j = 0, 1, 2, 3)$, for each time period *t*, we can form a *t*-ratio for each of these as being the average over *t*, denoted $\hat{\lambda}_j$, divided by its standard error (which is the standard deviation over time divided by the square root of the number of time series estimates of the $\hat{\lambda}_{j,t}$).

Thus the average value over *t* of $\hat{\lambda}_{j,t}$ can be calculated as

$$\hat{\lambda}_j = \frac{1}{T_{FMB}} \sum_{t=1}^{T_{FMB}} \hat{\lambda}_{j,t}, \quad j = 0, 1, 2, 3 \qquad \textbf{(14.29)}$$

where T_{FMB} is the number of cross-sectional regressions used in the second stage of the test, and the standard deviation is

(14.30)

$$\hat{\sigma}_j = \sqrt{\frac{1}{T_{FMB} - 1} \sum_{t=1}^{T_{FMB}} (\hat{\lambda}_{j,t} - \hat{\lambda}_j)^2}$$

The test statistic is then simply $\sqrt{T_{FMB}}\hat{\lambda}_j/\hat{\sigma}_j$, which is asymptotically standard normal, or follows a t distribution with $T_{FMB} - 1$ degrees of freedom in finite samples. The key results from Fama and MacBeth (1973) corroborate other early evidence by Black, Jensen and Scholes (1972), and are summarised in Table 14.1.

Table 14.1 Fama and MacBeth's results on testing the CAPM

Model	$\hat{\lambda}_0$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$
Model 1: CAPM	0.0061* (3.24)	0.0085* (2.57)		
Model 2: Augmented CAPM	0.0020 (0.55)	0.0114 (1.85)	−0.0026 (−0.86)	0.0516 (1.11)

Notes: t-ratios in parentheses; * denotes significance at the 5% level.

Source: Fama and MacBeth (1973), numbers extracted from their Table 3.

We can compare the estimated values of the intercept and slope with the actual values of the risk-free rate (R_f) and the market risk premium $[\bar{R}_m - \bar{R}_f]$, which are, for the full-sample corresponding to the results presented in the table, 0.013 and 0.143, respectively. The parameter estimates $\hat{\lambda}_0$ and $\hat{\lambda}_1$ have the correct signs (both are positive). Thus the implied risk-free rate is positive and so is the relationship between returns and beta – both parameters are significantly different from zero, although they become insignificant when the other risk measures are included as in the second row of the table. Hence it has been argued that there is *qualitative* support for the CAPM but not *quantitative* support as the intercept and slope are not of the appropriate sizes, although the differences between the estimated parameters and their expected values are not statistically significant for Fama and MacBeth's whole sample. It is also worth noting from the second row of the table that squared beta and idiosyncratic risk have parameters that are even less significant than beta itself in explaining the cross-sectional variation in returns.

14.2.2 Asset Pricing Tests: the Fama–French Approach

Of all of the approaches to asset pricing tests that have been developed, the range of techniques pioneered by Fama and French in a series of papers is by far the most commonly employed. The ‘Fama–French methodology’ is not really a single technique but rather a family of related approaches based on the notion that market risk is insufficient to explain the cross-section of stock returns – in other words, why some stocks generate higher average returns than others.

The Fama–French and Carhart models, described in detail below, seek to measure abnormal returns after allowing for the impact of the characteristics of the firms or portfolios under consideration. It is well-established in the finance literature that certain types of stocks yield, on average, considerably higher returns than others. For example, the stocks of small companies, value stocks (those with low price-to-earnings ratios), and stocks with momentum (that have experienced recent price increases), typically yield higher returns than those having the opposite characteristics. This has important implications for asset pricing and for the way that we think about risk and expected returns. If, for example, we wanted to evaluate the performance of a fund manager, it would be important to take the characteristics of these portfolios into account to avoid incorrectly labelling a manager as having stock-picking skills when he routinely followed a strategy of buying small, value stocks with momentum, which will on average outperform the equity market as a whole.

Fama–French (1992)

The Fama–French (1992) approach, like Fama and MacBeth (1973), is based on a time series of cross-sections model. Here, we run a set of cross-sectional regressions of the form

$$R_{i,t} = \alpha_{0,t} + \alpha_{1,t}\beta_{i,t} + \alpha_{2,t}MV_{i,t} + \alpha_{3,t}BTM_{i,t} + u_{i,t} \quad \textbf{(14.31)}$$

where $R_{i,t}$ are again the monthly returns, $\beta_{i,t}$ are the CAPM betas, $MV_{i,t}$ are the market capitalisations, and $BTM_{i,t}$ are the book-to-price ratios, each for firm i and month t. So the explanatory variables in the regressions here are the firm characteristics themselves. Fama and French show that when we employ size and book-to-market in the cross-sectional regressions, these are highly significantly related to returns (with negative and positive signs respectively) so that small and value stocks earn higher returns all else equal than growth or large stocks. They also show that market beta is not significant in the regression (and even has the wrong sign), providing very strong evidence against the CAPM.

Fama–French (1993)

Fama and French (1993) use a factor-based model in the context of a time series regression which is now run separately on each portfolio *i*

$$R_{i,t} = \alpha_i + \beta_{i,M} RMRF_t + \beta_{i,S} SMB_t + \beta_{i,V} HML_t + \epsilon_{i,t} \tag{14.32}$$

where $R_{i,t}$ is the return on stock or portfolio *i* at time *t*, *RMRF*, *SMB*, and *HML* are the *factor mimicking portfolio* returns for market excess returns, firm size, and value, respectively.[9]

The factor mimicking portfolios are designed to have unit exposure to the factor concerned and zero exposure to all other factors. In more detail, the factors in the Fama and French (1993) model are constructed as follows. The excess market return is measured as the difference in returns between the S&P500 index and the yield on Treasury bills (*RMRF*); *SMB* is the difference in returns between a portfolio of small stocks and a portfolio of large stocks, termed 'small minus big' portfolio returns; *HML* is the difference in returns between a portfolio of value stocks with high book-value to market-value ratios and a portfolio of growth stocks with low book-value to market-value ratios, termed 'high minus low' portfolio returns. One of the main reasons they use factor-mimicking portfolios rather than continuing their (1992) approach is that they want to also include bonds in the set of asset returns considered, and these do not have obvious analogues to market capitalisation or the book-to-market ratio.

In Fama and French's (1993) case, these time series regressions are run on portfolios of stocks that have been two-way sorted according to their book-to-market ratios and their market capitalisations. It is then possible to compare the parameter estimates qualitatively across the portfolios *i*. The parameter estimates from these time series regressions are known as *factor loadings* that measure the sensitivity of each individual portfolio to each of the factors. We will obtain a separate set of factor loadings for each portfolio *i* since each portfolio is the subject of a different time series regression and will have different sensitivities to the risk factors. Fama and French (1993) qualitatively compare these factor loadings across a set of twenty-five portfolios that have been two-way sorted on their size and book-to-market ratios.

Then, the second stage in this approach is to use the factor loadings from the first stage as explanatory variables in a cross-sectional regression

$$\bar{R}_i = \alpha + \lambda_M \beta_{i,M} + \lambda_S \beta_{i,S} + \lambda_V \beta_{i,V} + e_i \tag{14.33}$$

We can interpret the second stage regression parameters, λ_M, λ_S, λ_V as *factor risk premia* – in other words, they show the amount of extra return that is generated on average from taking on an additional unit of that source of risk.

Since the factor loadings and risk premia have a tendency to vary over time, the model is estimated using a rolling window. For example, the time-series model in equation (14.32) is typically estimated using five years of monthly data, and then the λs would be estimated from equation (14.33) using separate cross-sectional regressions with a monthly return for each of the following twelve months. The sample would then be rolled forward by a year with a new set of βs being estimated from equation (14.32) and then a new set of twelve estimates of λ produced and so on. Alternatively, the rolling update could occur monthly. Either way, there will be one estimate of each of the λs for every month after the initial five-year beta estimation window, which we would then average to get the overall estimates of the risk premia.

Fama and French (1993) apply the model to their twenty-five size- and value-sorted portfolios and argue that the statistical significance of the lambdas in the second stage regressions and the high R^2 values are indicative of the importance of size and value as explanators of the cross-sectional variation in returns. See also Gregory *et al.* (2013) for an application of the Fama–French and Carhart models in the UK context.

Carhart (1997)

Since Carhart's (1997) study on mutual fund performance persistence, it has become customary to add a fourth factor to the equations above based on momentum, measured as the difference between the returns on the best performing stocks over the past year and the worst performing stocks – this factor is known as *UMD* – 'up-minus-down'. Equation (14.32) then becomes

$$R_{i,t} = \alpha_i + \beta_{i,M} RMRF_t + \beta_{i,S} SMB_t + \beta_{i,V} HML_t + \beta_{i,U} UMD_t + \epsilon_{i,t} \tag{14.34}$$

And, if desired, equation (14.33) becomes[10]

$$\bar{R}_i = \alpha + \lambda_M \beta_{i,M} + \lambda_S \beta_{i,S} + \lambda_V \beta_{i,V} + \lambda_U \beta_{i,U} + e_i \quad \textbf{(14.35)}$$

Carhart forms decile portfolios of mutual funds based on their one-year lagged performance and runs the time series regression of equation (14.34) on each of them. He finds that the mutual funds which performed best last year (in the top decile) also had positive exposure to the momentum factor (*UMD*) while those which performed worst had negative exposure. Hence a significant portion of the momentum that exists at the fund level arises from momentum in the stocks that those funds are holding.

14.3 Extreme Value Theory

14.3.1 Extreme Value Theory: An Introduction

Much of classical statistics is focused upon accurate estimation of the ‘average’ value of a series (i.e., the mean) or the ‘average’ relationship between two or more series (the OLS regression line), and the central limit theorem is all about the sampling distribution of the means of a set of draws from a series.

However, in many situations it is not the average value that is of interest but rather an extreme and rare event. For instance, in climate forecasting applications, we might be interested in estimating the highest a tide is likely to be this year, or what might be the maximum amount of rainfall expected in a one-day period this spring. Using standard models based on an assumption that the underlying data follow a normal distribution is likely to lead to very inaccurate forecasts for such extreme events because the normal distribution often fits poorly to the tails of the distribution of actual observed values of a series. If we are interested in estimating the likelihood of *extreme* events rather than *typical* ones, it makes sense to use an approach that is focused on modelling the tails such as extreme value theory (EVT).

EVT became more widely adopted in finance from the 1990s onwards as a result of the increasingly prevalent realisation that asset return distributions systematically depart from normality due to their fat tails. Assuming a normal distribution can therefore lead to severe underestimates of the probability of large price movements in either direction, and so importantly can lead to a vast underprediction of the probability of severe losses.

As an illustration, Levine (2009) provides an example based on estimating the parameters of an extreme value distribution using a set of percentage returns on monthly medium maturity A-rated corporate bonds from January 1980 to August 2008. It turns out that the worst return in the sample was −10.84%. The extreme value distribution calculates a probability of a monthly return as low as this or lower (i.e., more negative) during a 30-year period of 1.4%. Yet the corresponding probability if instead we had assumed that the bond returns followed a normal distribution is approximately 8×10^{-7}, which is more than 16,000 times smaller than the probability according to the extreme value distribution. This is an event, albeit a very rare one, which actually happened during the sample period and yet if the series were normally distributed it would be unexpected even in ten millennia.

It is inevitably very hard to estimate tail probabilities accurately since by definition there will be very few extremes and thus very little data upon which to base estimates of the parameters of a tail distribution but EVT will usually get us much closer than the normal distribution.

It is also worth stating that the reverse of the above argument applies: while EVT can provide more a more accurate characterisation of the tail behaviour of a series than a normal distribution, the former should only be applied to the tails and it will not provide accurate estimates nearer the centre of the distribution.

Before we proceed to some notation and relevant formulae, we need to note an important point. Clearly, distributions have two tails (the upper and lower ones), so we need to be careful about which tail we are referring to in any given application. Unfortunately, most textbook treatments present their derivations and models as applied to the upper tail, although in financial risk management it is the lower tail (comprising all of the observations on the biggest losses) that is usually of interest. Since the model is applied separately to each tail and the distribution of actual data is unlikely to be perfectly symmetric, the parameter estimates applied to the upper tail are probably not going to be the same as those arising from an identical estimation on the lower tail. However, fortunately it is easy to flip the data from one tail to another by simply taking the negative of all the data points in that particular tail. Hence the extreme losses become extreme profits, and vice versa. So this is not a big issue but we need to be careful in each case to ensure that we are referring to the tail that we are interested in. For the purposes of the remainder of this section, we will use the upper-tail notation although it should be clear in each case to which tail we are referring from the context.

There are two approaches to parameter estimation that can be adopted under the broad umbrella heading of EVT: the *block maximum* framework and the *peak over threshold* framework. Each of these will now be discussed in turn.

14.3.2 The Block Maximum Approach

To implement this technique, suppose that we have a series, y, of total length T (i.e., we have T observations). We would separate the overall series into m blocks of data, each of length n (thus $m \times n = T$). Let the maximum observed value of the series in each block be denoted by M_k, $k = 1, \ldots, m$. So we have the blocks

$$
\begin{aligned}
M_1 &= max(y_1, y_2, \ldots, y_n) \\
M_2 &= max(y_{n+1}, y_{n+2}, \ldots, y_{2n}) \\
&\vdots \qquad\qquad \vdots \\
M_m &= max(y_{(n(m-1))+1}, y_{(n(m-1))+2}, \ldots, y_{mn})
\end{aligned}
\qquad \textbf{(14.36)}
$$

The results in Fisher and Tippett (1928) and Gnedenko (1943) show that, provided some additional assumptions hold, the distribution of a normalised (rescaled) version of the maxima in each block (i.e., the distribution of (M_1, M_2, …, M_m)) converges asymptotically to a generalised extreme value (GEV) distribution as $m, n \to \infty$.

There are only three classes of extreme value distributions into which the normalised M_k must fall: the Weibull, Gumbel and Frechét. The properties of each of these are discussed in Box 14.1 (p. 596) and their pdfs are plotted in Figure 14.1 (assuming for the plot that $\mu = 0$ and $\sigma = 1$ in all three cases, while $\xi = 0$ for the Gumbel, $\xi = -0.2$ for the Weibull and $\xi = 0.2$ for the Frechét).

BOX 14.1 The three generalised extreme value distributions

The cdfs for all three of the GEVs (Weibull, Gumbel and Frechét) can be described by the following equations (which are, more strictly, the limiting distributions as the sample size tends to infinity)

$$
H_{\xi,\mu,\sigma}(y_t) = \begin{cases} exp[-(1+\xi(y_t-\mu)/\sigma)^{-1/\xi}] & \text{if} \quad \xi \neq 0 \\ exp[-exp(-y_t+\mu)/\sigma)] & \text{if} \quad \xi = 0 \end{cases} \qquad \textbf{(14.37)}
$$

Note that H only exists for values of y_t that satisfy $1 + \xi(y_t - \mu)/\sigma > 0$.

The GEV has three parameters that describe it: ξ is the shape parameter, which defines how fat the tails of the distribution are; μ is the location parameter; and σ is the scale parameter. Sometimes, the shape parameter is expressed in its inverted form, i.e., $1/\xi$, which is known as the *tail index*. Although it would perhaps be slightly misleading to call μ and σ the mean and variance of the distribution given that we are only focusing on the tail, they are analogous to the usual interpretations in describing the central tendency (where most of the data are located) and the dispersion (the spread), respectively. Which one of the three GEVs applies in any particular empirical application depends on the shape parameter, ξ

- The Frechét distribution applies when $\xi > 0$ and has fat tails. Thus it is the most appropriate class for modelling in finance to capture this empirical property of most financial time series that they are leptokurtic.
- The Gumbel distribution applies when $\xi = 0$ and has a medium thickness of tails which decay at an exponential rate.
- The Weibull distribution applies when $\xi < 0$ and has short tails with a finite end point – i.e., a fixed upper limit beyond which the pdf is exactly zero (and the cdf has already reached the value of one). This distribution would thus be appropriate for modelling the tails of some platykurtic series.

These three GEV distributions each encompass many other distributions as special cases. For example, the Frechét includes the Student *t*; the Gumbel includes the normal and log-normal; the Weibull includes some more specialist distributions such as the beta. In addition, we can view the Gumbel as the limiting distribution of the other two as the shape parameter tends to zero from above (for the Frechét) or from below (for the Weibull).

For some members of the GEV family, not all of the moments exist. To illustrate, in the context of the Student's *t* distribution with v degrees of freedom, the kth moment only exists for $v > k$. So, for example, if $v = 3.5$ (corresponding to $\xi = 0.3$ approximately), only the first three moments would exist and none above that.

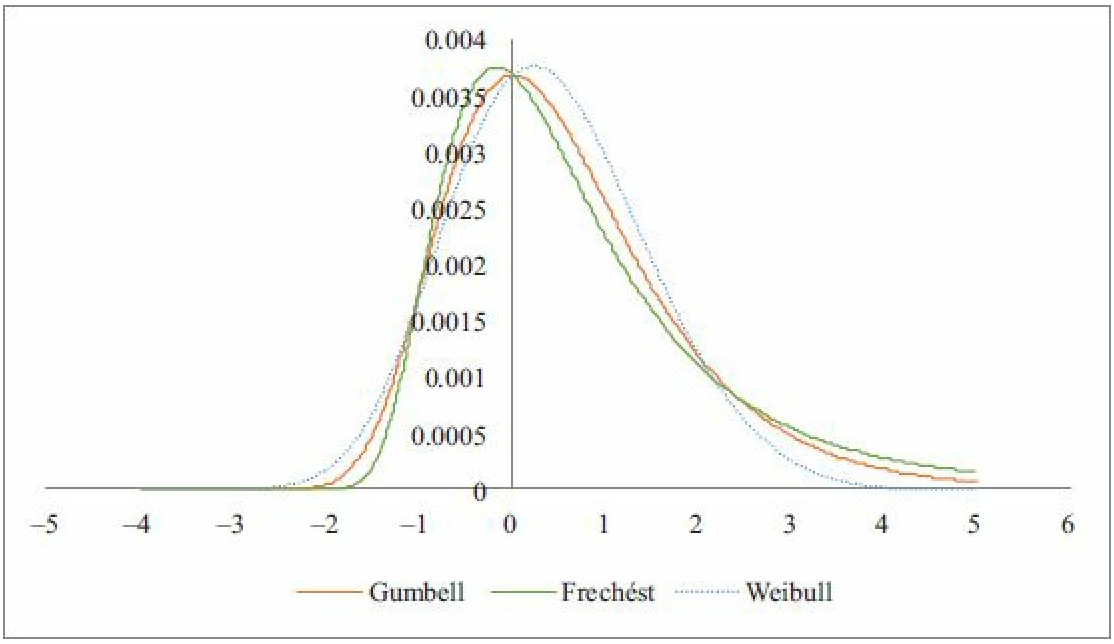

Figure 14.1 Pdfs for the Weibull, Gumbel and Frechét distributions

A tricky issue is, for a given total amount of data *T*, how to split the data into blocks as we could either have more smaller blocks or a lower number of longer blocks. If the blocks are too long (i.e., a small number of long blocks), the number of maxima will be very small and estimation inaccurate leading to parameter estimates with high variance (high standard errors). On the other hand, if the blocks are too short (a large number of short blocks), there is the potential for the maxima in some blocks to not be extreme values, which would cause a bias in the shape parameter estimate. Thus, there is a trade-off between bias and inefficiency here: long blocks = less bias but more inefficiency; short blocks = more bias but less inefficiency.

There are formal approaches that attempt to determine the block length optimally. But these are not discussed here since they are quite involved and in any case, it is clear that formally separating the data into blocks and using only the maximum value in each block is highly inefficient since there may be several extremes in some blocks and yet only the maximum will be counted in each case and the others ignored. An alternative method which avoids this arbitrary split of the data into blocks is known as the *peaks over threshold* (POT) technique. This approach has been preferred in most empirical applications and is considered in the following sub-section.

14.3.3 The Peaks Over Threshold Approach

Under this approach to extreme value modelling, an arbitrary high threshold U is specified, and any observed value of the series y_t exceeding this is defined as being in the extreme. Then in fact the exceedences over the threshold (call these $\bar{y}_t = y_t - U|y_t > U,$ where | means 'given') are actually modelled rather than the y_t themselves. As the threshold U tends to (+/−)∞, the distribution of a normalised (rescaled) version of $\bar{y}$ tends to the *generalised Pareto distribution* (GPD). We can write the cdf of the GPD as

$$G_{\xi,\sigma}(\bar{y}_t) = \begin{cases} 1-(1+\xi\bar{y}_t/\sigma)^{-1/\xi} & \text{if} \quad \xi \neq 0 \\ 1-exp[-\bar{y}_t/\sigma] & \text{if} \quad \xi = 0 \end{cases} \qquad \textbf{(14.38)}$$

where again ξ is the shape parameter and σ is the scale parameter.[11]

As for the block maximum approach, there are three cases corresponding to the Weibull, Gumbel and Frechét distributions, respectively, which in the context of the GPD are the ordinary Pareto, exponential and beta distributions, respectively. Furthermore, as above, the key parameter is ξ, which defines its shape. The first, when $\xi > 0$, corresponds to the fat-tailed case that is common for financial returns data. In fact, the tail index is the inverse of the number of degrees of freedom of the Student t distribution, v, so that $\xi = 1/v$. Common estimates of v are around 4–6, which would correspond to a tail index of 0.1–0.2.[12]

It should be clear that the generalised extreme value and the generalised Pareto distributions are closely related: in the limit in both cases, the former is the distribution of the standardised maxima while the latter is the distribution of the standardised data over a given threshold. The shape parameter, ξ, and therefore its inverse the tail index, are the same for both the extreme value and generalised Pareto distributions and thus the parameters from one approach can be obtained from an estimation under the other.

Choice of Threshold *U*

The choice of threshold U turns out to be important and, akin to the choice of block length, is tricky and involves a trade-off. If the threshold is too low in absolute value terms (and not far enough into the tail) then although this would provide a higher number of data points with which to estimate the distribution's parameters, some observations will be classified extreme when in fact they are not, leading to biased parameter estimates.

On the other hand, if the threshold is set too high, the number of data points classified as extreme will be small and thus although the bias in the estimates will be low, they will have a very high sampling variance as the effective number of observations being used to estimate the parameters will be small.

There are several approaches to reconciling these issues and selecting an 'optimal' threshold – some involve explicit calculations of the bias and variance, while others are more *ad hoc*. The simplest approach is to specify an arbitrary threshold that places, for example, 1% of the distribution over the threshold. However, another fairly straightforward idea that is better matched to the data and seems to work reasonably well is to estimate the parameters using increasingly large values of U until the tail index estimate becomes stable (i.e., it stops changing as U is increased).

14.3.4 Parameter Estimation for Extreme Value Distributions

The parameters ξ, μ and σ can be estimated by maximum likelihood. As usual, this involves setting up a log-likelihood function based on an assumed distribution and then finding the parameter values that maximise it.

Equations (14.37) and (14.38) given above are both cdfs. To obtain the likelihood function, we would need to use the corresponding pdfs, take the natural logarithms and then sum them over the relevant sample observations. So first, to get the pdf, we would differentiate the function $G_{\xi,\sigma}(\tilde{y}_t)$ with respect to $\tilde{y}_t$. This would give the pdf for $\xi \neq 0$ as

$$g_{\xi,\sigma}(\tilde{y}_t) = \frac{1}{\sigma}\left(1 + \frac{\xi \tilde{y}_t}{\sigma}\right)^{-\left(\frac{1}{\xi}+1\right)} \tag{14.39}$$

Then the joint density for all N_U observations on $\tilde{y}_t$ over the threshold U will be the likelihood function given the data, which can be written as

$$LF(\xi, \sigma, \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{N_U}) = \prod_{i=1}^{N_U} g_{\xi,\sigma}(\tilde{y}_t) = \prod_{i=1}^{N_U} \frac{1}{\sigma}\left(1 + \frac{\xi \tilde{y}_t}{\sigma}\right)^{-\left(\frac{1}{\xi}+1\right)} \tag{14.40}$$

The log-likelihood function for a sample of N_U observations on the exceedences of a series y_t over a threshold U, $\tilde{y}_t$ (i.e., an upper tail estimation), is given by taking the natural log of the previous expression and expanding the parentheses

$$LLF(\xi, \sigma, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{N_U}) = -N_U \ln(\sigma) - \left(\frac{1}{\xi} + 1\right) \sum_{i=1}^{N_U} \ln\left(1 + \frac{\xi \bar{y}}{\sigma}\right) \qquad \textbf{(14.41)}$$

where all notation is as above.

Provided that $\xi > -0.5$, maximum likelihood estimators are consistent and asymptotically normally distributed (Hosking and Wallis, 1987), and thus maximum likelihood estimators possesses these desirable properties. However, for extreme value distributions, unlike the normal case, there are no analytical solutions – in other words, there are no maximum likelihood formulae for the estimators given the data and thus a numerical approach (a search procedure) must be used instead. This is a significant disadvantage.

An alternative method is to directly estimate the tail parameter from the actual data using a *non-parametric* approach. A comparison between the two methods is provided in McNeil and Frey (2000). They show that, in some circumstances, the parametric approach may be preferable since it is applicable to a wider range of extreme value distributions and is less affected by the choice of threshold value. The non-parametric approach is much simpler to implement, however, since it does not require any optimisation.

The most common non-parametric approach is the Hill (1975) estimator, which is straightforward to implement and is applicable when the Frechét distribution is the relevant one (i.e., it only works for fat tails and not thin tails). Under some assumptions, the Hill estimator is consistent and asymptotically normally distributed (see Dowd, 2002; Rocco, 2011). The Hill estimator requires the raw data over the threshold $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_T$ to be ordered. Note that the notation becomes much simpler if we reverse order the sample from largest first to smallest last (following, for example, Brooks, Clare, Dalle Molle and Persand, hereafter BCDP, 2005) rather than the standard smallest-to-largest ordering in most of the literature. So if we order the data on $\bar{y}$, the exceedences over the threshold, from the largest, $\bar{y}_{(1)}$, to the smallest, $\bar{y}_{(T)}$, thus $\bar{y}_{(1)} \geq \bar{y}_{(2)} \geq \cdots \geq \bar{y}_{(T)}$, then the Hill estimator of the shape parameter ξ is given by

$$\hat{\xi} = \frac{1}{k-1} \sum_{i=1}^{k-1} [\ln(\bar{y}_{(i)}) - \ln(\bar{y}_{(k)})] \qquad \textbf{(14.42)}$$

where k is an integer to be selected, equal to the number of observations identified as being in the tail and is akin to the number of data points N_U exceeding the threshold of the POT method.

The most common approach to select k is again to estimate ξ over a range of plausible values of k and then to select the lowest value of it for which the estimate of ξ becomes stable. A graph of $\hat{\xi}$ against k is known as a *Hill plot*. Note also that in equations (14.42) and equations (14.43) and (14.44), if we wanted to calculate the tail index, v, rather than the shape parameter ξ, we would need to take the inverse of the expressions $[.]^{-1}$ in each case.

Recalling the law of logs that $\ln(A) - \ln(B) = \ln(A/B)$, we can see that effectively the Hill estimator is taking an average over all observations within $k - 1$ of the logs of the ratio of the value of one extreme point to the next most extreme one. In other words, it is estimating how quickly the tails fade away (the gradient of the pdf in the tail).

Two alternative non-parametric estimators that are effectively variants of the Hill estimator and that also estimate the speed of tail decay are due to Pickands (1975) and De Haan and Resnick (1980). The Pickands estimator is given by

$$\hat{\xi} = \frac{1}{\ln(2)} \ln\left(\frac{\tilde{y}_{(k)} - \tilde{y}_{(2k)}}{\tilde{y}_{(2k)} - \tilde{y}_{(4k)}}\right) \qquad \textbf{(14.43)}$$

The Pickands estimator is also consistent and asymptotically normal but is less efficient than the Hill estimator (Dowd, 2002, p. 212). The De Haan and Resnick estimator is given by

$$\hat{\xi} = \frac{\ln(\tilde{y}_{(1)}) - \ln(\tilde{y}_{(k)})}{\ln(k)} \qquad \textbf{(14.44)}$$

In order to provide a non-parametric tail estimator that removes small sample biases in the estimation of ξ, Huisman et al. (2001) develop a second stage regression of the estimates of ξ on a constant and the corresponding value of k used to estimate it

$$\hat{\xi}_i = \beta_0 + \beta_1 k_i + u_i \qquad \textbf{(14.45)}$$

The modified estimate of ξ is then the intercept from this regression (β_0) which, from the basic definition of a regression intercept, is effectively the value that ξ will take in the limit as $k \to 0$.

Maximum likelihood estimation of the generalised Pareto distribution for a given set of data will yield estimates of both ξ and σ. However, it can be seen from the above that the non-parametric approaches estimate ξ directly, so what about σ (the scale parameter) – how should that be estimated? One approach would be to take the ξ estimate from the non-parametric (e.g., Hill) approach, plug this value into the log-likelihood function as a given constant and then use maximum likelihood to estimate σ as the sole remaining free parameter. This might result in the estimate of the scale parameter being more stable than if it were determined jointly with the shape parameter, but it hardly makes the use of the Hill estimator worthwhile if we then need to use maximum likelihood anyway.

BCDP use the result that if the $\tilde{y}$ follow a Frechét distribution, the scale parameter σ can be calculated given the shape parameter ($\hat{\xi}$, estimated from a non-parametric approach) and the data over the threshold as

$$\hat{\sigma} = \left(\frac{1}{k} \sum_{i=1}^{k} \tilde{y}_i^{1/\hat{\xi}} \right)^{\hat{\xi}} \qquad \textbf{(14.46)}$$

14.3.5 Introduction to Value at Risk

Value at Risk (VaR) is a popular method for measuring financial risks – in that sense, it is a rival to others such a volatility (standard deviation), maximum drawdown, expected shortfall, etc. In general terms, VaR can be defined as an estimation of the financial losses which would be expected to arise from changes in market prices. More precisely, it is defined as the loss in monetary terms that is expected to occur over a pre-determined horizon and with a pre-determined degree of confidence. For example, a company might state that its one-day 99% VaR is ten million dollars. This would be interpreted as implying that the company is 99% confident that the maximum amount that it expects to lose on its portfolio of assets in a one-day period is ten million dollars.

VaR became highly popular as a risk measurement technique in the 1990s, and although its popularity has waned somewhat more recently and expected shortfall has emerged as the preferred approach, it is nonetheless worth studying as a good illustration of how extreme value distributions can be used in a practical finance setting. The evidence suggests that EVT can provide more accurate estimates of value at risk than the delta-normal method (both described below) since it provides a more precise characterisation of the shape of the tails of the distribution of losses, especially for extreme quantiles (such as the 99th percentile).

VaR became prevalent as a result of the simplicity of its calculation, its ease of interpretation, and from the fact that VaR can be suitably aggregated across an entire firm to produce a single number which broadly encompasses the risk of the positions of the firm as a whole.[13]

The calculated VaR is then used as a way to select the appropriate minimum capital risk requirement (MCRR), which is the value of liquid assets that a firm needs to hold in order to ensure that it can cover the expected losses should they materialise. It should be clear that the calculation of VaR and the corresponding selection of capital risk requirements involves a trade-off. If capital requirements are set too low, there is a danger that they will run out when needed most, resulting in financial distress or possibly even insolvency for the bank or securities firm involved. On the other hand, if the amount of capital is too high, then it is locked into an unprofitable usage as liquid assets including cash and Treasury bills usually provide very low returns.

There are several simple ways to compute VaR. The first is to assume that the distribution of portfolio losses is normal, and then we can simply take the appropriate critical value from the normal distribution at the α significance level multiplied by the standard deviation σ of the data y_1, y_2, …, y_N

$$VaR_{normal} = \sigma Z_\alpha \quad \textbf{(14.47)}$$

To then obtain the VaR in cash value terms, we would multiply the figure resulting from this equation by the value of the portfolio. This is sometimes called the *delta-normal* method for VaR calculation. This approach gives equal weight to all observations in the sample period in calculating the standard deviation, σ. But it would also be possible to replace it by an estimate or forecast of σ from any other parametric model – such as an EWMA or GARCH model as described in Chapter 9.

A second approach is to sort the portfolio returns and then simply select the appropriate quantile from the empirical distribution of ordered returns. This is a fully non-parametric approach that does not assume any specific distribution for the returns. This is arguably the simplest method for VaR calculation, and is sometimes termed ‘historical simulation’, a slight misnomer as a label that actually means to collect a sample of the historical returns (on the asset or portfolio under consideration), rank them and then take the fifth or first percentile of the empirical distribution. Then this number, multiplied by the value of the portfolio, is used as the VaR at the 95% or 99% confidence level, respectively. Historical simulation VaR is extremely simple to calculate, and often performs better than the delta-normal approach since it can at least potentially capture the fat tails of actual distributions of losses. However, a key disadvantage is that the technique effectively uses only one data point and ignores all of the other information in the distribution of points that are both less and more extreme, whereas EVT uses all of the data points that are defined as being in the tail.

Using extreme value distributions to calculate VaR proceeds as follows, as summarised nicely in Jorion (2006, Chapter 10). Given the POT approach, assuming that we have estimated the parameters (ξ and σ) from the sample data, the cdf G in equation (14.38) on p. 597 would give the quantile α (from 0 to 1) corresponding to a particular value of $\bar{y}$, $G(\bar{y}) = \alpha$. We thus effectively invert the cdf to determine, given a quantile of the assumed extreme value distribution, what is the corresponding value of y and this would be the VaR

$$VaR = U + \frac{\hat{\sigma}}{\hat{\xi}}\left[\left(\frac{N}{N_U}\alpha\right)^{-\hat{\xi}} - 1\right] \tag{14.48}$$

Usually, $\alpha = 0.01$ or 0.05 would be of most interest, corresponding to 99% and 95% confidence, respectively. Note that the formula includes N/N_U, which is the ratio of the total number of observations in the whole sample (N) to the total number exceeding the threshold (N_U).

For the block maxima approach, VaR can be calculated in an analogous fashion as

$$VaR = \hat{\mu} + \frac{\hat{\sigma}}{\hat{\xi}}\left[1 - (-m\ln(\alpha))^{-\hat{\xi}}\right] \tag{14.49}$$

for $\xi \neq 0$ and where m is the block length and all other notation is as above.

The VaR arising from the Hill estimator is calculated using the formula

$$VaR = \bar{y}_{(k)}\left[\frac{N}{N_U}\alpha\right]^{-\xi} \tag{14.50}$$

where $\bar{y}_k$ is the kth observation in the ordered series $\bar{y}$, which is the same observation corresponding to the threshold when estimating ξ using formula (14.42) on p. 599.

Again, we need slight care here since if we have adopted the convention of multiplying all the returns by −1 to turn the negative tail containing the losses into positives to simplify the calculations, then the appropriate values of α would be 0.99 and 0.95 and U would similarly be a positive number.

14.3.6 Some Final Further Issues in Implementing Extreme Value Theory

- It is possible to construct confidence intervals for VaR estimates arising from extreme value distributions, which would be useful to gauge whether they are precise or not. This is quite challenging to do in a valid and reliable way, however – see McNeil (1998).
- Embedded within the approach to estimating the parameters of an extreme value distribution is an assumption that the data are independently and identically distributed. If the observations are not independent, this can lead to misleading estimates and therefore potentially inaccurate VaR calculations. Given the kind of dependence structure that is usually found in financial data, a common approach that seems to work is to estimate an ARMA-GARCH-type model, collect the standardised residuals (which are more likely to be iid than the original data) and then to estimate the parameters of the extreme value distribution on those – see Rocco (2011) and the references therein for further details.
- A very useful extension of EVT is to the multivariate case where, for example, joint distributions can be employed to measure common dependence and spillovers between extreme events in series. Here, copula functions are used to ‘connect’ the individual distributions for each series. Both the block maxima and POT approaches can be extended to the multivariate context. The multivariate extension, however, brings considerable increased complexity and there is not a unique definition of multivariate extremes. An additional problem is that, due to diversification or hedging effects, simultaneous extreme movements in several individual asset positions may not constitute an extreme for the portfolio comprising those assets as individual movements may be attenuated somewhat or even cancelled out entirely.

14.3.7 An Application of Extreme Value Theory to VaR Estimation

This section will now present an application of extreme value theory for the estimation of value at risk based on research in Gençay and Selçuk (2004). They employ several approaches (the delta-normal method, a Student's *t* distribution, historical simulation and EVT) to calculate and evaluate the VaRs for a set of emerging market equity index returns. They obtain daily data from Datastream for the following countries: Argentina, Brazil, Hong Kong, Indonesia, Korea, Mexico, Singapore, Taiwan and Turkey. The Philippines is also used for some of the analysis but not throughout. The sample period varies by country (presumably because of differences in data availability) but roughly covers the 1993–2000 period.

The returns for Brazil and Turkey have the highest standard deviations, but Hong Kong and Singapore have the highest kurtosis at 36.64 and 61.25, respectively. All series show excess kurtosis, suggesting that a fat-tailed distribution would be appropriate; almost all countries have considerable skewness as well, although strangely its sign varies. Their summary statistics also indicate that a daily loss of over 10% is perfectly possible, but this is up to 1000 times the daily standard deviation and thus a normal distribution would indicate that such a movement is so unlikely as to be almost impossible.

Gençay and Selçuk use the Hill estimator and other diagnostics to determine the appropriate threshold value that specifies whether a particular observation is classified as extreme or not. This information is presented here in Table 14.2, which also shows the corresponding quantile within the distribution for the threshold observation and the number of data points that lie in excess of this threshold (the number of exceedences). In the interests of brevity and since it is of much more concern to most financial market participants (because they correspond to the extreme losses on long positions in those markets), we report only the lower tail part of their results here. The results show that the threshold is further out from the mean observation for Hong Kong (−7), Taiwan (−6.5) and Turkey (−9). In such cases, naturally the number of data points beyond that threshold is lower.

Table 14.2 Threshold percentage returns, corresponding empirical quantiles and the number of exceedences

	Threshold (%)	**Quantile (%)**	**Exceedences (*k*)**
Argentina	−2.7	6.7	129
Brazil	−3.8	7.0	130

Hong Kong	−7.0	0.6	41
Indonesia	−1.0	10.0	21
Korea	−3.5	4.5	130
Mexico	−3.0	5.0	69
Philippines	−4.0	1.8	19
Singapore	−2.5	2.6	101
Taiwan	−6.5	0.6	41
Turkey	−9.0	1.0	28

Source: Gençay and Selçuk (2004), Table 2. Reprinted with permission from Elsevier. Note that only the left tail results are reproduced here for brevity.

Table 14.3 presents the results from maximum likelihood estimation of the shape and scale parameters of the GDP and their associated standard errors. The former vary from a low of 0.03 for Korea and 0.15 for Brazil to a high of 0.60 for Taiwan. Inverting these numbers to obtain the tail indices gives figures of approximately 33.3, 6.67, and 1.67 respectively. This indicates that the Taiwanese series has particularly fat tails, with even the second moment (the variance) not existing ($v < 2$). The same is also almost true for Hong Kong and Singapore followed by the Philippines and Mexico. These results are slightly at variance with the summary statistics that Gençay and Selçuk present, where the Philippines has only a modest kurtosis and a small degree of skewness, perhaps attributable to the fact that the sample moments are estimated based on the whole distribution whereas the GPD parameters are based only on observations beyond the threshold.

Table 14.3 Maximum likelihood estimates of the parameters of the generalised Pareto distribution

	$\hat{\xi}$	SE($\hat{\xi}$)	$\hat{\sigma}$	SE($\hat{\sigma}$)
Argentina	0.20	0.01	1.1	0.2
Brazil	0.15	0.12	1.8	0.3
Hong Kong	0.48	0.22	1.6	0.4
Indonesia	0.32	0.09	0.6	0.1
Korea	0.03	0.11	1.5	0.2
Mexico	0.42	0.18	1.0	0.2
Philippines	0.44	0.37	0.5	0.2
Singapore	0.48	0.14	1.0	0.2
Taiwan	0.60	0.26	0.7	0.2
Turkey	0.22	0.27	1.6	0.5

Source: Gençay and Selçuk (2004), Table 3. Reprinted with permission from Elsevier. Note that only the left tail results are reproduced here for brevity.

The estimates of the scale parameter $(\hat{\sigma})$ vary from 0.5 and 0.6 for the Philippines and Indonesia to 1.8 for Brazil. Although again these estimates are based only on tail observations, in this case they do accord with the whole distribution estimate of the standard deviation, which is high for Turkey and Brazil but low for the Philippines and Indonesia.

Gençay and Selçuk then perform an out-of-sample evaluation of the effectiveness of the VaRs chosen by the various approaches described above. They do this by determining the number of days for which the VaR estimate is violated (i.e., the actual return is more negative than the calculated VaR level). For a good model, we would expect that the percentage of out-of-sample violations (where the VaR is exceeded by the loss) should be roughly the same as the nominal confidence level embedded in the VaR calculation. So, for example, if we have a VaR estimated from a model with 99% confidence, we would expect the VaR to be sufficient on 99% of the out-of-sample test days.

Clearly, if the VaR is higher, we would expect fewer violations but there is a trade-off as described above: if the VaR is too high (an over-estimation of the risk) and thus capital requirements are too high, this conservative approach would lead to capital being tied up unnecessarily. On the other hand, if the VaR is too low (an underestimation of the risk), insufficient liquid capital would be held leading to the possibility of bankruptcy for the firm concerned. There may be an asymmetry within this trade-off, where the implications of insufficient capital are more serious than those of having too much.

Gençay and Selçuk use a rolling window for estimation with the VaR being estimated, for example, on observations 1 to 500, and then the level being compared to the actual return for day 501. The sample is then rolled forward one observation so that 2 to 501 are used for VaR estimation and 502 is used for testing and so on. They provide separate tables presenting the model, at each given confidence level and for each country stock index, which provides the least underestimation of the risk, and the model that provides the least overestimation of the risk. Finally, they provide a table that combines the two and presents the model with the VaR proportion of exceedences closest to the expected proportion given the confidence level. This latter table is shown here as Table 14.4.

Table 14.4 Models that predict the actual left tail quantile most accurately

	5%	**2.5%**	**1%**	**0.5%**	**0.1%**
Argentina	H	N	E	E	E
Brazil	E	H	E	E	E
Hong Kong	H	T	E	E	E
Indonesia	N	T	E	E	E
Korea	T	T	T	T	T
Mexico	N	T	T	E	E
Singapore	E	N	E	E	E
Taiwan	N	T	T	T	E
Turkey	T	T	E	E	E

Source: Gençay and Selçuk (2004), Table 7. Reprinted with permission from Elsevier. Note that only the left tail results are reproduced here for brevity. H, N, T and E denote the situations where historical simulation, the delta-normal approach, the Student's t distribution and the GPD are the best model, respectively, for a given country and quantile, α.

The results in Table 14.4 clearly show that EVT becomes increasingly the approach of choice to calculate VaR as we move further and further into the tails. For VaR nominal exceedence level of 5% or 2.5% (corresponding to confidence levels of 95% and 97.5%, respectively), the Student's t distribution or historical simulation are the most successful overall, but at the 0.1% level (99.9% confidence), EVT performs best for all but one country. In fact, the separate underestimation and overestimation results (not shown here) demonstrate the superiority of EVT in both cases and thus we can conclude that it is more accurate than alternatives in capturing the shape of the tail of financial market return distributions, leading to better VaR estimation.

A further conclusion of Gençay and Selçuk is that the lower and upper tails have very different shapes and behaviour, and therefore it is necessary to use an approach that allows for this difference rather than one assuming a symmetric distribution. They find overall that models based on EVT outperform the more conventional approaches based on an assumption that returns follow a normal distribution or based on historical simulation (i.e., the actual quantile of the historical distribution of returns) for quantiles well within the tails; for quantiles closer to the centre of the distribution – e.g., 5% or 95%, the winning approach is less clear-cut.

14.3.8 Additional Further Reading on Extreme Value Theory

In addition to the in-text citations above, the classic reference on extreme value distributions is the book by Embrechts, Klüppelberg Mikosch (2013), which is at a high technical level. For more general discussions of models for value at risk including short sections on EVT, see Dowd (2002) and Jorion (2006). A particularly accessible review of both the theoretical background to EVT and empirical applications in finance is by Rocco (2011). Empirical applications using several different techniques are compared in Danielsson and de Vries (1997). The study by BCDP (2005) is also worth looking at since it compares various approaches to calculating VaR – both conventional EVT techniques of the type discussed above and the seminonparametric approach proposed by Hsieh (1993) and outlined in the bootstrapping example of Section 13.9 in Chapter 13 of this book.

14.4 The Generalised Method of Moments

14.4.1 Introduction to the Method of Moments

In Chapters 3-5 of this book, we have discussed how the method of least squares can be used to estimate the parameters of a model by setting up a loss function (the residual sum of squares) and minimising it. While least squares has many advantages, including its tractability and our depth of knowledge about how and when it works (and how and when it doesn't), there are two further broad approaches to model parameter estimation that are available and widely used. One of these is *maximum likelihood*, which was discussed in detail in Section 9.9 of Chapter 9 with some mathematical results covered in the Appendix to that chapter; the final estimation technique is known as the *method of moments*, and this will now be discussed in detail in the remainder of this section.

The generalised method of moments (GMM), as the name suggests, provides a generalisation of the conventional method of moments estimator which has found widespread applicability for finance in areas as diverse as asset pricing (including factor models and utility functions), interest rate models, and market microstructure – see Jaganathan, Skoulakis and Wang (2002) for a high-level survey. GMM can be applied in the context of time-series, cross-sectional or panel data. In fact, many other estimators that we have seen at various points in this book are special cases of the GMM estimator: OLS, GLS, instrumental variables, two-stage least squares, and maximum likelihood.

The method of moments technique dates back to Pearson (1895) and in essence it works by computing the moments of the sample data and setting them equal to their corresponding population values based on an assumed probability distribution for the latter. If we have k parameters to estimate, we need k sample moments. So, for example, if the observed data (y) are assumed to follow a normal distribution, there are two parameters we would need to estimate: the mean and the variance. To estimate the population mean (call this μ_0), we know that $E[y_t] - \mu_0 = 0$. We also know that the sample moments will converge to their population counterparts asymptotically by the law of large numbers. So, as the number of data points T increases, we have that

$$\frac{1}{T}\sum_{t=1}^{T} y_t - \mu_0 \to 0 \quad \text{as} \quad T \to \infty$$

Thus the first sample moment condition is found by taking the usual sample average of y_t, $\bar{y}$

$$\frac{1}{T}\sum_{t=1}^{T} y_t - \mu_0 = 0 \qquad \textbf{(14.51)}$$

We would then adopt the same approach to match the second moment

$$\sigma^2 = E[(y_t - \mu_0)^2] \qquad \textbf{(14.52)}$$

and thus

$$\frac{1}{T}\sum_{t=1}^{T} y_t^2 - \sigma^2 = 0 \quad \textbf{(14.53)}$$

and so we have

$$\hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T} y_t^2 - \bar{y}^2 \quad \textbf{(14.54)}$$

If we had a more complex distribution with more than two parameters, we would simply continue to compute the third, fourth, … moments until we had the same number as parameters to estimate.

In the context of estimation of the parameters in a regression model, the method of moments relies principally on the assumption that the explanatory variables are orthogonal to the disturbances in the model

$$E[u_t x_t] = 0 \quad \textbf{(14.55)}$$

for all t where x_t is a $T \times k$ matrix of observations on the explanatory variables and there are $k + 1$ unknowns including an intercept term to keep the notation consistent with the previous chapters.

Given this assumption, if we let β^* denote the true value of β, a vector of parameters, then we can write a moment condition as

$$E[(y_t - x_t'\beta^*)x_t] = 0 \quad \textbf{(14.56)}$$

Solving these moment conditions would lead to the familiar OLS estimator for $\hat{\beta}$ given in equation (4.8) of Chapter 4. Again, in practice we use the sample analogue of the moments of $E(y)$.

In terms of its properties, the method of moments is a consistent estimator but sometimes not efficient. The maximum likelihood technique (see Chapter 9 of this book for details) uses information from the entire assumed distribution function, and OLS requires an assumption about the independence of the error terms while the method of moments (and GMM) use only information on specific moments and thus the latter is more flexible and less restrictive.

14.4.2 The Generalised Method of Moments

The main disadvantage of the conventional method of moments estimator is that it is only applicable in situations where we have exactly the same number of moment conditions (i.e., equations) as unknowns (parameters to estimate) – in other words, we could say that the system is *exactly identified*.[14] However, in most situations we will have more moment conditions than unknowns, in which case the system would be *overidentified*; GMM was developed by Hansen (1982) precisely for this purpose.

If the number of moment conditions is the same as the number of unknowns, then there will be a unique solution that optimises the moment conditions and in the case of the moment condition in equation (14.56) above, it will be exactly satisfied. However, if the number of moment conditions exceeds the number of unknowns, there will be multiple solutions and it is necessary to select the 'best' from among them. A natural way to do this would be to choose the parameter estimates that minimise the variance of the moment conditions. Effectively, via a weighting matrix W, this gives higher weight to moment conditions with a lower variance (in other words, those that are closer to being satisfied).

To establish some more general notation, suppose that we have $l = 1, \ldots, L$ moment conditions and we wish to estimate k parameters in a model, and all of these parameters are stacked into a vector β. We would write the moment conditions as

$$E[m_l(y_t, x_t; \beta)] = 0 \tag{14.57}$$

The sample analogue of this equation is effectively the mean of each moment condition

$$\hat{m}_l(y_t, x_t; \hat{\beta}) = \sum_{t=1}^{T} m_l(y_t, x_t; \hat{\beta}) = 0 \tag{14.58}$$

Note that it does not matter here whether we divide by $1/T$ or not since this term would cancel out anyway. As discussed above, if $L = k$, these L equations will have a unique solution and thus for such exactly identified systems, all of the moment conditions in the equation above will be exactly zero, but there will be more than one solution when $L > k$. In such cases we would choose the parameters that come as near as possible to solving this, which would mean that the sample moment vector is as close to zero as possible. This would be written as

$$\hat{\beta}_{GMM} = \text{argmin}_{\beta}\ \hat{m}(\hat{\beta})' W \hat{m}(\hat{\beta}) \tag{14.59}$$

where $\hat{m}(\hat{\beta}) = (\hat{m}_1, \ldots, \hat{m}_L)$ are the L moment conditions (which will be a function of the estimated parameters, $\hat{\beta}$, and W is the weighting matrix which must be positive definite. It is possible to show that the optimal W is the inverse of the variance–covariance matrix of the moment conditions

$$W = \left[\frac{1}{T}\left(\sum_{t=1}^{T} \hat{m}(\hat{\beta})\hat{m}(\hat{\beta})'\right)\right]^{-1} \tag{14.60}$$

The necessity to choose a weighting matrix is a disadvantage of GMM. Although, as stated, the optimal weighting matrix will be the inverse of the covariance of the moment equations, this depends on the true but unknown parameter vector. The most common approach to dealing with this problem is to use a two-step estimation procedure where in the first stage the weighting matrix is substituted by an arbitrary choice that does not depend on the parameters (such as the identity matrix of appropriate order) and then in the second stage it is substituted by an estimate of the variance based on the parameter estimates given in the first stage. If the weighting matrix is the identity matrix, then minimisation has OLS as a special case. More generally, it can be seen that the form of equation (14.59) is redolent of the GLS approach.

A more sophisticated variant of this technique employs these steps repeatedly, continually updating the parameter estimates and the variance of the moment conditions until the collective change in the parameter estimates from one iteration to the next falls below some pre-specified threshold.

For overidentified systems where there are more moment conditions than parameters to estimate, we can use these degrees of freedom to test the overidentifying restrictions through what is known as the Sargan–Hansen *J*-test, or sometimes just the Sargan *J*-test. The null hypothesis is that all of the moment conditions are exactly satisfied so if the null is rejected it would be indicative that the estimated parameters are not supported by the data. The test statistic is given by

$$\hat{m}(\hat{\beta})'[\text{EAV}(\hat{m}(\hat{\beta}))]^{-1}\hat{m}(\hat{\beta})$$

where EAV is the estimated asymptotic variance, and is asymptotically distributed as a chi-squared with $L - k$ degrees of freedom.

The sampling theory that lies behind GMM, and the test for overidentifying restrictions, are only valid asymptotically, and this might provide particular issues when the number of observations available is small. Monte Carlo simulation evidence in Ferson and Foerster (1994) has suggested that GMM estimators may be oversized for modest numbers of data points.

Under some assumptions, it is possible to show that the GMM estimator is asymptotically normal with mean equal to the true parameter vector and a variance that is an inverse function of the sample size and of the partial derivatives of the moments with respect to the parameters – see Hansen (1982).

14.4.3 GMM in the Asset Pricing Context

One of the most common uses of GMM is in the context of asset pricing models that seek to simultaneously estimate the exposures of the returns on stocks to a set of risk factors and the risk premium per unit of each source of risk. We therefore briefly discuss the setup in this context, loosely following the description and notation in Jaganathan, Skoulakis and Wang (2010). If we define R_t as an $N \times 1$ vector of excess returns (over the risk-free rate) on N stocks at time t, Λ as a $K \times 1$ vector of risk premia and B as a $K \times N$ matrix of factor loadings on a $K \times 1$ vector f_t of K risk factors. In the context of the empirical arbitrage pricing model of Chen, Roll and Ross (1986), these would be broad economic factors such as market risk, unexpected changes in inflation or oil prices or GDP, etc.

Each element in B, which we might term $B_{k,n}$, defines the amount of exposure to factor k that each stock n has. Then a straightforward linear pricing model that defines the expected returns follows as

$$E[R_t] = B\Lambda \tag{14.61}$$

The Fama–MacBeth procedure involves two steps to implement the model (see Section 14.2 earlier in this chapter): first, a set of time-series regressions to estimate the factor exposures, B, and second, a set of cross-sectional regressions to estimate the risk premia, Δ. If we further define μ to be a $K \times 1$ vector of means of each of the factors, the first of these stages to estimate the factor loadings would involve the regressions

$$R_t = A + Bf_t + u_t \tag{14.62}$$

where A is a $N \times 1$ vector of intercept terms, and u_t is a $N \times 1$ vector of disturbances. However, the GMM approach would be able to estimate both B and Λ in a single stage. We could define the moment restrictions as

$$E[R_t - B(\Lambda - \mu + f_t)] = 0 \tag{14.63}$$

$$E[(R_t - B(\Lambda - \mu + f_t))f_t'] = 0 \tag{14.64}$$

$$E[f_t - \mu] = 0 \tag{14.65}$$

Equations (14.63) comprises N moment restrictions, equation (14.64) has $N \times K$ restrictions and equation (14.65) has K restrictions; there will be a total of $N - K$ degrees of freedom which can be used as overidentifying restrictions in a J-test. A natural extension of this framework is to allow either the factor exposures B or the risk premia to be time-varying – see Jaganathan *et al.* (2010) for further details.

14.4.4 A GMM Application to the Link Between Financial Markets and Economic Growth

We now discuss an application of GMM in the context of the link between financial markets and economic growth by Beck and Levine (2004). Their key research question is to what extent the development of the banking sector and the stock market can positively affect the level of economic growth. The theoretical literature proposes that effectively functioning financial intermediation can help the flow of information regarding the quality of investment projects and can reduce transactions costs between investors/savers on the one hand and borrowers/issuers on the other. This would support higher economic growth by ensuring an optimal allocation of resources. Yet there also exist contrary arguments suggesting that greater financial development may harm long-run economic growth, and thus the link between the two is a live issue to be tested empirically.

Beck and Levine examine the roles of both bank lending and the stock market, since they represent quite different forms of financing for firms and may therefore help to overcome different forms of information deficiencies or transactions costs. They establish a 40-country panel of data measured using non-overlapping five-year averages over the 1976 to 1998 period and thus comprising a total of 146 data points. Five-year averages are used rather than annual data to enable the authors to focus on the long run and since several of their variables do not show much variation from one year to the next for each given country.

The variables employed in the model are as follows. Stock market development is proxied by the turnover ratio, which is the total value of shares traded divided by the total value of shares listed on the exchange. The higher this ratio, the deeper is the market and the more frequently the stock is turned over, suggesting higher liquidity and lower transactions costs. Banking sector development is proxied by the ratio of total loans to the private sector divided by gross domestic product (GDP). Several control variables are also employed in the model: the initial level of GDP is included to allow for the 'catching up effect' where countries with lower GDP tend to grow faster and GDP figures converge cross-sectionally; average years of schooling (measures the country's stock of investment in human capital); government consumption; the ratio of imports and exports to GDP (a measure of trade openness); the inflation rate; and the 'black market premium'.[15] The dependent variable in all of their specifications is real per capita GDP growth (or, in some specifications, its first difference).

The basic model is

$$y_{i,t} - y_{i,t-1} = \alpha y_{i,t-1} + \beta' x_{i,t} + \eta_i + u_{i,t} \quad \textbf{(14.66)}$$

where $y_{i,t}$ represents the log of real GDP per capita in country i at time t, $x_{i,t}$ includes all of the explanatory variables except the previous level of GDP per capita (which is separated out), β is a vector of slope parameters and $u_{i,t}$ is a disturbance term. The additional term, η has an i subscript but no t subscript, indicating that it varies by country and not over time. This is a vector of parameters that allows the intercept to be different for each country. These are known as *country fixed effects*, and are discussed in detail in Chapter 11. The authors turn equation (14.66) into a first difference form

$$\begin{aligned}(y_{i,t} - y_{i,t-1}) - (y_{i,t-1} - y_{i,t-2}) &= \alpha_1 (y_{i,t-1} - y_{i,t-2}) \\ &\quad + \beta_1' (x_{i,t} - x_{i,t-1}) + (u_{i,t} - u_{i,t-1})\end{aligned} \quad \textbf{(14.67)}$$

In this equation, the country-specific effects (η_i) have dropped out when using a difference form since they do not vary over time. GMM is employed as the core estimation approach rather than OLS. If we write the error term in this equation (14.67) as $v_{i,t} = (u_{i,t} - u_{i,t-1})$ for simplicity, then we could use the following moment conditions

$$E[y_{i,t-s}v_{i,t}] = 0 \qquad \textbf{(14.68)}$$

$$E[x_{i,t-s}v_{i,t}] = 0 \qquad \textbf{(14.69)}$$

for $s \geq 2$; $t = 3, \ldots, T$ in both cases.

Beck and Levine use several specifications, but for brevity I only report in Table 14.5 here the results from their Table 5, which are based on the above GMM differences specification.

Table 14.5 GMM estimates of the effect of stock markets and bank lending on economic growth

Regressors	**(1)**	**(2)**	**(3)**	**(4)**	**(5)**
Constant	2.089 (0.014)**	2.067 (0.001)***	(0.008)***	2.028 (0.054)*	2.06 (0.005)*
Lagged log(GDP)	−13.59 (0.001)***	−8.517 (0.001)***	−7.374 (0.019)**	−15.956 (0.001)***	−10.547 (0.001)*
Av. years of school	1.554 (0.717)	−1.395 (0.690)	−10.605 (0.012)**	2.557 (0.495)	3.76 (0.271)
Government consumption		2.992 (0.229)			
Trade openness			5.676 (0.001)***		
Inflation rate				0.866 (0.336)	
Black market premium					−0.788 (0.738)
Bank credit	0.749 (0.388)	0.683 (0.426)	−0.471 (0.644)	0.370 (0.656)	0.626 (0.552)
Turnover ratio	−0.36 (0.674)	−0.145 (0.803)	0.699 (0.129)	−0.225 (0.828)	−0.496 (0.506)
Sargan test	0.259	0.120	0.315	0.305	0.155
Serial correlation	0.859	0.530	0.102	0.710	0.800

test					
Wald joint significance test	0.361	0.483	0.189	0.787	0.323

Notes: The dependent variable is the change in the growth of GDP. The first column states the explanatory variables while the numbered columns give the parameter estimates with *p*-values in parentheses. *, ** and *** denote significance at the 10%, 5% and 1%, levels respectively. Av. years of school is measured as ln(1 + number of years). All regressions are conducted with data spanning 40 countries and with a total of 146 observations. The numbers presented in the second panel for the diagnostic tests are all *p*-values.

Source: Beck and Levine (2004). Reprinted with permission from Elsevier.

In the differences regression presented above, neither the bank credit variable nor the turnover ratio have consistent positive signs and are not statistically significant in any of the five specifications, although they are consistently so in the levels GMM regression (not reported here, where the dependent variable is the level of GDP growth rather than the change in GDP growth). In the levels regression, the authors give the example of Mexico, whose stock market turnover ratio and bank lending to the private sector were particularly low, but if the values of the two variables had instead been at the OECD's average level then GDP would have been expected to grow by 0.6 and 0.8 percentage points more per year, respectively.

The log of initial GDP per capita parameter estimates are statistically significant and negative for all five specifications, indicating a 'regression to the mean' due to a convergence effect where countries with already high GDP grow more slowly, as expected; the parameter estimate on the trade openness variable also has a positive sign and is statistically significant in model (3) in the table where it is included.

The second panel of Table 14.5 reports *p*-values for tests of three summary diagnostic measures for the model. The first of these is the Sargan *J*-test and since the *p*-value for all five models is greater than 0.1, we would conclude that the overidentifying restrictions are satisfied and that the moments are close to zero and therefore the proposed models are adequate. Likewise, for the autocorrelation test reported in the second row of that panel, the *p*-values are all greater than 0.1 (albeit only marginally in specification (3)), and therefore there is no evidence of autocorrelation in the residuals from the fitted model. The final row presents the *p*-values from Wald tests, akin to the regression *F*-statistic which measures the joint significance of all parameters in the model. In this case, we would want to reject the null hypothesis that all of the parameters are zero, but we are unable to do so in any of the specifications and thus the model in differences form displayed here fails this test. This is not the case, however, for the models in other forms or for the more complex hybrid between the levels and differences form not presented here – see Beck and Levine (2004) for further details.

The main conclusion from the study is that both stock market depth and bank lending – and thus overall financial development – enhance economic growth as both have positive and statistically significant parameter estimates in the majority of specifications that Beck and Levine examine. As is often the case, while GMM is demonstrably superior from an econometric perspective, as the authors note, the conclusions are mostly not qualitatively altered compared with the case where OLS is used.

14.4.5 Additional Further Reading

In addition to the in-text citations above, the core reference for GMM as a technique is the book by Hall (2005); a further mathematical treatment is available in Hamilton (1994, Chapter 14). Within the finance area specifically, Cochrane's (2005) book is also very useful.

KEY CONCEPTS

The key terms to be able to define and explain from this chapter are

- event study
- cross-sectional dependence
- buy-and-hold abnormal return
- time-series of cross-sections
- extreme value distribution
- block maximum
- generalised Pareto distribution
- value at risk
- weighting matrix
- cumulative abnormal return
- event window
- Fama–MacBeth approach
- factor risk premia
- peak over threshold
- Weibull, Gumbel and Frechét

- Hill estimator
- generalised method of moments
- moment conditions

SELF-STUDY QUESTIONS

1. (a) What is an event study? Present and explain one potential use of an event study.

(b) Explain the difference between the cumulative abnormal return and buy-and-hold abnormal return methods of constructing a test statistic. What are the advantages and disadvantages of each approach?

(c) Why might we use a non-parametric test in event studies and what would be the advantages and disadvantages of doing so compared with a parametric test?

2. (a) Explain the steps in the Fama–MacBeth method for testing the CAPM.

(b) What is the time-series of cross-sections?

3. (a) Why is a member of the extreme value family of distributions usually more appropriate for capturing tail behaviour than assuming normality for financial time-series?

(b) Explain the differences between the block maxima and peak over threshold frameworks for estimating the parameters of an extreme value distribution.

(c) What are the Weibull, Gumbel and Frechét distributions and which is the more appropriate for financial data?

(d) Outline the Hill and De Haan–Resnick approaches for estimating the parameters of extreme value distributions.

(e) What is the link between the tail index and the shape parameter for an extreme value distribution and what is the range of plausible values of the former?

(f) Explain how value at risk can be calculated using the delta normal method, historical simulation, and EVT. Compare and contrast the three approaches

4. (a) Explain the principle behind the method of moments estimator.

(b) What is the difference between the method of moments and the generalised method of moments (GMM)?

(c) Explain what overidentifying restrictions are and how they are used in the context of GMM estimation.

(d) Contrast the Fama–MacBeth and GMM procedures for empirical asset pricing – what are the advantages and disadvantages of each approach?

1 We need to be aware of the potential impacts that thin trading of stocks may have, leading to stale prices and unrepresentative abnormal returns; however, this issue is not discussed further here.

2 Note that in some studies, since the sample variance has to be estimated, the test statistic is assumed to follow a Student's t distribution with $T - k$ degrees of freedom in finite samples, where k is the number of parameters estimated in constructing the measure of expected returns (k = 2 for the market model). Provided that the estimation window is of a reasonable length (e.g., six months of trading days or more), it will be inconsequential whether the t or normal distributions are employed.

3 The number of days during the period T_1 to T_2 including both the end points is $T_2 - T_1 + 1$.

4 Although Lyon, Barber and Tsai (1999) propose a skewness-adjusted t-statistic with bootstrapping to mitigate this problem.

5 The example below uses a small sample of real data from a real event, but no details are given as to the nature of the event so that they can be distributed freely with the book.

6 Note of course that it is not possible to calculate Z for the event date by itself since the proportion of negative returns, p would be either exactly zero or exactly one.

7 Note that many studies use the market-to-book ratio, which is simply one divided by the book-to-market ratio – so value stocks have a low number for the former and a high number for the latter.

8 The main reason that the updating was only undertaken every four years was due to the lack of computing power available at that time. More recent studies would do this annually or even monthly.

9 While this model could be applied to individual stocks, it makes more sense in the context of portfolios, although the principles are the same.

10 Note that Carhart's (1997) paper does not use this second-stage cross-sectional regression containing the factor sensitivities.

[11] Note that β is often used to denote the latter parameter. In the context of $\bar{y}_t$ being exceedences over a threshold, it would not make sense to also have a location parameter.

[12] Note that the notation and terminology differs somewhat across studies in a very confusing manner. For example, Brooks, Clare, Dalle Molle and Persand (2005), amongst others, use the term 'tail index' when strictly they mean the shape parameter that we denote by ξ here.

[13] We need to note, however, that VaR is not *sub-additive*, meaning that the VaR of a portfolio is not a fixed combination of the VaRs of the component assets, and indeed in some circumstances the former can be larger than the sum of the latter.

[14] See Chapter 7 for a detailed discussion of this concept in a different context.

[15] Neither this variable nor government consumption appear to be defined in the paper.

15

Conducting Empirical Research or Doing a Project or Dissertation in Finance

LEARNING OUTCOMES

In this chapter, you will learn how to

- Choose a suitable topic for an empirical research project in finance
- Draft a research proposal
- Find appropriate sources of literature and data
- Determine a sensible structure for the dissertation

15.1 What is an Empirical Research Project and What is it For?

Many courses, at both the undergraduate and postgraduate levels, require or allow the student to conduct a project. This may vary from being effectively an extended essay to a full-scale dissertation or thesis of 10,000 words or more.

Students often approach this part of their degree with much trepidation, although in fact doing a project gives students a unique opportunity to select a topic of interest and to specify the whole project themselves from start to finish. The purpose of a project is usually to determine whether students can define and execute a piece of fairly original research within given time, resource and report-length constraints. In terms of econometrics, conducting empirical research is one of the best ways to get to grips with the theoretical material, and to find out what practical difficulties econometricians encounter when conducting research. Conducting the research gives the investigator the opportunity to solve a puzzle and potentially to uncover something that nobody else has; it can be a highly rewarding experience. In addition, the project allows students to select a topic of direct interest or relevance to them, and is often useful in helping students to develop time-management and report-writing skills. The final document can in many cases provide a platform for discussion at job interviews, or act as a springboard to further study at the taught postgraduate or doctoral level.

This chapter seeks to give suggestions on how to go about the process of conducting empirical research in finance. Only general guidance is given, and following this advice cannot necessarily guarantee high marks, for the objectives and required level of the project will vary from one institution to another.[1]

15.2 Selecting the Topic

Following the decision or requirement to do a project, the first stage is to determine an appropriate *subject area*. This is, in many respects, one of the most difficult and most crucial parts of the whole exercise. Some students are immediately able to think of a precise topic, but for most, it is a process that starts with specifying a very general and very broad subject area, and subsequently narrowing it down to a much smaller and manageable problem.

Inspiration for the choice of topic may come from a number of sources. A good approach is to think rationally about your own interests and areas of expertise. For example, you may have worked in the financial markets in some capacity, or you may have been particularly interested in one aspect of a course unit that you have studied. It is worth spending time talking to some of your instructors in order to gain their advice on what are interesting and plausible topics in their subject areas. At the same time, you may feel very confident at the quantitative end of finance, pricing assets or estimating models for example, but you may not feel comfortable with qualitative analysis where you are asked to give an opinion on particular issues (e.g., ‘should financial markets be more regulated?’). In that case, a highly technical piece of work may be appropriate.

Equally, many students find econometrics both difficult and uninteresting. Such students may be better suited to more qualitative topics, or topics that involve only elementary statistics, but where the rigour and value added comes from some other aspect of the problem. A case-study approach that is not based on any quantitative analysis may be entirely acceptable and indeed an examination of a set of carefully selected case studies may be more appropriate for addressing particular problems, especially in situations where hard data are not readily available, or where each entity is distinct so that generalising from a model estimated on one set of data may be inadvisable. Case studies are useful when the case itself is unusual or unique or when each entity under study is very heterogeneous. They involve more depth of study than quantitative approaches. Highly mathematical work that has little relevance and which has been applied inappropriately may be much weaker than a well constructed and carefully analysed case study.

Combining all of these inputs to the choice of topic should enable you at the least to determine whether to conduct quantitative or non-quantitative work, and to select a general subject area (e.g., pricing securities, market microstructure, risk management, asset selection, operational issues, international finance, financial econometrics, etc.). The project may take one of a number of forms as illustrated in Box 15.1.

BOX 15.1 Possible types of research project

- An empirical piece of work involving quantitative analysis of data
- A survey of business practice in the context of a financial firm
- A new method for pricing a security, or the theoretical development of a new method for hedging an exposure
- A critical review of an area of literature
- An analysis of a new market or new asset class.

Each of these types of project requires a slightly different approach, and is conducted with varying degrees of success. The remainder of this chapter focuses upon the type of study which involves the formulation of an empirical model using the tools developed in this book. This type of project seems to be the one most commonly selected. It also seems to be a lower risk strategy than others. For example, projects which have the bold ambition to develop a new financial theory, or a whole new model for pricing options, are likely to be unsuccessful and to leave the student with little to write about. Also, critical reviews often lack rigour and are not critical enough, so that an empirical application involving estimating an econometric model appears to be a less risky approach, since the results can be written up whether they are ‘good’ or not.

A good project or dissertation must have an element of *originality*, i.e., a ‘contribution to knowledge’. It should add, probably a very small piece, to the overall picture in that subject area, so that the body of knowledge is larger at the end than before the project was started. This statement often scares students, for they are unsure from where the originality will arise. In empirically based projects, this usually arises naturally. For example, a project may employ standard techniques on data from a different country or a new market or asset, or a project may develop a new technique or apply an existing technique to a different area. Interesting projects can often arise when ideas are taken from another field and applied to finance – for example, you may be able to identify ideas or approaches from the material that you studied from a different discipline as part of your undergraduate degree.

A good project will also contain an in-depth analysis of the issues at hand, rather than a superficial, purely descriptive presentation, as well as an individual contribution. A good project will be interesting, and it will have relevance for one or more user groups (although the user group may be other academic researchers and not necessarily practitioners); it may or may not be on a currently fashionable and newsworthy topic. The best research challenges prior beliefs and changes the way that the reader thinks about the problem under investigation. Good projects can be primarily of interest to other academics and they do not necessarily have to be of direct practical applicability. On the other hand, highly practical work must also be well grounded in the academic approach to doing research.

The next stage is to transform this broad direction into a workably sized topic that can be tackled within the constraints laid down by the institution. It is important to ensure that the aims of the research are not so broad or substantive that the questions cannot be addressed within the constraints on available time and word limits. The objective of the project is usually not to solve the entire world’s financial puzzles, but rather to form and address a small problem.

It is often advisable at this stage to browse through recent issues of the main journals relevant to the subject area. This will show which ideas are relatively fashionable, and how existing research has tackled particular problems. A list of relevant journals is presented in Table 15.1. They can be broadly divided into two categories: practitioner-oriented and academic journals. Practitioner-oriented journals are usually very focused in a particular area, and articles in these often centre on very practical problems, and are typically less mathematical in nature and less theory-based, than are those in academic journals. Of course, the divide between practitioner and academic journals is not a total one, for many articles in practitioner journals are written by academics and vice versa! The list given in Table 15.1 is by no means exhaustive and, particularly in finance, new journals appear on a monthly basis.

Table 15.1 Journals in finance and econometrics

Journals in finance	**Journals in econometrics and related fields**
Applied Financial Economics	Biometrika
Applied Mathematical Finance	Econometrica
European Financial Management	Econometric Reviews
European Journal of Finance	Econometric Theory
Finance and Stochastics	Econometrics Journal
Financial Analysts Journal	International Journal of Forecasting
Financial Management	Journal of Applied Econometrics
Financial Review	Journal of Business and Economic Statistics
Global Finance Journal	Journal of Econometrics
International Journal of Finance & Economics	Journal of Forecasting
International Journal of Theoretical and Applied Finance	Journal of the American Statistical Association Journal of Financial Econometrics
Journal of Applied Corporate Finance	Journal of the Royal Statistical Society (A to C)
International Review of Financial Analysis	Journal of Time Series Analysis
Journal of Applied Finance	Society for Nonlinear Dynamics

	and Econometrics
Journal of Asset Management	
Journal of Banking and Finance	
Journal of Business	
Journal of Business Finance & Accounting	
Journal of Computational Finance	
Journal of Corporate Finance	
Journal of Derivatives	
Journal of Empirical Finance	
Journal of Finance	
Journal of Financial & Quantitative Analysis	
Journal of Financial Economics	
Journal of Financial Markets	
Journal of Financial Research	
Journal of Fixed Income	
Journal of Futures Markets	
Journal of International Financial Markets, Institutions and Money	
Journal of International Money and Finance	
Journal of Money, Credit, and Banking	
Journal of Portfolio Management	
Journal of Risk	
Journal of Risk and Insurance	
Journal of Risk and Uncertainty	
Mathematical Finance	

Pacific Basin Finance Journal	
Quarterly Review of Economics and Finance	
Review of Asset Pricing Studies	
Review of Behavioural Finance	
Review of Corporate Finance Studies	
Review of Finance	
Review of Financial Studies	
Risk	

Many web sites contain lists of journals in finance or links to finance journals. Some useful ones are

- www.cob.ohio-state.edu/dept/fin/overview.htm–the Virtual Finance Library, with good links and a list of finance journals
- www.helsinki.fi/WebEc/journals.html – provides a list of journals in the economics area, including finance, plus a number of finance-related resources
- www.people.hbs.edu/pgompers/finjourn.htm – provides a list of links to finance journals
- www.numa.com/ref/journals.htm – the Numa directory of derivatives journals – lots of useful links and contacts for academic and especially practitioner journals on derivatives
- www.aeaweb.org/econlit/journal_list.php – provides a comprehensive list of journals in the economics area, including finance

15.3 Sponsored or Independent Research?

Some business schools are sufficiently well connected with industry that they are able to offer students the opportunity to work on a specific research project with a ‘sponsor’. The sponsor may choose the topic and offer additional expert guidance from a practical perspective. Sponsorship may give the student an insight into the kind of research problems that are of interest to practitioners, and will probably ensure that the work is practically focused and of direct relevance in the private sector. The sponsor may be able to provide access to proprietary or confidential data, which will broaden the range of topics that could be tackled. Most importantly, many students hope that if they impress the firm that they are working with, a permanent job offer will follow.

The chance to work on a sponsored project is usually much sought after by students but it is very much a double-edged sword, so that there are also a number of disadvantages. First, most schools are not able to offer such sponsorships, and even those that can are usually able to provide them to only a fraction of the class. Second, the disappointing reality is that the problems of most interest and relevance to practitioners are often (although admittedly not always) of less interest to an academic audience – fundamentally, the objectives of the sponsor and of a university may be divergent. For example, a stereotypical investment bank might like to see a project that compares a number of technical trading rules and evaluates their profitability; but many academics would argue that this area has been well researched before and that finding a highly profitable rule does not constitute a contribution to knowledge and is therefore weak as a research project. So if you have the opportunity to undertake a sponsored project, ensure that your research is of academic as well as practical value – after all, it will almost certainly be the academic who grades the work.

15.4 The Research Proposal

Some schools will require the submission of a research proposal which will be evaluated and used to determine the appropriateness of the ideas and to select a suitable supervisor. While the requirements for the proposal are likely to differ widely from one institution to another, there are some general points that may be universally useful. In some ways, the proposal should be structured as a miniature version of the final report, but without the results or conclusions!

- The required length of the proposal will vary, but will usually be between one and six sides of A4, typed with page numbering.
- The proposal should start by briefly motivating the topic – why is it interesting or useful?
- There should be a **brief** review of the relevant literature, but this should not cover more than around a third to one half of the total length of the proposal.
- The research questions or hypotheses to be tested should then be clearly stated.

- There should be a discussion of the data and methodology that you intend to use.
- Some proposals also include a time-scale – i.e. which parts of the project do you expect to have completed by what dates?

15.5 Working Papers and Literature on the Internet

Unfortunately, the lag between a paper being written and it actually being published in a journal is often two–three years (and increasing fast), so that research in even the most recent issues of the published journals will be somewhat dated. Additionally, many securities firms, banks and central banks across the world, produce high quality research output in report form, which they often do not bother to try to publish. Much of this is now available on the internet, so it is worth conducting searches with keywords using readily available web search engines. A few suggestions for places to start are given in Table 15.2.

Table 15.2 Useful internet sites for financial literature

Universities
Almost all universities around the world now make copies of their discussion papers available electronically.
A few examples from finance departments are:
stern.nyu.edu/finance – Department of Finance, Stern School, New York University
fic.wharton.upenn.edu/fic/papers.html – Wharton Financial Institutions Center
haas.berkeley.edu/finance/WP/rpf.html – University of California at Berkeley
www.icmacentre.ac.uk/research/discussion-papers – ICMA Centre, University of Reading, of course!
US Federal Reserve Banks and the Bank of England
www.bankofengland.co.uk – Bank of England – containing their working papers, news and discussion
www.frbatlanta.org – Federal Reserve Bank of Atlanta – including information on economic and research data and publications

www.stls.frb.org/fred – Federal Reserve Bank of St. Louis – a great deal of useful US data, including monetary, interest rate, and financial data, available daily, weekly, or monthly, including long time histories of data
www.chicagofed.org – Federal Reserve Bank of Chicago – including interest data and useful links
www.dallasfed.org – Federal Reserve Bank of Dallas – including macroeconomic, interest rate, monetary and bank data
www.federalreserve.gov/pubs/ifdp – Federal Reserve Board of Governors International Finance Discussion Papers
www.ny.frb.org/research – Federal Reserve Bank of New York
International bodies
dsbb.imf.org – the International Monetary Fund (IMF) – including working papers, forecasts, and IMF primary commodity price series
www.worldbank.org/reference – World Bank working papers in finance
www.oecd-ilibrary.org – Organisation for Economic Cooperation and Development (OECD) working papers, data etc., searchable
Miscellaneous
www.nber.org – National Bureau of Economic Research (NBER) – huge database of discussion papers and links including data sources
econpapers.repec.org – Econpapers (formerly WoPEc) – huge database of working papers in areas of economics, including finance
www.ssrn.com – The Social Science Research Network – a huge and rapidly growing searchable database of working papers and the abstracts of published papers
The free data sources used in this book
www.nationwide.co.uk/default.htm – UK house price index, quarterly back to 1952, plus house prices by region and by property type
www.oanda.com/convert/fxhistory – historical exchange rate series for an incredible range of currency pairs
www.bls.gov – US Bureau of Labor Statistics – US macroeconomic series

www.federalreserve.gov/econresdata/default.htm – US Federal Reserve Board – more US macroeconomic series, interest rates, etc. and working papers	
research.stlouisfed.org/fred2 – a vast array of US macroeconomic series	
finance.yahoo.com – Yahoo! Finance – an incredible range of free financial data, information, research and commentary	

15.6 Getting the Data

Although there is more work to be done before the data are analysed, it is important to think prior to doing anything further about *what data are required* to complete the project. Many interesting and sensible ideas for projects fall flat owing to a lack of availability of relevant data. For example, the data required may be confidential, they may be available only at great financial cost, they may be too time-consuming to collect from a number of different paper sources, and so on. So before finally deciding on a particular topic, make sure that the data are going to be available.

The data may be available at your institution, either in paper form (for example, from the IMF or World Bank reports), or preferably electronically. Many universities have access to Reuters, Datastream or the Bloomberg. Many of the URLs listed above include extensive databases and furthermore, many markets and exchanges have their own web pages detailing data availability. One needs to be slightly careful, however, in ensuring the accuracy of freely available data; 'free' data also sometimes turn out not to be!

15.7 Choice of Computer Software

Clearly, the choice of computer software will depend on the tasks at hand. Projects that seek to offer opinions, to synthesise the literature and to provide a review, may not require any specialist software at all. However, even for those conducting highly technical research, project students rarely have the time to learn a completely new programming language from scratch while conducting the research. Therefore, it is usually advisable, if possible, to use a standard software package. It is also worth stating that marks will hardly ever be awarded for students who 'reinvent the wheel'. Therefore, learning to program a multivariate GARCH model estimation routine in C++ may be a valuable exercise for career development for those who wish to be quantitative researchers, but is unlikely to attract high marks as part of a research project unless there is some other value added. The best approach is usually to conduct the estimation as quickly and accurately as possible to leave time free for other parts of the work.

15.8 Methodology

Good research is rarely purely empirical – the empirical model should arise from an economic or financial *theory* and this theory should be presented and discussed before the investigative work begins. We could define a theory as a system of statements that encompass a number of hypotheses. Theory shows what features in the data and what relationships would be expected based on some underlying principles. Theory can give order and meaning to empirical results, and can ensure that the findings are not the result of a data-mining exercise.

Assuming that the project is empirical in nature (i.e., it seeks to test a theory or answer a particular question using actual data), then an important question will concern the type of model to employ. Hopefully this book has provided a solid foundation from which that choice can be made.

15.9 How Might the Finished Project Look?

Different projects will of course require different structures, but it is worth outlining at the outset the form that a good project or dissertation will take. Unless there are good reasons for doing otherwise (for example, because of the nature of the subject), it is advisable to follow the format and structure of a full-length article in a scholarly journal. In fact, many journal articles are, at approximately 5,000 words long, roughly the same length as a student research project. A suggested outline for an empirical research project in finance is presented in Table 15.3. We shall examine each component in Table 15.3 in turn.

Table 15.3 Suggested structure for a typical dissertation or project

Title page
Abstract or executive summary
Acknowledgements
Table of contents

Section 1: Introduction
Section 2: Literature review
Section 3: Data
Section 4: Methodology
Section 5: Results
Section 6: Conclusions
References
Appendices

Title Page

The *title page* is usually not numbered, and will contain only the title of the project, the name of the author, and the name of the department, faculty, or centre in which the research is being undertaken.

The Abstract

The *abstract* is usually a short summary of the problem being addressed and of the main results and conclusions of the research. The maximum permissible length of the abstract will vary, but as a general guide, it should not be more than 300 words in total. The abstract should usually not contain any references or quotations, and should not be unduly technical, even if the subject matter of the project is.

Acknowledgements

The *acknowledgements* page is a list of people whose help you would like to note. For example, it is courteous to thank your instructor or project supervisor (even if he/she was useless and didn't help at all), any agency that gave you the data, friends who read and checked or commented upon the work, etc. It is also 'academic etiquette' to put a disclaimer after the acknowledgements, worded something like 'Responsibility for any remaining errors lies with the author(s) alone'. This also seems appropriate for a dissertation, for it symbolises that the student is completely responsible for the topic chosen, and for the contents and the structure of the project. It is your project, so you cannot blame anyone else, either deliberately or inadvertently, for anything wrong with it! The disclaimer should also remind project authors that it is not valid to take the work of others and to pass it off as one's own. Any ideas taken from other papers should be adequately referenced as such, and any sentences lifted directly from other research should be placed in quotations and attributed to their original author(s).

Table of Contents

The *table of contents* should list the sections and sub-sections contained in the report. The section and sub-section headings should reflect accurately and concisely the subject matter that is contained within those sections. It should also list the page number of the first page of each section, including the references and any appendices.

The abstract, acknowledgements and table of contents pages are usually numbered with lower case Roman numerals (e.g., i, ii, iii, iv, etc.), and the introduction then starts on page 1 (reverting back to Arabic numbers), with page numbering being consecutive thereafter for the whole document, including references and any appendices.

Introduction

The *introduction* should give some very general background information on the problem considered, and why it is an important area for research. A good introductory section will also give a description of what is *original* in the study – in other words, how does this study help to advance the literature on this topic or how does it address a new problem, or an old problem in a new way? What are the aims and objectives of the research? If these can be clearly and concisely expressed, it usually demonstrates that the project is well defined. The introduction should be sufficiently non-technical that the intelligent non-specialist should be able to understand what the study is about, and it should finish with an outline of the remainder of the report.

Literature Review

Before commencing any empirical work, it is essential to thoroughly review the existing literature, and the relevant articles that are found can be summarised in the *literature review* section. This will not only help to give ideas and put the proposed research in a relevant context, but may also highlight potential problem areas. Conducting a careful review of existing work will ensure that up-to-date techniques are used and that the project is not a direct (even if unintentional) copy of an already existing work.

The literature review should follow the style of an extended literature review in a scholarly journal, and should always be *critical in nature*. It should comment on the relevance, value, advantages and shortcomings of the cited articles. Do not simply provide a list of authors and contributions – the review should be written in continuous prose and not in note form. It is important to demonstrate understanding of the work and to provide a critical assessment – i.e., to point out important weaknesses in existing studies. Being ‘critical’ is not always easy but is a delicate balance; the tone of the review should remain polite. The review should synthesise existing work into a summary of what is and is not known and should identify trends, gaps and controversies.

Some papers in the literature are *seminal*: they change the way that people have thought about a problem or have had a major influence on policy or practice They might be introducing a new idea or an idea new to that subject area. Reviews can sometimes be organised around such papers and certainly any literature review should cite the seminal works in the field.

The process of writing a literature review can be made much easier if there exists a closely related *survey* or *review* paper. Review papers are published and (usually) high quality and detailed reports on a particular area of research. However, it goes without saying that you should not simply copy the review for several reasons. First, your topic may not match exactly that of the survey paper. Second, there may be more recent studies that are not included in the review paper. Third, you may wish to have a different emphasis and a wider perspective.

An interesting question is whether papers from low ranking journals, poorly written papers, those that are methodologically weak, and so on, be included in the review? This is, again, a difficult balance. In general the answer is probably not, but they should be included if they are directly relevant to your own work, but you should be sure to highlight the weaknesses of the approaches used.

Data

The *data* section should describe the data in detail – the source, the format, the features of the data and any limitations which are relevant for later analysis (for example, are there missing observations? Is the sample period short? Does the sample include large potential structural breaks, e.g. caused by a stock market crash?). If there are a small number of series which are being primarily investigated, it is common to plot the series, noting any interesting features, and to supply summary statistics – such as the mean, variance, skewness, kurtosis, minimum and maximum values of each series, tests for non-stationarity, measures of autocorrelation, etc.

Methodology

'*Methodology*' should describe the estimation technique(s) used to compute estimates of the parameters of the model or models. The models should be outlined and explained, using equations where appropriate. Again, this description should be written *critically*, noting any potential weaknesses in the approach and, if relevant, why more robust or up-to-date techniques were not employed. If the methodology employed does not require detailed descriptions, this section may usefully be combined with the data section.

Results

The *results* will usually be tabulated or graphed, and each table or figure should be described, noting any interesting features – whether expected or unexpected, and in particular, inferences should relate to the original aims and objectives of the research outlined in the introduction. Results should be *discussed and analysed*, not simply presented blandly. Comparisons should also be drawn with the results of similar existing studies if relevant – do your results confirm or contradict those of previous research? Each table or figure should be mentioned explicitly in the text (e.g., 'Results from estimation of equation (11) are presented in Table 4'). Do not include in the project any tables or figures which are not discussed in the text. It is also worth trying to present the results in as interesting and varied a way as possible – for example, including figures and charts as well as just tables.

Conclusions

The *conclusions* section should re-state the original aim of the dissertation and outline the most important results. Any weaknesses of the study as a whole should be highlighted, and finally some suggestions for further research in the area should be presented.

References

A list of *references* should be provided, in alphabetical order by author. Note that a list of *references* (a list of all the papers, books or web pages referred to in the study, irrespective of whether you read them, or found them cited in other studies), as opposed to a bibliography (a list of items that you read, irrespective of whether you referred to them in the study), is usually required.

Although there are many ways to show citations and to list references, one possible style is the following. The citations given in the text can be given as ‘Brooks (1999) demonstrated that …’ or ‘A number of authors have concluded that …(see, for example, Brooks, 1999).

All works cited can be listed in the references section using the following style:

Books

Harvey, A. C. (1993) *Time Series Models*, second edition, Harvester Wheatsheaf, Hemel Hempstead, England

Published articles

Hinich, M. J. (1982) Testing for Gaussianity and Linearity of a Stationary Time Series, *Journal of Time Series Analysis* 3(3), 169–176

Unpublished articles or theses

Bera, A. K. and Jarque, C. M. (1981) An Efficient Large-Sample Test for Normality of Observations and Regression Residuals, *Australian National University Working Papers in Econometrics* 40, Canberra.

Appendices

Finally, an *appendix* or *appendices* can be used to improve the structure of the study as a whole when placing a specific item in the text would interrupt the flow of the document. For example, if you want to outline how a particular variable was constructed, or you had to write some computer code to estimate the models, and you think this could be interesting to readers, then it can be placed in an appendix. The appendices should not be used as a dumping ground for irrelevant material, or for padding, and should not be filled with printouts of raw output from computer packages!

15.10 Presentational Issues

There is little sense in making the final report longer than it needs to be. Even if you are not in danger of exceeding the word limit, superfluous material will generate no additional credit and may be penalised. Assessors are likely to take into account the presentation of the document, as well as its content. Hence students should ensure that the structure of their report is orderly and logical, that equations are correctly specified, and that there are no spelling or other typographical mistakes, or grammatical errors.

Some students find it hard to know when to stop the investigative part of their work and get to the tidying up stage. Of course, it is always possible to make a piece of work better by working longer on it but there comes a point when further work on the project seems counterproductive because the remaining time is better spent on improving the writing and presentational aspects. It is definitely worth reserving a week at the end of the allocated project time if possible to read the draft paper carefully at least twice. Also, your supervisor or advisor may be willing to read through the draft and to offer comments upon it prior to final submission. If not, maybe friends who have done similar courses can give suggestions. All comments are useful – after all, any that you do not like or agree with can be ignored!

1 Note that there is only one review question for this chapter and that is to write an excellent research project.

Appendix 1

Sources of Data Used in This Book and the Accompanying Software Manuals

I am grateful to the following people and organisations, who all kindly agreed to allow their data to be used as examples in this book (plus the accompanying software manuals) and for the files to be copied onto the book's web site: Alan Gregory/Rajesh Tharyan, the Bureau of Labor Statistics, Federal Reserve Board, Federal Reserve Bank of St. Louis, Nationwide, Oanda, and Yahoo! Finance. The following table gives details of the data used and of the provider's web site.

Provider	Data	Web
Alan Gregory/Rajesh Tharyan	Size/value-sorted portfolios and Fama–French factors	business-school.exeter.ac.uk/research/areas/centres
Bureau of Labor Statistics	CPI	www.bls.gov
Federal Reserve Board	US T-bill yields, money supply, industrial production, consumer credit	www.federalreserve.gov
Federal	average	research.stlouisfed.org/fred2

Reserve Bank of St. Louis	AAA & BAA corporate bond yields	
Nationwide	UK average house prices	www.nationwide.co.uk/hpi/datadownload
Oanda	euro–dollar, pound–dollar & yen–dollar exchange rates	www.oanda.com/convert/fxhistory
Yahoo! Finance	S&P500 and various US stock and futures prices	finance.yahoo.com

Appendix 2

Tables of Statistical Distributions

Table A2.1 Normal critical values for different values of α

α	0.4	0.25	0.2	0.15	0.1	0.05	0.025	0.0
Z_α	.2533	.6745	.8416	1.0364	1.2816	1.6449	1.9600	2.3

Source: Author's computation using the NORMDIST function in Excel.

Table A2.2 Critical values of Student's *t*-distribution for different probability levels, α and degrees of freedom, v

α	**0.4**	**0.25**	**0.15**	**0.1**	**0.05**	**0.025**	**0.01**
v							
1	0.3249	1.0000	1.9626	3.0777	6.3138	12.7062	31.8205
2	0.2887	0.8165	1.3862	1.8856	2.9200	4.3027	6.9646
3	0.2767	0.7649	1.2498	1.6377	2.3534	3.1824	4.5407
4	0.2707	0.7407	1.1896	1.5332	2.1318	2.7764	3.7469
5	0.2672	0.7267	1.1558	1.4759	2.0150	2.5706	3.3649
6	0.2648	0.7176	1.1342	1.4398	1.9432	2.4469	3.1427
7	0.2632	0.7111	1.1192	1.4149	1.8946	2.3646	2.9980
8	0.2619	0.7064	1.1081	1.3968	1.8595	2.3060	2.8965
9	0.2610	0.7027	1.0997	1.3830	1.8331	2.2622	2.8214
10	0.2602	0.6998	1.0931	1.3722	1.8125	2.2281	2.7638
11	0.2596	0.6974	1.0877	1.3634	1.7959	2.2010	2.7181
12	0.2590	0.6955	1.0832	1.3562	1.7823	2.1788	2.6810

13	0.2586	0.6938	1.0795	1.3502	1.7709	2.1604	2.6503
14	0.2582	0.6924	1.0763	1.3450	1.7613	2.1448	2.6245
15	0.2579	0.6912	1.0735	1.3406	1.7531	2.1314	2.6025
16	0.2576	0.6901	1.0711	1.3368	1.7459	2.1199	2.5835
17	0.2573	0.6892	1.0690	1.3334	1.7396	2.1098	2.5669
18	0.2571	0.6884	1.0672	1.3304	1.7341	2.1009	2.5524
19	0.2569	0.6876	1.0655	1.3277	1.7291	2.0930	2.5395
20	0.2567	0.6870	1.0640	1.3253	1.7247	2.0860	2.5280
21	0.2566	0.6864	1.0627	1.3232	1.7207	2.0796	2.5176
22	0.2564	0.6858	1.0614	1.3212	1.7171	2.0739	2.5083
23	0.2563	0.6853	1.0603	1.3195	1.7139	2.0687	2.4999
24	0.2562	0.6848	1.0593	1.3178	1.7109	2.0639	2.4922
25	0.2561	0.6844	1.0584	1.3163	1.7081	2.0595	2.4851
26	0.2560	0.6840	1.0575	1.3150	1.7056	2.0555	2.4786
27	0.2559	0.6837	1.0567	1.3137	1.7033	2.0518	2.4727
28	0.2558	0.6834	1.0560	1.3125	1.7011	2.0484	2.4671
29	0.2557	0.6830	1.0553	1.3114	1.6991	2.0452	2.4620
30	0.2556	0.6828	1.0547	1.3104	1.6973	2.0423	2.4573
35	0.2553	0.6816	1.0520	1.3062	1.6896	2.0301	2.4377
40	0.2550	0.6807	1.0500	1.3031	1.6839	2.0211	2.4233
45	0.2549	0.6800	1.0485	1.3006	1.6794	2.0141	2.4121
50	0.2547	0.6794	1.0473	1.2987	1.6759	2.0086	2.4033
60	0.2545	0.6786	1.0455	1.2958	1.6706	2.0003	2.3901
70	0.2543	0.6780	1.0442	1.2938	1.6669	1.9944	2.3808
80	0.2542	0.6776	1.0432	1.2922	1.6641	1.9901	2.3739
90	0.2541	0.6772	1.0424	1.2910	1.6620	1.9867	2.3685
100	0.2540	0.6770	1.0418	1.2901	1.6602	1.9840	2.3642

120	0.2539	0.6765	1.0409	1.2886	1.6577	1.9799	2.3578
150	0.2538	0.6761	1.0400	1.2872	1.6551	1.9759	2.3515
200	0.2537	0.6757	1.0391	1.2858	1.6525	1.9719	2.3451
300	0.2536	0.6753	1.0382	1.2844	1.6499	1.9679	2.3388
∞	0.2533	0.6745	1.0364	1.2816	1.6449	1.9600	2.3263

Source: Author's own computation using the TINV function in Excel.

Table A2.3 Upper 5% critical values for F-distribution

Degrees of freedom for denominator ($T-k$)	Degrees of freedom for numerator (m)																		
	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161	200	216	225	230	234	237	239	241	242	244	246	248	249	250	251	252	253	254
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.5	19.5	19.5	19.5	19.5	19.5
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2:33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08.	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

Source: Author's own computation using the Excel FINV function.

Table A2.4 Upper 1% critical values for F-distribution

	Degrees of freedom for numerator (m)																		
	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
Degrees of freedom for denominator ($T-k$)																			
1	4,052	5,000	5,403	5,625	5,764	5,859	5,928	5,982	6,023	6,056	6,106	6,157	6,209	6,235	6,261	6,287	6,313	6,339	6,366
2	98.5	99.0	99.2	99.3	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.4	99.5	99.5	99.5	99.5	99.5	99.5	99.5
3	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	27.1	26.9	26.7	26.6	26.5	26.4	26.4	26.2	26.1
4	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.5	14.4	14.2	14.0	13.9	13.8	13.7	13.7	13.6	13.5
5	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
6	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
7	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
8	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
9	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
10	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
14	8.86	6.51	5.56	5.04	4.70	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.00
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75
17	8.40	6.11	5.19	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
19	8.19	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
25	7.77	5.57	4.68	4.18	3.86	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.62	2.53	2.45	2.36	2.27	2.17
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00

Source: Author's own computation using the Excel CHIINV function.

Table 2.5 Chi-squared critical values for different values of α and degrees of freedom, v

v	0.995	0.990	0.975	0.950	0.900	0.750	0.500	0.250	0.100	0.050	0.025	0.010	0.005
1	0.00004	0.00016	0.00098	0.00393	0.01579	0.1015	0.4549	1.323	2.706	3.841	5.024	6.635	7.879
2	0.01003	0.02010	0.05065	0.1026	0.2107	0.5754	1.386	2.773	4.605	5.991	7.378	9.210	10.597
3	0.07172	0.1148	0.2158	0.3518	0.5844	1.213	2.366	4.108	6.251	7.815	9.348	11.345	12.838
4	0.2070	0.2971	0.4844	0.7107	1.064	1.923	3.357	5.385	7.779	9.488	11.143	13.277	14.860
5	0.4117	0.5543	0.8312	1.145	1.610	2.675	4.351	6.626	9.236	11.070	12.833	15.086	16.750
6	0.6757	0.8721	1.237	1.635	2.204	3.455	5.348	7.841	10.645	12.592	14.449	16.812	18.548
7	0.9893	1.239	1.690	2.167	2.833	4.255	6.346	9.037	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	5.071	7.344	10.219	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	5.899	8.343	11.389	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	6.737	9.342	12.549	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	7.584	10.341	13.701	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	8.438	11.340	14.845	18.54	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.041	9.299	12.340	15.984	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	10.165	13.339	17.117	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	11.036	14.339	18.245	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	11.912	15.338	19.369	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	12.792	16.338	20.489	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	13.675	17.338	21.605	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	14.562	18.338	22.718	27.204	30.143	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	15.452	19.337	23.828	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	16.344	20.337	24.935	29.615	32.670	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	17.240	21.337	26.039	30.813	33.924	36.781	40.289	42.796

23	9.260	10.196	11.688	13.090	14.848	18.137	22.337	27.141	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	19.037	23.337	28.241	33.196	36.415	39.364	42.080	45.558
25	10.520	11.524	13.120	14.611	16.473	19.939	24.337	29.339	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	20.843	25.336	30.434	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	21.749	26.336	31.528	36.741	40.113	43.194	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	22.657	27.336	32.620	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	23.567	28.336	33.711	39.087	42.557	45.722	49.588	52.336
30	13.787	14.954	16.791	18.493	20.599	24.478	29.336	34.800	40.256	43.773	46.979	50.892	53.672
35	17.192	18.509	20.569	22.465	24.797	29.054	34.336	40.223	46.059	49.802	53.203	57.342	60.275
40	20.707	22.164	24.433	26.509	29.050	33.660	39.335	45.616	51.805	55.758	59.342	63.691	66.766
45	24.311	25.901	28.366	30.612	33.350	38.291	44.335	50.985	57.505	61.656	65.410	69.957	73.166
50	27.991	29.707	32.357	34.764	37.689	42.942	49.335	56.334	63.167	67.505	71.420	76.154	79.490
55	31.735	33.571	36.398	38.958	42.060	47.611	54.335	61.665	68.796	73.311	77.381	82.292	85.749
60	35.535	37.485	40.482	43.158	46.459	52.294	59.335	66.981	74.397	79.082	83.298	85.379	91.952
70	43.275	45.442	48.758	51.739	55.329	61.698	69.334	77.577	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	71.144	79.334	88.130	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	80.625	89.334	98.650	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	90.133	99.334	109.141	118.498	124.342	129.561	135.807	140.169
120	83.829	86.909	91.568	95.705	100.627	109.224	119.335	130.051	140.228	146.565	152.214	158.963	163.670
150	109.122	112.655	117.980	122.692	126.278	137.987	149.334	161.258	172.577	179.579	185.803	193.219	198.380
200	152.224	156.421	162.724	168.279	174.825	156.175	199.334	213.099	226.018	233.993	241.060	249.455	255.281
250	196.145	200.929	208.095	214.392	221.809	234.580	249.334	264.694	279.947	287.889	295.691	304.948	311.361

Source: Author's own computation using the Excel CHIINV function.

Table A2.6 Lower and upper 1% critical values for the Durbin–Watson statistic

	$k' = 1$		$k' = 2$		$k' = 3$		$k' = 4$		$k' = 5$	
T	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U
15	0.81	1.07	0.70	1.25	0.59	1.46	0.49	1.70	0.39	1.96
16	0.84	1.09	0.74	1.25	0.63	1.44	0.53	1.66	0.44	1.90
17	0.87	1.10	0.77	1.25	0.67	1.43	0.57	1.63	0.48	1.85
18	0.90	1.12	0.80	1.26	0.71	1.42	0.61	1.60	0.52	1.80
19	0.93	1.13	0.83	1.26	0.74	1.41	0.65	1.58	0.56	1.77
20	0.95	1.15	0.86	1.27	0.77	1.41	0.68	1.57	0.60	1.74
21	0.97	1.16	0.89	1.27	0.80	1.41	0.72	1.55	0.63	1.71
22	1.00	1.17	0.91	1.28	0.83	1.40	0.75	1.54	0.66	1.69
23	1.02	1.19	0.94	1.29	0.86	1.40	0.77	1.53	0.70	1.67
24	1.04	1.20	0.96	1.30	0.88	1.41	0.80	1.53	0.72	1.66
25	1.05	1.21	0.98	1.30	0.90	1.41	0.83	1.52	0.75	1.65

26	1.07	1.22	1.00	1.31	0.93	1.41	0.85	1.52	0.78	1.64
27	1.09	1.23	1.02	1.32	0.95	1.41	0.88	1.51	0.81	1.63
28	1.10	1.24	1.04	1.32	0.97	1.41	0.90	1.51	0.83	1.62
29	1.12	1.25	1.05	1.33	0.99	1.42	0.92	1.51	0.85	1.61
30	1.13	1.26	1.07	1.34	1.01	1.42	0.94	1.51	0.88	1.61
31	1.15	1.27	1.08	1.34	1.02	1.42	0.96	1.51	0.90	1.60
32	1.16	1.28	1.10	1.35	1.04	1.43	0.98	1.51	0.92	1.60
33	1.17	1.29	1.11	1.36	1.05	1.43	1.00	1.51	0.94	1.59
34	1.18	1.30	1.13	1.36	1.07	1.43	1.01	1.51	0.95	1.59
35	1.19	1.31	1.14	1.37	1.08	1.44	1.03	1.51	0.97	1.59
36	1.21	1.32	1.15	1.38	1.10	1.44	1.04	1.51	0.99	1.59
37	1.22	1.32	1.16	1.38	1.11	1.45	1.06	1.51	1.00	1.59
38	1.23	1.33	1.18	1.39	1.12	1.45	1.07	1.52	1.02	1.58
39	1.24	1.34	1.19	1.39	1.14	1.45	1.09	1.52	1.03	1.58
40	1.25	1.34	1.20	1.40	1.15	1.46	1.10	1.52	1.05	1.58
45	1.29	1.38	1.24	1.42	1.20	1.48	1.16	1.53	1.11	1.58
50	1.32	1.40	1.28	1.45	1.24	1.49	1.20	1.54	1.16	1.59
55	1.36	1.43	1.32	1.47	1.28	1.51	1.25	1.55	1.21	1.59
60	1.38	1.45	1.35	1.48	1.32	1.52	1.28	1.56	1.25	1.60
65	1.41	1.47	1.38	1.50	1.35	1.53	1.31	1.57	1.28	1.61
70	1.43	1.49	1.40	1.52	1.37	1.55	1.34	1.58	1.31	1.61
75	1.45	1.50	1.42	1.53	1.39	1.56	1.37	1.59	1.34	1.62
80	1.47	1.52	1.44	1.54	1.42	1.57	1.39	1.60	1.36	1.62
85	1.48	1.53	1.46	1.55	1.43	1.58	1.41	1.60	1.39	1.63
90	1.50	1.54	1.47	1.56	1.45	1.59	1.43	1.61	1.41	1.64
95	1.51	1.55	1.49	1.57	1.47	1.60	1.45	1.62	1.42	1.64
100	1.52	1.56	1.50	1.58	1.48	1.60	1.46	1.63	1.44	1.65

Note: T, number of observations; k', number of explanatory variables (excluding a constant term).

Source: Durbin and Watson (1951): 159–77. Reprinted with the permission of Oxford University Press.

Table A2.7 Dickey–Fuller critical values for different significance levels, α

Sample size T	0.01	0.025	0.05	0.10
τ				
25	−2.66	−2.26	−1.95	−1.60
50	−2.62	−2.25	−1.95	−1.61
100	−2.60	−2.24	−1.95	−1.61
250	−2.58	−2.23	−1.95	−1.62
500	−2.58	−2.23	−1.95	−1.62
∞	−2.58	−2.23	−1.95	−1.62
τ_μ				
25	−3.75	−3.33	−3.00	−2.63
50	−3.58	−3.22	−2.93	−2.60
100	−3.51	−3.17	−2.89	−2.58
250	−3.46	−3.14	−2.88	−2.57
500	−3.44	−3.13	−2.87	−2.57
∞	−3.43	−3.12	−2.86	−2.57
τ_τ				
25	−4.38	−3.95	−3.60	−3.24
50	−4.15	−3.80	−3.50	−3.18
100	−4.04	−3.73	−3.45	−3.15
250	−3.99	−3.69	−3.43	−3.13
500	−3.98	−3.68	−3.42	−3.13
∞	−3.96	−3.66	−3.41	−3.12

Source: Fuller (1976). Reprinted with the permission of John Wiley and Sons.

Table A2.8 Critical values for the Engle–Granger cointegration test on regression residuals with no constant in test regression

Number of variables in system	Sample size T	0.01	0.05	0.10
	50	−4.32	−3.67	−3.28
2	100	−4.07	−3.37	−3.03
	200	−4.00	−3.37	−3.02
	50	−4.84	−4.11	−3.73
3	100	−4.45	−3.93	−3.59
	200	−4.35	−3.78	−3.47
	50	−4.94	−4.35	−4.02
4	100	−4.75	−4.22	−3.89
	200	−4.70	−4.18	−3.89
	50	−5.41	−4.76	−4.42
5	100	−5.18	−4.58	−4.26
	200	−5.02	−4.48	−4.18

Source: Engle and Granger (1987). Reprinted with the permission of Elsevier.

Table A2.9 Quantiles of the asymptotic distribution of the Johansen cointegration rank test statistics (constant in cointegrating vectors only)

$p-r$	50%	80%	90%	95%	97.5%	99%	Mean	Var
λ_{max}								
1	3.40	5.91	7.52	9.24	10.80	12.97	4.03	7.07
2	8.27	11.54	13.75	15.67	17.63	20.20	8.86	13.08
3	13.47	17.40	19.77	22.00	24.07	26.81	14.02	19.24
4	18.70	22.95	25.56	28.14	30.32	33.24	19.23	23.83
5	23.78	28.76	31.66	34.40	36.90	39.79	24.48	29.26
6	29.08	34.25	37.45	40.30	43.22	46.82	29.72	34.63
7	34.73	40.13	43.25	46.45	48.99	51.91	35.18	38.35
8	39.70	45.53	48.91	52.00	54.71	57.95	40.35	41.98
9	44.97	50.73	54.35	57.42	60.50	63.71	45.55	44.13
10	50.21	56.52	60.25	63.57	66.24	69.94	50.82	49.28
11	55.70	62.38	66.02	69.74	72.64	76.63	56.33	54.99
λ_{Trace}								
1	3.40	5.91	7.52	9.24	10.80	12.97	4.03	7.07
2	11.25	15.25	17.85	19.96	22.05	24.60	11.91	18.94
3	23.28	28.75	32.00	34.91	37.61	41.07	23.84	37.98
4	38.84	45.65	49.65	53.12	56.06	60.16	39.50	59.42
5	58.46	66.91	71.86	76.07	80.06	84.45	59.16	91.65
6	81.90	91.57	97.18	102.14	106.74	111.01	82.49	126.94
7	109.17	120.35	126.58	131.70	136.49	143.09	109.75	167.91
8	139.83	152.56	159.48	165.58	171.28	177.20	140.57	208.09
9	174.88	198.08	196.37	202.92	208.81	215.74	175.44	257.84
10	212.93	228.08	236.54	244.15	251.30	257.68	213.53	317.24
11	254.84	272.82	282.45	291.40	298.31	307.64	256.15	413.35

Source: Osterwald-Lenum (1992, Table 1*). Reprinted with the permission of Blackwell Publishers.

Table A2.10 Quantiles of the asymptotic distribution of the Johansen cointegration rank test statistics (constant, i.e., a drift only in VAR and in cointegrating vector)

$p-r$	50%	80%	90%	95%	97.5%	99%	Mean	Var
λ_{max}								
1	0.44	1.66	2.69	3.76	4.95	6.65	0.99	2.04
2	6.85	10.04	12.07	14.07	16.05	18.63	7.47	12.42
3	12.34	16.20	18.60	20.97	23.09	25.52	12.88	18.67
4	17.66	21.98	24.73	27.07	28.98	32.24	18.26	23.47
5	23.05	27.85	30.90	33.46	35.71	38.77	23.67	28.82
6	28.45	33.67	36.76	39.37	41.86	45.10	29.06	33.57
7	33.83	39.12	42.32	45.28	47.96	51.57	34.37	37.41
8	39.29	45.05	48.33	51.42	54.29	57.69	39.85	42.90
9	44.58	50.55	53.98	57.12	59.33	62.80	45.10	44.93
10	49.66	55.97	59.62	62.81	65.44	69.09	50.29	49.41
11	54.99	61.55	65.38	68.83	72.11	75.95	55.63	54.92
λ_{Trace}								
1	0.44	1.66	2.69	3.76	4.95	6.65	0.99	2.04
2	7.55	11.07	13.33	15.41	17.52	20.04	8.23	14.38
3	18.70	23.64	26.79	29.68	32.56	35.65	19.32	32.43
4	33.60	40.15	43.95	47.21	50.35	54.46	34.24	52.75
5	52.30	60.29	64.84	68.52	71.80	76.07	52.95	79.25
6	75.26	84.57	89.48	94.15	98.33	103.18	75.74	114.65
7	101.22	112.30	118.50	124.24	128.45	133.57	101.91	158.78
8	131.62	143.97	150.53	156.00	161.32	168.36	132.09	201.82
9	165.11	178.90	186.39	192.89	198.82	204.95	165.90	246.45
10	202.58	217.81	225.85	233.13	239.46	247.18	203.39	300.80
11	243.90	260.82	269.96	277.71	284.87	293.44	244.66	379.56

Source: Osterwald-Lenum (1992, Table 1). Reprinted with the permission of Blackwell Publishers.

Table A2.11 Quantiles of the asymptotic distribution of the Johansen cointegration rank test statistics (constant in cointegrating vector and VAR, trend in cointegrating vector)

$p-r$	50%	80%	90%	95%	97.5%	99%	Mean	Var
λ_{max}								
1	5.55	8.65	10.49	12.25	14.21	16.26	6.22	10.11
2	10.90	14.70	16.85	18.96	21.14	23.65	11.51	16.38
3	16.24	20.45	23.11	25.54	27.68	30.34	16.82	22.01
4	21.50	26.30	29.12	31.46	33.60	36.65	22.08	27.74
5	26.72	31.72	34.75	37.52	40.01	42.36	27.32	31.36
6	32.01	37.50	40.91	43.97	46.84	49.51	32.68	37.91
7	37.57	43.11	46.32	49.42	51.94	54.71	38.06	39.74
8	42.72	48.56	52.16	55.50	58.08	62.46	43.34	44.83
9	48.17	54.34	57.87	61.29	64.12	67.88	48.74	49.20
10	53.21	59.49	63.18	66.23	69.56	73.73	53.74	52.64
11	58.54	64.97	69.26	72.72	75.72	79.23	59.15	56.97
λ_{Trace}								
1	5.55	8.65	10.49	12.25	14.21	16.26	6.22	10.11
2	15.59	20.19	22.76	25.32	27.75	30.45	16.20	24.90
3	29.53	35.56	39.06	42.44	45.42	48.45	30.15	45.68
4	47.17	54.80	59.14	62.99	66.25	70.05	47.79	74.48
5	68.64	77.83	83.20	87.31	91.06	96.58	69.35	106.56
6	94.05	104.73	110.42	114.90	119.29	124.75	94.67	143.33
7	122.87	134.57	141.01	146.76	152.52	158.49	123.51	182.85
8	155.40	169.10	176.67	182.82	187.91	196.08	156.41	234.11
9	192.37	207.25	215.17	222.21	228.05	234.41	193.03	288.30
10	231.59	247.91	256.72	263.42	270.33	279.07	232.25	345.23
11	276.34	294.12	303.13	310.81	318.02	327.45	276.88	416.98

Source: Osterwald-Lenum (1992, Table 2*). Reprinted with the permission of Blackwell Publishers.

Glossary

This glossary gives brief definitions of all the key terms used in the book. For more details, go back and read the chapters or the references therein!

abnormal return: a measure of the performance (return) of a financial asset calculated by subtracting the expected performance (based on a model or benchmark) from the actual return.

adjusted R^2: a measure of how well a model fits the sample data that automatically penalises models with large numbers of parameters.

Akaike information criterion (AIC): a metric that can be used to select the best fitting from a set of competing models and that incorporates a weak penalty term for including additional parameters.

alternative hypothesis: a formal expression as part of a hypothesis testing framework that encompasses all of the remaining outcomes of interest aside from that incorporated into the null hypothesis.

arbitrage: a concept from finance that refers to the situation where profits can be made without taking any risk (and without using any wealth).

arithmetic progression: a number sequence where the change from one entry in the series to the next is a fixed number.

asymptotic: a property that applies as the sample size tends to infinity.

autocorrelation: a standardised measure, which must lie between −1 and +1, of the extent to which the current value of a series is related to its own previous values.

autocorrelation function: a set of estimated values showing the strength of association between a variable and its previous values as the lag length increases.

autocovariance: an unstandardised measure of the extent to which the current value of a series is related to its own previous values.

autoregressive conditional heteroscedasticity (ARCH) model: a time series model for volatilities.

autoregressive (AR) model: a time-series model where the current value of a series is fitted with its previous values.

autoregressive moving average (ARMA) model: a time-series model where the current value of a series is fitted with its previous values (the autoregressive part) and the current and previous values of an error term (the moving average part).

autoregressive volatility (ARV) model: a time-series model where the current volatility is fitted with its previous values.

auxiliary regression: a second stage regression that is usually not of direct interest in its own right, but rather is conducted in order to test the statistical adequacy of the original regression model.

backshift operator: *see* lag operator.

balanced panel: a dataset where the variables have both time-series and cross-sectional dimensions, and where there are equally long samples for each cross-sectional entity (i.e., no missing data).

Bayes information criterion: *see* Schwarz's Bayesian information criterion (SBIC).

Bayesian statistics: a branch of statistics based on Bayes theorem where the evidence regarding a hypothesis is formulated into a probability which is then updated according to new information arising.

BDS test: a test for whether there are patterns in a series, predominantly used for determining whether there is evidence for non-linearities.

***BEKK* model**: a multivariate model for volatilities and covariances between series that ensures the variance–covariance matrix is positive definite.

Bera–Jarque test: a widely employed test for determining whether a series closely approximates a normal distribution.

best linear unbiased estimator (BLUE): is one that provides the lowest sampling variance and which is also unbiased.

between estimator: is used in the context of a fixed effects panel model, involving running a cross-sectional regression on the time-averaged values of all the variables in order to reduce the number of parameters requiring estimation.

BHHH algorithm: a technique that can be used for solving optimisation problems including maximum likelihood.

biased estimator: where the expected value of the parameter to be estimated is not equal to the true value.

bid–ask spread: the difference between the amount paid for an asset (the ask or offer price) when it is purchased and the amount received if it is sold (the bid).

binary choice: a discrete choice situation with only two possible outcomes.

bivariate regression: a regression model where there are only two variables – the dependent variable and a single independent variable.

block maximum: an approach to estimating the parameters of an extreme value distribution based on separating the data into blocks and modelling the maximum values from each of the blocks.

bootstrapping: a technique for constructing standard errors and conducting hypothesis tests that requires no distributional assumptions and works by resampling from the data.

Box–Jenkins approach: a methodology for estimating ARMA models.

Box–Pierce *Q*-statistic: a general measure of the extent to which a series is autocorrelated.

break date: the date at which a structural change occurs in a time series or in a model's parameters.

Breusch–Godfrey test: a test for autocorrelation of any order in the residuals from an estimated regression model, based on an auxiliary regression of the residuals on the original explanatory variables plus lags of the residuals.

broken trend: a process which is a deterministic trend with a structural break.

calendar effects: the systematic tendency for a series, especially stock returns, to be higher at certain times than others.

capital asset pricing model (CAPM): a financial model for determining the expected return on stocks as a function of their level of market risk.

capital market line (CML): a straight line showing the risks and returns of all combinations of a risk-free asset and an optimal portfolio of risky assets.

Carhart model: a time series model for explaining the performance of mutual funds or trading rules based on four factors: excess market returns, size, value and momentum.

causality tests: a way to examine whether one series leads or lags another.

censored dependent variable: where values of the dependent variable above or below a certain threshold cannot be observed, while the corresponding values for the independent variables are still available.

central limit theorem: the mean of a sample of data having any distribution converges upon a normal distribution as the sample size tends to infinity.

chaos theory: an idea taken from the physical sciences whereby although a series may appear completely random to the naked eye or to many statistical tests, in fact there is an entirely deterministic set of non-linear equations driving its behaviour.

Chow test: an approach to determine whether a regression model contains a change in behaviour (structural break) part-way through based on splitting the sample into two parts, assuming that the break-date is known.

Cochrane–Orcutt procedure: an iterative approach that corrects standard errors for a specific form of autocorrelation.

coefficient of multiple determination: *see* R^2.

coefficient of variation: a unit-free measure of the spread across the observations in a series where the standard deviation is divided by the mean and thus valid comparisons can be made across series even if they have different scales.

cointegration: a concept whereby time series have a fixed relationship in the long run.

cointegrating vector: the set of parameters that describes the long-run relationship between two or more time series.

common factor restrictions: these are the conditions on the parameter estimates that are implicitly assumed when an iterative procedure such as Cochrane–Orcutt is employed to correct for autocorrelation.

continuously compounded return: the proportion or percentage reward to an investor calculated by assuming that interest payments are calculated at infinitesimally small intervals and re-invested.

conditional expectation: the value of a random variable that is expected for time $t + s$ ($s = 1, 2, \ldots$) given information available until time t.

conditional mean: the mean of a series at a point in time t fitted given all information available until the previous point in time $t - 1$.

conditional variance: the variance of a series at a point in time t fitted given all information available until the previous point in time $t - 1$.

confidence interval: a range of values within which we are confident to a given degree (e.g., 95% confident) that the true value of a given parameter lies.

confidence level: one minus the significance level (expressed as a proportion rather than a percentage) for a hypothesis test.

consistency: the desirable property of an estimator whereby the calculated value of a parameter converges upon the true value as the sample size increases.

contemporaneous terms: those variables that are measured at the same time as the dependent variable – i.e., both are at time t.

continuous variable: a random variable that can take on any value (possibly within a given range).

convergence criterion: a pre-specified rule that tells an optimiser when to stop looking further for a solution and to stick with the best one it has already found.

copulas: a flexible way to link together the distributions for individual series in order to form joint distributions.

correlation: a standardised measure, bounded between −1 and +1, of the strength of association between two variables.

correlogram: *see* autocorrelation function.

cost of carry (COC) model: shows the equilibrium relationship between spot and corresponding futures prices where the spot price is adjusted for the cost of 'carrying' the spot asset forward to the maturity date.

covariance matrix: *see* variance–covariance matrix.

covariance stationary process: *see* weakly stationary process.

covered interest parity (CIP): states that exchange rates should adjust so that borrowing funds in one currency and investing them in another would not be expected to earn abnormal profits.

credit rating: an evaluation made by a ratings agency of the ability of a borrower to meet its obligations to meet interest costs and to make capital repayments when due.

critical values (CV): key points in a statistical distribution that determine whether, given a calculated value of a test statistic, the null hypothesis will be rejected or not.

cross-equation restrictions: a set of restrictions needed for a hypothesis test that involves more than one equation within a system.

cross-sectional regression: a regression involving series that are measured only at a single point in time but across many entities.

cumulative distribution: a function giving the probability that a random variable will take on a value lower than some pre-specified value.

CUSUM and CUSUMSQ tests: tests for parameter stability in an estimated model based on the cumulative sum of residuals (CUSUM) or cumulative sum of squared residuals (CUSUMSQ) from a recursive regression.

daily range estimator: a crude measure of volatility calculated as the difference between the day's lowest and highest observed prices.

damped sine wave: a pattern, especially in an autocorrelation function plot, where the values cycle from positive to negative in a declining manner as the lag length increases.

data generating process (DGP): the true relationship between the series in a model.

data mining: looking very intensively for patterns in data and relationships between series without recourse to financial theory, possibly leading to spurious findings.

data revisions: changes to series, especially macroeconomic variables, that are made after they are first published.

data snooping: *see* data mining.

day-of-the-week effect: the systematic tendency for stock returns to be higher on some days of the week than others.

degrees of freedom: a parameter that affects the shape of a statistical distribution and therefore its critical values. Some distributions have one degree of freedom parameter, while others have more.

degree of persistence: the extent to which a series is positively related to its previous values.

delta normal method: an approach to calculating value at risk based on assumption of normality where the calculation involves multiplying the standard deviation by the appropriate quantile from the standard normal distribution.

dependent variable: the variable, usually denoted by *y* that the model tries to explain.

deterministic: a process that has no random (stochastic) component.

Dickey–Fuller (DF) test: an approach to determining whether a series contains a unit root, based on a regression of the change in that variable on the lag of the level of that variable.

differencing: a technique used to remove a (stochastic) trend from a series that involves forming a new series by taking the lagged value of the original series away from the current one.

differentiation: a mathematical technique to find the derivative, which is the slope of a function, or in other words the rate at which *y* changes in response to changes in *x*.

discrete choice: a model where the key variable takes only integer values that capture the selections made between alternatives – for example, between modes of transport for a particular journey.

discrete variable: a random variable that can only take specific values.

distributed lag models: contain lags of the explanatory variables but no lags of the explained variable.

disturbance term: *see* error term.

double logarithmic form: a specification of a model where logarithms are taken of both the dependent variable (*y*) and the independent variable(s) (*x*).

dummy variables: artificially constructed variables that capture qualitative information – for example, for male/female, days of the week, emerging/developed markets, etc. They are usually binary variables (0 or 1).

Durbin–Watson (DW) statistic: a test for first order autocorrelation, i.e., a test for whether a (residual) series is related to its immediately preceding values.

dynamic conditional correlation: a model that explicitly models correlations in a timevarying, autoregressive fashion.

dynamic model: a model that includes lagged or differenced terms of the dependent or independent variables (or both).

efficient estimator: an approach to parameter estimation that is optimal in some sense. In econometrics, this is usually taken to mean a formula for calculating the parameters that leads to minimum sampling variance; in other words, the estimates vary as little as possible from one sample to another.

efficient frontier: a curve that traces out all possible optimal portfolios.

efficient market hypothesis: the notion that asset prices will rapidly reflect all relevant and available information.

eigenvalues: the characteristic roots of a matrix.

eigenvectors: a set of vectors that, when multiplied by a square matrix, give a set of vectors that differ from the originals by a multiplicative scalar.

elasticity: the responsiveness of a percentage change in one variable to percentage changes in another.

encompassing principle: the notion that a good model will be able to explain all that competing models can and more.

encompassing regression: a hybrid model that incorporates the variables contained in two or more competing models as a method of selecting which is the best between them. The parameters of the best model will be significant in the hybrid model.

endogenous variable: a variable whose value is determined within the system of equations under study. In the context of a simultaneous system, each endogenous variable has its own equation specifying how it is generated.

Engle–Granger (EG) test: a unit root test applied to the residuals of a potentially cointegrating regression.

Engle–Ng test: a test for appropriate specification of a GARCH model in terms of whether there are any uncaptured asymmetries.

equilibrium correction model: *see* error correction model.

error correction model (ECM): a model constructed using variables that are employed in stationary, first–differenced forms together with a term that captures movements back towards long run equilibrium.

error term: part of a regression model that sweeps up any influences on the dependent variable that are not captured by the independent variables.

errors-in-variables regression: a valid approach to estimating the parameters of a regression when the explanatory variables are measured with error and are thus stochastic.

estimate: the calculated value of a parameter obtained from the sample data.

estimator: an equation that is employed together with the data in order to calculate the parameters that describe the regression relationship.

event study: an approach to financial research where the impact of an identifiable event (e.g., a dividend announcement) is measured on a firm characteristic (e.g., its stock price) to evaluate the market reaction to that event.

exogeneity: the extent to which a variable is determined outside of the model under study.

exogenous variables: variables whose values are taken as given and are determined outside of the equation or system of equations under study and are thus not correlated with the error term.

expectations hypothesis: related particularly to the term structure of interest rates. It states that the expected return from investing in a long-term bond will be equal to the return from investing in a series of short-term bonds plus a risk premium. In other words, the long-term interest rate is a geometric average of the current and expected future short term rates (plus a risk premium).

explained sum of squares (ESS): the part of the variation in y that is explained by the model.

explained variable: *see* dependent variable.

explanatory variables: those variables which are on the RHS of an equation, whose values are usually taken as fixed, and which are purported to be explaining the values of the dependent variable y.

exponential (EGARCH): a model where volatility is modelled in an exponential form so that no non-negativity conditions need to be applied to the parameters. This specification also that allows for asymmetries in the relationship between volatility and returns of different signs.

exponential growth model: a model where the dependent variable is an exponential function of one or more independent variables.

exponential smoothing: a simple approach to modelling and forecasting where the current smoothed value is a geometrically declining function of all previous values of the series.

exponentially weighted moving average (EWMA) model: a simple method for modelling and forecasting volatility where the current estimate is simply a weighted combination of previous values, with the weightings exponentially declining back through time.

extreme value distribution: a broad family of statistical distributions that are suitable for modelling the group of observations in a series furthest away from the mean.

F-statistic: a measure that follows an F-distribution used for testing multiple hypotheses.

factor model: a framework used in asset pricing where the returns are decomposed into parts explained by a set of variables (factors) that may either be observed or latent.

factor loading: has several meanings but in particular in the context of principal component analysis, it gives the amount of a variable that appears in each component.

Fama–MacBeth procedure: a two-step procedure for testing asset pricing models such as the CAPM. In the first stage the betas are estimated in a set of time series regressions and then a second stage cross-sectional regression examines the explanatory power of these betas.

financial options: securities that give the holder the right but not the obligation to buy or sell another asset at a pre-specified price on a pre-specified date.

first differences: new series constructed by taking the immediately previous value of a series from its current value.

fitted value: the value of *y* that the model fits for a given data point, i.e., for given values of the explanatory variable.

fixed effects: most commonly a type of model used for panel data that employs dummies to account for variables that affect the dependent variable *y* cross-sectionally but do not vary over time. Alternatively, the dummies can capture variables that affect *y* over time but do not vary cross-sectionally.

forcing variable: sometimes used synonymously with explanatory variable; alternatively it can mean the unobservable state-determining variable that governs the regime in a Markov switching regression model.

forecast encompassing test: a regression of the actual values of a series on several corresponding sets of forecasts. The idea is that if a parameter estimate is statistically significant, then the forecasts from the corresponding model encompass (i.e., contain more information than) those of the other model(s).

forecast error: the difference between the actual value of a series and the value that has been forecast for it.

forward rate unbiasedness (FRU): the hypothesis that the forward rate of foreign exchange should be an unbiased prediction of the future spot rate of interest.

fractionally integrated models: a way to represent series that are stationary but highly persistent and thus have long memory.

Frechét distribution: the member of the family of three limiting extreme value distributions that has fat tails and is therefore most suitable for modelling financial time series. It includes the Student's *t* distribution as a special case.

function: an expression that maps members of one set to members of another and describes the relationship between the two.

functional form misspecification: *see* RESET test.

futures prices: the price of a specific quantity of a good or asset for delivery at some pre-specified date in the future.

GARCH-in-mean (GARCH-M): a dynamic model for volatility where the standard deviation (or variance) enters into the generating process for returns.

Gauss–Markov theorem: a derivation using algebra showing that, providing a certain set of assumptions holds, the OLS estimator is the best linear unbiased estimator (BLUE).

general-to-specific methodology: a philosophical approach to constructing econometric models where the researcher commences with a very broad model and then, through hypothesis testing, reduces the model down to a smaller one.

generalised autoregressive conditional heteroscedasticity (GARCH) models: a common specification of dynamic model for volatility.

generalised least squares (GLS): an approach to the estimation of econometric models that is more flexible than OLS and can be used to relax one or more of its limiting assumptions.

generalised method of moments: an approach to estimating the parameters of a model based on specifying a set of moment restrictions which use information from the sample; the technique can handle situations where there are more restrictions than parameters to be estimated.

generalised unrestricted model (GUM): the initial, broad model that is specified as the first step of the general-to-specific approach to model construction.

geometric progression: a number sequence where the change from one entry in the series to the next arises from multiplication by a fixed number, which may be a fraction.

gilt–equity yield ratio (GEYR): the ratio of the yield on long term Treasury bonds to the dividend yield on stocks.

GJR model: a model for time-varying volatilities developed by Glosten, Jaganathan and Runkle (1993) to allow for asymmetries in the relationship between volatility and returns of different signs.

Goldfeld–Quandt test for heteroscedasticity: one of several available tests for whether the residuals from an estimated model have constant variance.

goodness of fit statistic: a measure of how well the model that has been estimated fits the sample data.

Granger representation theorem: states that if there exists a dynamic linear model with stationary disturbances but where the component variables are non-stationary, then they must be cointegrated.

Gumbel distribution: the member of the family of three limiting extreme value distributions that has a medium tail thickness and a zero shape parameter. It includes the normal and log-normal distributions as special cases.

Hamilton's filter: a form of Markov-switching model where an unobservable state variable switches between discrete regimes via a first-order Markov process.

Hannan–Quinn information criterion: a metric that can be used to select the best fitting from a set of competing models and that incorporates a moderate penalty term for including additional parameters.

Hausman test: a test for whether a variable can be treated as exogenous or whether in fact the researcher needs to specify a separate structural equation for that variable. It can also refer to a test for whether a random effects approach to panel regression is valid or whether a fixed effects model is necessary.

Heckman procedure: a two-step method that corrects for the selection bias that can be observed in the context of samples not selected randomly.

hedge ratio: in the context of hedging with futures contracts, this is the number of futures contracts that are sold per unit of the spot asset held.

hedonic pricing model: a modelling approach where the price of a physical asset is modelled as a function of its characteristics.

heteroscedasticity: where the variance of a series is not constant throughout the sample.

heteroscedasticity-robust: a set of standard errors (or test statistics) that have been calculated using an approach that is valid in the presence of heteroscedastic residuals.

Hill estimator: a non-parametric method for determining the shape parameter of the generalised Pareto distribution.

historical simulation: a technique for estimating value at risk based on measuring the appropriate quantile from the historical distribution of ordered returns.

hyperparameters: parameters, the values of which are set before estimation. In the Bayesian context, they are prior parameters, while in state space models, these are fixed before a final sweep of the Kalman filter.

hypothesis test: a framework for considering plausible values of the true population parameters given the sample estimates.

identification: a condition for whether all of the structural parameters in a particular equation from a simultaneous system can be retrieved from estimating the corresponding reduced form equation.

identity matrix: a square matrix containing ones on the main diagonal and zeros everywhere else.

implied volatility models: an approach whereby the volatility of an underlying asset is calculated from the traded price of an option and a pricing formula.

impulse responses: an examination of the impact of a unit shock to one variable on the other variables in a vector autoregressive (VAR) system.

independent variables: *see* explanatory variables.

information criteria: a family of methods for selecting between competing models that incorporate automatic correction penalties when larger numbers of parameters are included.

instrumental variables (instruments): can be used to replace endogenous variables on the RHS of a regression equation. The instruments are correlated with the variables they replace but not with the error term in the regression.

integrated GARCH (IGARCH): a model where the variance process is non-stationary so that the impact of shocks on volatility persists indefinitely.

integrated variable: one which requires differencing to make it stationary.

integration: a process used to calculate the area under a curve – the mathematically opposite operation to differentiation.

interactive dummy variable: when a dummy variable is multiplied by an explanatory variable to allow the regression slope to change according to the value of the dummy.

intercept: the point where a regression line crosses the y-axis, also known sometimes as ‘the coefficient on the constant term’, or sometimes just ‘the constant term’.

internal rate of return (IRR): the discount rate which would make the net present value from a project equal to zero.

interquartile range: the difference between the first and third quartiles i.e., the difference between the 25th and 75th percentiles in an ordered distribution, sometimes used as a measure of spread.

inverse (of a matrix): a transformed matrix which, when multiplied by the original matrix, yields the identity matrix.

invertibility: a condition for a moving average (MA) model to be representable as a valid infinite-order autoregressive model.

irrelevant variables: variables that are included in a regression equation but in fact have no impact on the dependent variable.

Jensen's alpha: the intercept estimate in a regression model of the returns to a portfolio or strategy on a risk factor or set of risk factors, especially in the context of the CAPM. Alpha measures the degree to which there was abnormally bad or good performance.

Johansen test: an approach to determining whether a set of variables is cointegrated – i.e., if they have a long-run equilibrium relationship.

joint hypothesis: a multiple hypothesis that involves making more than one restriction simultaneously.

just identified equation: occurs when the parameters in a structural equation from a system can be uniquely obtained by substitution from the reduced form estimates.

Kalman filter: a technique to estimate the state vector in a time varying parameters model.

KPSS test: a test for stationarity – in other words, a test where the null hypothesis is that a series is stationary against an alternative hypothesis that it is not.

kurtosis: the standardised fourth moment of a series; a measure of whether a series has 'fat tails'.

lag length: the number of lagged values of a series used in a model.

lag operator: an algebraic notation for taking the current value of a series and turning it into a past value of that series.

Lagrange multiplier (LM) test: used in the context of maximum-likelihood estimation, an LM test involves estimation of a restricted regression only. In practice, an LM test is often employed via the calculation of R^2 from an auxiliary regression to construct a test statistic that follows a χ^2 distribution.

law of large numbers: a theorem stating that the mean from a sample will approach the true population mean (i.e., the expected value) as the sample size increases.

least squares: *see* ordinary least squares.

least squares dummy variables (LSDV): an approach to estimating panel data models using 0–1 intercept dummy variables for each cross-sectional unit.

leptokurtosis: a phenomenon whereby a series has a higher peak at the mean and fatter tails than a normal distribution with the same mean and variance.

leverage effects: the tendency for stock volatility to rise more following a large stock price fall than a price rise of the same magnitude owing to the consequent impact on the firm's debt-to-equity (leverage) ratio.

likelihood function: a mathematical expression that relates to the data and the parameters. A likelihood function is constructed given an assumption about the distribution of the errors, and then the values of the parameters that maximise it are chosen.

likelihood ratio (LR) test: an approach to hypothesis testing arising from maximum likelihood estimation that revolves around a comparison of the maximised values of the log-likelihood functions for the restricted and unrestricted models.

limited dependent variable: when the values that the dependent variable can take are restricted in some way. In such cases, OLS cannot be validly used to estimate the model parameters.

linear probability model: a simple but flawed model for use when the dependent variable in a regression model is binary (0 or 1).

linearity: the extent to which a relationship between variables can be represented by a (possibly multi-dimensional) straight line.

Ljung–Box test: a general test for autocorrelation in a variable or residual series.

logarithm: or sometimes written as log, is the power to which the base must be raised to produce that number.

log-likelihood function (LLF): the natural logarithm of the likelihood function.

log-log model: *see* double logarithmic form.

logit model: an approach for use when the dependent variable in a regression model is binary (0 or 1), and which ensures that the estimated probabilities are bounded by 0 and 1.

long-memory models: *see* fractionally integrated models.

long-run static solution: the algebraic manipulation of a dynamic equation to construct the long-run relationship between the variables.

longitudinal data: *see* panel data analysis.

loss function: is constructed in order to evaluate the accuracy of a model fit or of forecasts. The parameters of a model are usually estimated by minimising or maximising a loss function.

Lyapunov exponent: a characteristic that can be used to determine whether a series can be described as chaotic.

marginal effects: the impacts of changes in the explanatory variables on changes in the probabilities for probit and logit models. They are calculated in order to intuitively interpret the models.

marginal probability: the probability of a single random variable.

market microstructure: a financial term, concerned with the way that markets work and the impact that the design and structure of the market can have on the outcomes of trade, including prices, volumes and execution costs.

market risk premium: the amount of additional return that an investor requires for accepting an additional unit of market risk, often calculated as the difference between the returns on a broad portfolio of stocks and a proxy for the risk free rate of interest.

market timing: the extent to which investors are able to select the optimal times to invest in different asset classes.

Markov switching model: a time series approach based on a dependent variable that alternates between regimes according to the value of an unobservable state variable that follows a Markov process.

Marquardt algorithm: an approach to optimisation that can be used, for example, as part of the procedure to estimate the parameter values in maximum likelihood estimation.

matrix: a two-dimensional array of numbers constructed in rows and columns.

maximum likelihood (ML): an approach that can be used for parameter estimation based on the construction and maximisation of a likelihood function, which is particularly useful for non-linear models.

measurement equation: describes the link between the time series of interest and the state vector. It is one of two equations (alongside the state equation) in a state space model.

median: the central observation in an ordered series, which provides a measure of its average value.

method of moments: an approach to estimating the parameters of a model based on specifying a set of moment restrictions which use information from the sample; the technique is only suitable for situations where the number of restrictions is equal to the number of parameters to be estimated.

minimum capital risk requirement (MCRR): *see* value-at-risk.

misspecification error: occurs when the model estimated is incorrect – for example, if the true relationship between the variables is non-linear but a linear model is adopted.

misspecification tests: are diagnostic tests that can provide the researcher with information concerning whether a model has desirable statistical properties, particularly regarding the residuals.

mode: the most commonly occurring observation in a series, which provides a measure of its average value

model interpretation: the examination of an estimated model in terms of whether the signs of the parameters (i.e. positive or negative) and sizes of the parameters (i.e., their values) make sense intuitively.

moments: the moments of a distribution describe its shape. The first moment of a distribution is the mean, the second moment is the variance, the third (standardised) moment is the skewness and the fourth (standardised) moment is the kurtosis. The fifth moments and higher are harder to interpret and in general are not calculated.

moving average (MA) process: a model where the dependent variable depends upon the current and past values of a white noise (error) process.

multicollinearity: a phenomenon where two or more of the explanatory variables used in a regression model are highly related to one another.

multimodal: a characteristic of a distribution whereby it does not have a single peak at the mean, but rather reaches a maximum in more than one place.

multinomial logit or probit: classes of models that are used for discrete choice problems, where we wish to explain how individuals make choices between more than two alternatives.

multivariate generalised autoregressive conditionally heteroscedastic (GARCH) models: a family of dynamic models for time-varying variances and covariances.

neural network models: a class of statistical models whose structure is loosely based on how computation is performed by the brain. They have been employed for time-series modelling and for classification purposes.

Newey–West estimator: a procedure that can be employed to adjust standard errors to allow for heteroscedasticity and/or autocorrelation in the residuals from a regression model.

news impact curve: a pictorial representation of the responsiveness of volatility to positive and negative shocks of different magnitudes.

Newton–Raphson procedure: an iterative approach to optimisation – in other words, for finding the values of a parameter or set of parameters that maximise or minimise a function.

nominal series: a series that has not been deflated (i.e., not been adjusted for inflation).

non-linear least squares (NLS): an estimation technique for use on non-linear models (models that are non-linear in the parameters) based on minimising the sum of the squared residuals.

non-negativity constraints: the conditions that it is sometimes necessary to impose on the parameter estimates from non-linear models to ensure that they are not negative in situations where it would not make sense for them to be so.

non-nested models: where there are at least two models, neither of which is a special (i.e., restricted) case of the other.

non-normality: not following a normal or Gaussian distribution.

non-parametric: an approach to modelling and inference that is not based on the assumption that the data follow a particular statistical distribution.

non-stationarity: a characteristic of a time series whereby it does not have a constant mean, a constant variance, and a constant autocovariance structure.

null hypothesis: a formal expression of the statement actually being tested as part of a hypothesis test.

observations: another name for the data points available for analysis.

omitted variable: a relevant variable for explaining the dependent variable has been left out of the estimated regression equation, leading to biased inferences on the remaining parameters.

one-sided hypothesis test: used when theory suggests that the alternative hypothesis should be of the greater than form only or of the less than form only (and not both).

optimal portfolio: a combination of risky assets that maximises return for a given risk or minimises risk for a given return.

order of integration: the number of times that a stochastically non-stationary series must be differenced to make it stationary.

ordered response variable: usually a situation where the dependent variable in a model is limited to only certain values but where there is a natural ordering of those values – for example, where the values represent sovereign credit rating assignments.

ordinal scale: where a variable is limited so that its values define a position or ordering only, and thus the precise values that the variable takes have no direct interpretation.

ordinary least squares (OLS): the standard and most common approach that is used to estimate linear regression models.

out-of-sample: sometimes, not all observations are employed to estimate the model (in-sample data), but instead some are retained for forecasting (the out-of-sample data).

outliers: data points that do not fit in with the pattern of the other observations and that are a long way from the fitted model.

overfitting: estimating too large a model with too many parameters.

overidentified equation: occurs when more than one estimate of each parameter in the structural equation from a system can be obtained by substitution from the reduced form estimates.

overreaction effect: the tendency for asset (especially stock) prices to overshoot their new equilibrium prices when news is released.

oversized test: a statistical test that rejects the null hypothesis too often when it is in fact correct.

p-value: the exact significance level, or the marginal significance level which would make us indifferent between rejecting and not rejecting the null hypothesis.

panel data analysis: the use of data having both cross-sectional and time series dimensions.

parsimonious model: one that describes the data accurately while using as few parameters as possible.

partial autocorrelation function (pacf): measures the correlation of a variable with its value k periods ago (k = 1, 2, …) after removing the effects of observations at all intermediate lags.

peak over threshold: an approach to estimating the parameters of an extreme value distribution based on specifying a fixed cutoff point and treating observations beyond that as extremes.

pecking order hypothesis: the notion from corporate finance that firms will select the cheapest method of financing (usually retained earnings) first before switching to increasingly more expensive forms.

perfect multicollinearity: occurs when an explanatory variable used in a regression model is a precise linear combination of one or more other explanatory variables from that model.

period effects: *see* time fixed effects.

piecewise linear model: a model that is linear (i.e., can be represented by a straight line) within restricted ranges of the data, but where taken overall the model is non-linear.

polynomial: an equation incorporating different powers of the same variable(s).

pooled sample: where there is a panel of data (i.e., having both time series and crosssectional dimensions), but where all of the observations are employed together without regard for the panel structure.

population: the collection of all objects or entities that are relevant to the idea being tested in a model.

population regression function (PRF): embodies the true but unobservable relationship between the dependent and independent variables.

portmanteau tests: general tests for non-linear patterns or model-misspecification; in other words, tests that have power over a broad range of alternative structures.

position risk requirement: *see* value-at-risk.

power of a test: the ability of a test to correctly reject a wrong null hypothesis.

pre-determined variables: are uncorrelated with past or current values of the error term in a regression equation but may be correlated with future values of the error term.

predicted value: *see* fitted value.

predictive failure test: a test for parameter stability or structural change in a regression model, which is based on estimating an auxiliary regression for a sub-sample of the data and then evaluating how well that model can 'predict' the other observations.

present value: the amount that a cashflow to be received in the future is worth in today's terms.

price deflator: a series that measures the general level of prices in an economy, used to adjust a nominal series to a real one.

principal components analysis (PCA): a technique that is sometimes used where a set of variables are highly correlated. More specifically, it is a mathematical operation that converts a set of correlated series into a new set of linearly independent series.

probability density function (pdf): a relationship or mapping that describes how likely it is that a random variable will take on a value within a given range.

probit model: an appropriate model for binary (0 or 1) dependent variables where the underlying function used to transform the model is a cumulative normal distribution.

product: when two or more terms are multiplied together.

pseudo-random numbers: a set of artificial random-looking numbers generated using a purely deterministic sequence (e.g., using a computer).

purchasing power parity (PPP): the hypothesis that, in equilibrium, exchange rates should adjust so that a representative basket of goods and services should cost the same when converted into a common currency irrespective of where it was purchased.

quadratic: an equation involving (linear and) squared terms only.

qualitative variables: *see* dummy variables.

Quandt likelihood ratio test: a test for structural breaks in a regression model, based on the Chow test but where the break date is assumed unknown.

quantile: the position (within the 0–1 interval) in an ordered series where an observation falls.

quantile regression: an approach to model specification that involves constructing a family of regression models, each for different quantiles of the distribution of the dependent variable.

quotient: when one term is divided by another.

R^2: a standardised measure, bounded between zero and one, of how well a sample regression model fits the data.

R-bar^2: *see* adjusted R^2.

random effects model: a particular type of panel data model specification where the intercepts vary cross-sectionally as a result of each cross-sectional entity having a different error term.

random walk: a simple model where the current value of a series is simply the previous value perturbed by a white noise (error) term. Therefore the optimal forecast for a variable that follows a random walk is the most recently observed value of that series.

random walk with drift: a random walk model that also includes an intercept, so that changes in the variable are not required to average zero.

rank (of a matrix): a measure of whether all the rows and columns of a matrix are independent of one another.

real series: a series that has been deflated (adjusted for inflation).

recursive model: an approach to estimation where a set of time series regressions are estimated using sub-samples of increasing length. After the first model is estimated, an additional observation is added to the end of the sample so that the sample size increases by one observation. This continues until the end of the sample is reached.

reduced form equations: the equations with no endogenous variables on the RHS that have been derived algebraically from the structural forms in the context of a simultaneous system.

redundant fixed effects test: a test for whether a fixed effects panel regression approach must be employed, or whether the data can simply be pooled and estimated using a standard ordinary least squares regression model.

regressand: *see* dependent variable.

regressors: *see* explanatory variables.

rejection region: if a test statistic falls within this area plotted onto a statistical distribution function then the null hypothesis under study is rejected.

re-sampling: creating a simulated distribution for computing standard errors or critical values via sampling with replacement from the original data.

RESET test: a non-linearity test, or a test for misspecification of functional form, i.e., a situation where the shape of the regression model estimated is incorrect – for example, where the model estimated is linear but it should have been non-linear.

residual diagnostics: an examination of the residuals for whether they have any patterns remaining that were present in the dependent variable and not captured by the fitted model.

residual sum of squares (RSS): the addition of all of the squared values of the differences between the actual data points and the corresponding model fitted values.

residual terms: the differences between the actual values of the dependent variable and the values that the model estimated for them – in other words, the parts of the dependent variable that the model could not explain.

restricted model: a regression where the parameters cannot be freely determined by the data, but instead some restrictions have been placed on the values that can be taken by one or more of the parameters.

risk premium: the additional return that investors expect for bearing risk.

riskless arbitrage opportunities: *see* arbitrage.

rolling window: an approach to estimation where a set of time-series regressions are estimated using sub-samples of fixed length. After the first model is estimated, the first observation is removed from the sample and one observation is added to the end. This continues until the end of the sample is reached.

root: the point(s) at which a function crosses the x-axis, or equivalently the solution(s) of an equation – i.e., the corresponding values of x when y is zero.

sample: a selection of some entities from the population which are then used to estimate a model.

sample regression function (SRF): the regression model that has been estimated from the actual data.

sample size: the number of observations or data points per series in the sample.

sampling error: the inaccuracy in parameter estimation that arises as a result of having only a sample and not the whole population; as a consequence of sampling error, the estimates vary from one sample to another.

Schwarz's Bayesian information criterion (SBIC): a metric that can be used to select the best fitting from a set of competing models and that incorporates a strict penalty term for including additional parameters.

second moment: the moments of a distribution define its shape; the second moment is another term for the variance of the data.

seemingly unrelated regression (SUR): a time-series regression approach for modelling the movements of several highly related dependent variables. The approach allows for the correlation between the error terms of the regressions, hence improving the efficiency of the estimation.

self-exciting threshold autoregression (SETAR): a TAR model where the statedetermining variable is the same as the variable under study.

semi-interquartile range: a measure of the spread of a set of data (an alternative to the variance) that is based on the difference between the quarter- and three-quarter points of the ordered data.

sensitive dependence on initial conditions (SDIC): this is the defining characteristic of a chaotic system that the impact on a system of an infinitesimally small change in the initial values will grow exponentially over time.

serial correlation: *see* autocorrelation.

Sharpe ratio: in finance, this is a risk-adjusted performance measure calculated by subtracting the risk-free return from the portfolio return, and then dividing this by the portfolio standard deviation.

shocks: another name for the disturbances in a regression model.

short-selling: selling a financial asset that you do not own, in anticipation of repurchasing it at a later date when the price has fallen.

significance level: the size of the rejection region for a statistical test, also equal to the probability that the null hypothesis will be rejected when it is correct.

sign and size bias tests: tests for asymmetries in volatility – i.e., tests for whether positive and negative shocks of a given size have the same effect on volatility.

simultaneous equations: a set of inter-linked equations each comprising several variables.

size of test: *see* significance level.

skewness: the standardised third moment of a distribution that shows whether it is symmetrical around its mean value.

slippage time: the amount of time that it is assumed to take to execute a trade after a rule is computer-generated.

slope: the gradient of a straight (regression) line, measured by taking the change in the dependent variable, *y* between two points, and dividing it by the change in the independent variable, *x* between the same points.

sovereign credit ratings: are assessments of the riskiness of debts issued by governments.

sovereign yield spreads: usually defined as the difference between the yield on the bonds of a government under study and the yield on US Treasury bonds.

specific-to-general modelling: a philosophical approach to building econometric models that involves starting with a specific model as indicated by theory and then sequentially adding to it or modifying it so that it gradually becomes a better description of reality.

spline techniques: piecewise linear models that involve the application of polynomial functions in a piecewise fashion to different portions of the data.

spot price: the price of a specific quantity of a good or asset for immediate delivery.

spurious regression: if a regression involves two or more independent non-stationary variables, the slope estimate(s) may appear highly significant to standard statistical tests and may have highly significant *t*-ratios even though in reality there is no relationship between the variables.

standard deviation: a measure of the spread of the data about their mean value, which has the same units as the data.

standard errors: measure the precision or reliability of the regression estimates.

state equation: describes the state of the system, and links the current state with the future state. It is one of two equations (alongside the measurement equation) in a state space model.

state space model: a model with two or more equations (including the state and measurement equations) which describe how a series varies over time.

stationary variable: one that does not contain a unit or explosive root and can thus be validly used directly in a regression model.

statistical inference: the process of drawing conclusions about the likely characteristics of the population from the sample estimates.

statistically significant: a result is statistically significant if the null hypothesis is rejected (usually using a 5% significance level).

stochastic regressors: it is usually assumed when using regression models that the regressors are non-stochastic or fixed; in practice, however, they may be random or stochastic – for example, if there are lagged dependent variables or endogenous regressors.

stochastic trend: some levels time series possess a stochastic trend, meaning that they can be characterised as unit root processes, which are non-stationary.

stochastic volatility (SV) model: a less common alternative to GARCH models, where the conditional variance is explicitly modelled using an equation containing an error term.

strictly exogenous variable: one that is uncorrelated with past, present and future values of the error term.

strictly stationary process: one where the entire probability distribution is constant over time.

structural break: a situation where the properties of a time series or of a model exhibit a substantial long-term shift in behaviour.

structural equations: the original equations describing a simultaneous system, which contain endogenous variables on the RHS.

sum of squared residuals: *see* residual sum of squares.

switching model: an econometric specification for a variable whose behaviour alternates between two or more different states.

t-ratio: the ratio of a parameter estimate to its standard error, forming a statistic to test the null hypothesis that the true value of the parameter is one.

Theil's U-statistic: a metric to evaluate forecasts, where the mean squared error of the forecasts from the model under study is divided by the mean squared error of the forecasts from a benchmark model. A *U*-statistic of less than one implies that the model is superior to the benchmark.

threshold autoregressive (TAR) models: a class of time-series models where the series under study switches between different types of autoregressive dynamics when an underlying (observable) variable exceeds a certain threshold.

time fixed effects: a panel data model that allows the regression intercept to vary over time and is useful when the average value of the variable under study changes over time but not cross-sectionally.

time-series regressions: models built using time-series data – i.e., data collected for a period of time for one or more variables.

tobit regression: a model that is appropriate when the dependent variable is censored – that is, where the values of the variable beyond a specific threshold cannot be observed, even though the corresponding values of the independent variables are observable.

total sum of squares (TSS): the sum of the squared deviations of the dependent variable about its mean value.

trace of a matrix: the sum of the elements on the main diagonal from the top left to the bottom right.

transition probabilities: a square matrix of estimates of the likelihood that a Markov switching variable will move from a given regime to each other regime.

truncated dependent variable: a situation where the values of this variable beyond a certain threshold cannot be observed, and neither can the corresponding values of the independent variables.

two-stage least squares (TSLS or 2SLS): an approach to parameter estimation that is valid for use on simultaneous equations systems.

unbalanced panel: a set of data having both time series and cross-sectional elements, but where some data are missing – i.e., where the number of time series observations available is not the same for all cross-sectional entities.

unbiased estimator: a formula or set of formulae that, when applied, will give estimates that are on average equal to the corresponding true population parameter values.

uncovered interest parity (UIP): holds if covered interest parity and forward rate unbiasedness both apply.

underidentified or unidentified equation: occurs when estimates of the parameters in the structural equation from a system cannot be obtained by substitution from the reduced form estimates as there is insufficient information in the latter.

unit root process: a series follows a unit root process if it is non-stationary but becomes stationary by taking first differences.

unparameterised: if a feature of the dependent variable y is not captured by the model, it is unparameterised.

unrestricted regression: a model that is specified without any restrictions being imposed so that the estimation technique can freely determine the parameter estimates.

value-at-risk (VaR): an approach to measuring risk based on the loss to a portfolio that may be expected to occur with a given probability over a specific horizon.

variance–covariance matrix: an array of numbers that comprises each of the variances of a set of random variables on the leading diagonal of the matrix and their covariances as the off-diagonal elements.

variance decomposition: a way to examine the importance of each variable in a vector autoregressive (VAR) model by calculating how much of the forecast error variance (for 1, 2, …, periods ahead) for each dependent variable can be explained by innovations in each independent variable.

variance reduction techniques: are employed in the context of Monte Carlo simulations in order to reduce the number of replications required to achieve a given level of standard errors of the estimates.

***VECH* model**: a relatively simple multivariate approach that allows for the estimation of time-varying volatilities and covariances that are stacked into a vector.

vector autoregressive (VAR) model: a multivariate time series specification where lagged values of (all) the variables appear on the right hand side in (all) the equations of the (unrestricted) model.

vector autoregressive moving average (VARMA) model: a VAR model where there are also lagged values of the error terms appearing in each equation.

vector error correction model (VECM): an error correction model that is embedded into a VAR framework so that the short- and long-run relationships between a set of variables can be modelled simultaneously.

vector moving average (VMA) model: a multivariate time-series model where a series is expressed as a combination of lagged values of a vector of white noise processes.

volatility: the extent to which a series is highly variable over time, usually measured by its standard deviation or variance.

volatility clustering: the tendency for the variability of asset returns to occur 'in bunches', so that there are prolonged periods when volatility is high and other prolonged periods when it is low.

Wald test: an approach to testing hypotheses where estimation is undertaken only under the alternative hypothesis; most common forms of hypothesis tests (e.g., t- and F-tests) are Wald tests.

weakly exogenous variables: *see* pre-determined variables.

weakly stationary process: has a constant mean, constant variance and constant autocovariances for each given lag.

Weibull distribution: the member of the family of three limiting extreme value distributions that has short tails and a fixed end point.

weighted least squares (WLS): *see* generalised least squares.

white noise process: has a fixed mean and variance but no other structure (e.g., it has zero autocorrelations for all lags). The error term in a regression model is usually assumed to be white noise.

White's correction: an adjustment to the standard errors of regression parameters that allows for heteroscedasticity in the residuals from the estimated equation.

White's test: an approach to determining whether the assumption of homoscedastic errors in a model is valid, based on estimating an auxiliary regression of the squared residuals on the regressors, their squared values, and their cross-products.

within transformation: used in the context of a fixed effects panel model, involving the subtraction of the time-series mean from each variable to reduce the number of dummy variable parameters requiring estimation.

Wold's decomposition theorem: states that any stationary series can be decomposed into the sum of two unrelated processes, a purely deterministic part and a purely stochastic part.

yield curves: show how the yields on bonds vary as the term to maturity increases.

Yule–Walker equations: a set of formulae that can be used to calculate the autocorrelation function coefficients for an autoregressive model.

References

Akaike, H. (1974) A New Look at the Statistical Model Identification, *IEEE Transactions on Automatic Control* AC-19(6), 716–23

Akgiray, V. (1989) Conditional Heteroskedasticity in Time Series of Stock Returns: Evidence and Forecasts, *Journal of Business* 62(1), 55–80

Amemiya, T. (1984) Tobit Models: A Survey, *Journal of Econometrics* 24, 3–61

Andersen, T. and Bollerslev, T. (1998) Answering the Skeptics: Yes, Standard Volatility Models do Provide Accurate Forecasts, *International Economic Review* 39, 885–905

Anselin, L. (1988) *Spatial Econometrics: Methods and Models*, Kluwer Academic, Dordrecht

Antoniou, A. and Garrett, I. (1993) To What Extent Did Stock Index Futures Contribute to the October 1987 Stock Market Crash?, *Economic Journal* 103, 1444–61

ap Gwilym, O., Clare, A. and Thomas, S. (1998) The Bid–Ask Spread on Stock Index Options: An Ordered Probit Analysis, *Journal of Futures Markets* 18(4), 467–85

Arellano, M. (2003) *Panel Data Econometrics*, Oxford University Press, Oxford

Armitage, S. (1995) Event Study Methods and Evidence on their Performance, *Journal of Economic Surveys* 8(4), 25–52

Babbs, S.H. and Nowman, K.B. (1999) Kalman Filtering of Generalized Vasicek Term Structure Models, *Journal of Financial and Quantitative Analysis* 34(1), 115–30

Bai, J. and Ng, S. (2004) A Panic Attack on Unit Roots and Cointegration, *Econometrica* 72, 1127–77

Baillie, R. T. (1989) Tests of Rational Expectations and Market Efficiency, *Econometric Reviews* 8, 151–86

Baillie, R. T. and Bollerslev, T. (1989) The Message in Daily Exchange Rates: A Conditional-Variance Tale, *Journal of Business and Economic Statistics* 7(3), 297–305

Baillie, R. T. and Myers, R. J. (1991) Bivariate GARCH Estimation of the Optimal Commodity Futures Hedge, *Journal of Applied Econometrics* 6, 109–24

Baks, K., Metrick, A. and Wachter, J. A. (2001) Should Investors Avoid All Actively Managed Mutual Funds? A Study in Bayesian Performance Evaluation, *Journal of Finance* 56(1), 45–85

Ball, R. and Brown, P. (1968) An Empirical Evaluation of Accounting Numbers, *Journal of Accounting Research* 6(2), 159–78

Ball, R. and Kothari, S. P. (1989) Nonstationary Expected Returns: Implications for Tests of Market Efficiency and Serial Correlation in Returns, *Journal of Financial Economics* 25, 51–74

Baltagi, B. H. (2005) *Econometric Analysis of Panel Data*, John Wiley, Chichester

Banerjee, A., Lumsdaine, R. L. and Stock, J. H. (1992) Recursive and Sequential Tests of the Unit-Root and Trend-Break Hypotheses: Theory and International Evidence, *Journal of Business and Economic Statistics* 10, 271–87

Barber, B. and Lyon, J. (1997) Detecting Long-Run Abnormal Stock Returns: The Empirical Power and Specifications of Test Statistics, *Journal of Financial Economics* 43, 341–72

Bassett, G. W. and Chen, H-L. (2001) Portfolio Style: Return-based Attribution Using Quantile Regression, *Empirical Economics* 26, 293–305

Bauwens, L. and Laurent, S. (2002) A New Class of Multivariate Skew Densities with Application to GARCH Models, *CORE Discussion Paper* 2002/20

Bauwens, L., Laurent, S. and Rombouts, J. V. K. (2006) Multivariate GARCH Models: A Survey, *Journal of Applied Econometrics* 21, 79–109

Bauwens, L. and Lubrano, M. (1998) Bayesian Inference on GARCH Models Using the Gibbs Sampler, *Econometrics Journal* 1(1), 23–46

Beck, T. and Levine, R. (2004) Stock Markets, Banks and Growth: Panel Evidence, *Journal of Banking and Finance* 28, 423–42

Benninga, S. (2017) *Principles of Finance with Microsoft Excel*, 3rd ed, Oxford University Press, New York

Bera, A. K. and Jarque, C. M. (1981) An Efficient Large-Sample Test for Normality of Observations and Regression Residuals, *Australian National University Working Papers in Econometrics* 40, Canberra

Bera, A. K. and Kim, S. (2002) Testing Constancy of Correlation and Other Specifications of the BGARCH Model with an Application to International Equity Returns, *Journal of Empirical Finance* 9, 171–95

Bergman, U. M. and Hansson, J. (2005) Real Exchange Rates and Switching Regimes, *Journal of International Money and Finance* 24, 121–38

Berndt, E. K., Hall, B. H., Hall, R. E. and Hausman, J. A. (1974) Estimation and Inference in Nonlinear Structural Models, *Annals of Economic and Social Measurement* 4, 653–65

Black, A., Fraser P. and Power, D. (1992) UK Unit Trust Performance 1980–1989: A Passive Time-varying Approach, *Journal of Banking and Finance* 16, 1015–

33

Black, F., Jensen, M. C. and Scholes, M. (1972) The Capital Asset Pricing Model: Some Empirical Tests, in M. C. Jensen (ed.) *Studies in the Theory of Capital Markets*, Praeger, New York

Black, F. and Scholes, M. (1973) The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81(3), 637–54

Bodie, Z., Kane, A. and Marcus, A. J. (2014) *Investments and Portfolio Management* 10th ed., McGraw-Hill, New York

Boehmer, E., Musumeci, J. and Poulsen, A. (1991) Event Study Methodology Under Conditions of Event Induced Variance, *Journal of Financial Economics* 30, 253–72

Bollerslev, T. (1986) Generalised Autoregressive Conditional Heteroskedasticity, *Journal of Econometrics* 31, 307–27

– (1990) Modelling the Coherence in Short-Run Nominal Exchange Rates: A Multivariate Generalised ARCH Model, *Review of Economics and Statistics* 72, 498–505

Bollerslev, T., Chou, R. Y. and Kroner, K. F. (1992) ARCH Modelling in Finance: A Review of the Theory and Empirical Evidence, *Journal of Econometrics* 52(5), 5–59

Bollerslev, T., Engle, R. F. and Wooldridge, J. M. (1988) A Capital-Asset Pricing Model with Time-Varying Covariances, *Journal of Political Economy* 96(1), 116–31

Bollerslev, T. and Mikkelsen, H. O. (1996) Modelling and Pricing Long Memory in Stock Market Volatility, *Journal of Econometrics* 73, 151–84

Bollerslev, T. and Wooldridge, J. M. (1992) Quasi-Maximum Likelihood Estimation and Inference in Dynamic Models with Time-Varying Covariances, *Econometric Reviews* 11(2), 143–72

Box, G. E. P. and Jenkins, G. M. (1976) *Time Series Analysis: Forecasting and Control*, 2nd ed., Holden-Day, San Francisco

Box, G. E. P. and Pierce, D. A. (1970) Distributions of Residual Autocorrelations in Autoregressive Integrated Moving Average Models, *Journal of the American Statistical Association* 65, 1509–26

Boyle, P. P. (1977) Options: A Monte Carlo Approach, *Journal of Financial Economics* 4(3), 323–38

Brailsford, T. J. and Faff, R. W. (1996) An Evaluation of Volatility Forecasting Techniques, *Journal of Banking and Finance* 20, 419–38

Brealey, R. A. and Myers, S. C. (2013) *Principles of Corporate Finance*, Global ed, McGraw-Hill, New York

Breitung, J. (2000) The Local Power of Some Unit Root Tests for Panel Data, in B. Baltagi (ed.) *Nonstationary Panels, Panel Cointegration and Dynamic Panels*, Advances in Econometrics 15, 161–78, JAI Press, Amsterdam

Breitung, J. and Das, S. (2005) Panel Unit Root Tests under Cross-Sectional Dependence, *Statistica Neerlandica* 59, 414–33

Breitung, J. and Pesaran, M. H. (2008) Unit Roots and Cointegration in Panels, in L. Matyas and P. Sevestre (eds.) *The Econometrics of Panel Data*, 3rd ed, Springer-Verlag, Berlin

Brock, W. A., Dechert, D., Scheinkman, H. and LeBaron, B. (1996) A Test for Independence Based on the Correlation Dimension, *Econometric Reviews* 15, 197–235

Brock, W. A., Hsieh, D. A. and LeBaron, B. (1991) *Nonlinear Dynamics, Chaos, and Instability: Statistical Theory and Economic Evidence*, MIT Press, Cambridge, MA

Brockwell, P.J. and Davis, R.A. (1991) *Time Series: Theory and Methods* Springer-Verlag, New York.

Brooks, C. (1996) Testing for Nonlinearity in Daily Pound Exchange Rates, *Applied Financial Economics* 6, 307–17

– (1997) GARCH Modelling in Finance: A Review of the Software Options, *Economic Journal* 107(443), 1271–6

– (1998) Forecasting Stock Return Volatility: Does Volume Help?, *Journal of Forecasting* 17, 59–80

– (2001) A Double Threshold GARCH Model for the French Franc/German Mark Exchange Rate, *Journal of Forecasting* 20, 135–43

Brooks, C., Burke, S. P. and Persand, G. (2001) Benchmarks and the Accuracy of GARCH Model Estimation, *International Journal of Forecasting* 17, 45–56

– (2003) Multivariate GARCH Models: Software Choice and Estimation Issues, *Journal of Applied Econometrics* 18, 725–34

Brooks, C., Burke, S. P., Heravi, S. and Persand, G. (2005) Autoregressive Conditional Kurtosis, *Journal of Financial Econometrics* 3(3), 399–421

Brooks, C., Černý, A. and Miffre, J. (2012) Optimal Hedging with Higher Moments, *Journal of Futures Markets* 32, 909–44

Brooks, C., Clare, A. D., Dalle Molle, J. W. and Persand, G. (2005) A Comparison of Extreme Value Approaches for Determining Value at Risk, *Journal of Empirical Finance* 12, 339–52

Brooks, C., Clare, A. D. and Persand, G. (2000) A Word of Caution on Calculating Market-Based Minimum Capital Risk Requirements, *Journal of Banking and Finance* 14(10), 1557–74

Brooks, C. and Garrett, I. (2002) Can We Explain the Dynamics of the UK FTSE 100 Stock and Stock Index Futures Markets?, *Applied Financial Economics* 12(1), 25–31

Brooks, C. and Henry, O. T. (2000) Can Portmanteau Model Nonlinearity Tests Serve as General Model Mis-Specification Diagnostics? Evidence from Symmetric and Asymmetric GARCH Models, *Economics Letters* 67, 245–51

Brooks, C., Henry, O. T. and Persand, G. (2002) The Effect of Asymmetries on Optimal Hedge Ratios, *Journal of Business* 75(2), 333–52

Brooks, C. and Heravi, S. (1999) The Effect of Mis-Specified GARCH Filters on the Finite Sample Distribution of the BDS Test, *Computational Economics* 13, 147–62

Brooks, C. and Hinich, M. J. (1999) Cross-Correlations and Cross-Bicorrelations in Sterling Exchange Rates, *Journal of Empirical Finance* 6(4), 385–404

Brooks, C. and Persand, G. (2001a) Seasonality in Southeast Asian Stock Markets: Some New Evidence on Day-of-the-Week Effect, *Applied Economics Letters* 8, 155–8

– (2001b) The Trading Profitability of Forecasts of the Gilt–Equity Yield Ratio, *International Journal of Forecasting* 17, 11–29

Brooks, C. and Rew, A. G. (2002) Testing for Non-Stationarity and Cointegration Allowing for the Possibility of a Structural Break: An Application to EuroSterling Interest Rates, *Economic Modelling* 19, 65–90

Brooks, C., Rew, A. G. and Ritson, S. (2001) A Trading Strategy Based on the Lead–Lag Relationship Between the FTSE 100 Spot Index and the LIFFE Traded FTSE Futures Contract, *International Journal of Forecasting* 17, 31–44

Brooks, C. and Tsolacos, S. (1999) The Impact of Economic and Financial Factors on UK Property Performance, *Journal of Property Research* 16(2), 139–52

Brown, S. J. and Warner, J. B. (1980) Measuring Security Price Performance, *Journal of Financial Economics* 8, 205–58

– (1985) Using Daily Stock Returns: The Case of Event Studies, *Journal of Financial Economics* 14, 3–31

Campbell, J. Y., Lo, A. W. and MacKinlay, A. C. (1997) *The Econometrics of Financial Markets*, Princeton University Press, Princeton, NJ

Campbell, J. Y. and Shiller, R. J. (1988) Interpreting Cointegrated Models, *Journal of Economic Dynamics and Control* 12, 503–22

– (1991) Yield Spreads and Interest Rate Movements: a Bird's Eye View, *Review of Economic Studies* 58, 495–514

Cantor, R. and Packer, F. (1996) Determinants and Impacts of Sovereign Credit Ratings, *Journal of Fixed Income* 6, 76–91

Carhart, M. (1997) On Persistence in Mutual Fund Performance, *Journal of Finance* 52, 57–82

Cecchetti, S. G., Cumby, R. E. and Figlewski, S. (1988) Estimation of the Optimal Futures Hedges, *Review of Economics and Statistics* 70(4), 623–30

Chappell, D., Padmore, J., Mistry, P. and Ellis, C. (1996) A Threshold Model for the French Franc/Deutschmark Exchange Rate, *Journal of Forecasting* 15, 155–64

Chen, B. (1995) Long-Run Purchasing Power Parity: Evidence from Some European Monetary System Countries, *Applied Economics* 27, 377–83

Chen, N-F., Roll, R. and Ross, S. A. (1986) Economic Forces and the Stock Market, *Journal of Business* 59(3), 383–403

Chernozhukov, V. and Umantsev, L. (2001) Conditional Value-at-Risk: Aspects of Modelling and Estimation, *Empirical Economics* 26(1), 271–92

Chib, S. and Greenberg, E. (1996) Markov Chain Monte Carlo Simulation Methods in Econometrics, *Econometric Theory* 12, 409–31

Choi, I. (2001) Unit Root Tests for Panel Data, *Journal of International Money and Finance* 20, 249–72

Christiano, L. J. (1992) Searching for a Break in GNP, *Journal of Business and Economic Statistics* 10, 237–50

Christopoulos, D. K. and Tsionas, E. G. (2004) Financial Development and Economic Growth: Evidence from Panel Unit Root and Cointegration Tests, *Journal of Development Economics* 73, 55–74

Chu, K.-Y. (1978) Short-Run Forecasting of Commodity Prices: An Application of Autoregressive Moving Average Models, *IMF Staff Papers* 25, 90–111

Chu, S.-H. and Freund, S. (1996) Volatility Estimation for Stock Index Options: A GARCH Approach, *Quarterly Review of Economics and Finance* 36(4), 431–50

Clare, A. D., Maras, M. and Thomas, S. H. (1995) The Integration and Efficiency of International Bond Markets, *Journal of Business Finance and Accounting* 22(2), 313–22

Clare, A. D. and Thomas, S. H. (1995) The Overreaction Hypothesis and the UK Stock Market, *Journal of Business Finance and Accounting* 22(7), 961–73

Cochrane, D. and Orcutt, G. H. (1949) Application of Least Squares Regression to Relationships Containing Autocorrelated Error Terms, *Journal of the American Statistical Association* 44, 32–61

Cochrane, J. H. (2005) *Asset Pricing*, Princeton University Press, Princeton, NJ

Corrado, C. J. (2011) Event Studies: A Methodology Review, *Accounting and Finance* 51, 207–34

Cuthbertson, K. and Nitzsche, D. (2004) *Quantitative Financial Economics*, 2nd ed, John Wiley, Chichester, UK

Dacco, R. and Satchell, S. E. (1999) Why do Regime Switching Models Forecast so Badly?, *Journal of Forecasting* 18, 1–16

Danielsson, J. (1998) Multivariate Stochastic Volatility Models: Estimation and

Comparison with VGARCH Models, *Journal of Empirical Finance* 5, 155–73

Danielsson, J.B.N. and de Vries, C.G. (1997) Tail Index and Quantile Estimation with Very High Frequency Data, *Journal of Empirical Finance* 4, 241–57

Davidson, R. and MacKinnon, J. G. (1981) Several Tests for Model Specification in the Presence of Alternative Hypotheses, *Econometrica* 49(3), 781–94

Davison, A. C. and Hinkley, D. V. (1997) *Bootstrap Methods and their Application*, Cambridge University Press, Cambridge, UK

Day, T. E. and Lewis, C. M. (1992) Stock Market Volatility and the Information Content of Stock Index Options, *Journal of Econometrics* 52, 267–87

DeBondt, W. F. M. and Thaler, R. H. (1985) Does the Stock Market Overreact?, *Journal of Finance* 40, 793–805

– (1987) Further Evidence on Investor Overreaction and Stock Market Seasonality, *Journal of Finance* 42, 567–80

De Haan, L. and Resnick, S.I. (1980) A Simple Estimate for the Index of a Stable Distribution, *Journal of the Royal Statistical Society B* 42, 83–7

de Haas, R. and van Lelyveld, I. (2006) Foreign Banks and Credit Stability in Central and Eastern Europe. A Panel Data Analysis, *Journal of Banking and Finance* 30, 1927–52

des Rosiers, F. and Thériault, M. (1996) Rental Amenities and the Stability of Hedonic Prices: A Comparative Analysis of Five Market Segments, *Journal of Real Estate Research* 12(1), 17–36

Dickey, D. A. and Fuller, W. A. (1979) Distribution of Estimators for Time Series Regressions with a Unit Root, *Journal of the American Statistical Association* 74, 427–31

– (1981) Likelihood Ratio Statistics for Autoregressive Time Series with a Unit Root, *Econometrica* 49(4), 1057–72

Dickey, D. A. and Pantula, S. (1987) Determining the Order of Differencing in Autoregressive Processes, *Journal of Business and Economic Statistics* 5, 455–61

Dielman, T. E. (1986) A Comparison of Forecasts from Least Absolute Value and Least Squares Regression, *Journal of Forecasting* 5, 189–95

Dimson, E. and Marsh, P. (1990) Volatility Forecasting Without Data-Snooping, *Journal of Banking and Finance* 14, 399–421

Ding, Z., Granger, C. W. J. and Engle, R. F. (1993) A Long Memory Property of Stock Market Returns and a New Model, *Journal of Empirical Finance* 1, 83–106

Doan, T. (1994) *Regression Analysis of Time Series User Manual*, 4th ed, Estima, Evanston, IL

Doan, T., Litterman, R. and Sims, C. (1984) Forecasting and Conditional

Projection Using Realistic Prior Distributions, *Econometric Reviews* 3, 1–100

Dougherty, C. (1992) *Introduction to Econometrics*, Oxford University Press, Oxford

Dowd, K. (1998) *Beyond Value at Risk: the New Science of Risk Management*, Wiley, Chichester, UK

– (2002) *An Introduction to Market Risk Measurement* Wiley, Chichester, UK

Duffie, D. (1996) *Dynamic Asset Pricing Theory*, 2nd ed, Princeton University Press, Princeton, NJ

Dufour, A. and Engle, R. F. (2000) Time and the Price Impact of a Trade, *Journal of Finance* 55(6), 2467–98

Durbin, J. and Koopman, S.J. (2001) *Time-Series Analysis by State Space Methods* Oxford Statistical Science Series, Oxford University Press, New York

Durbin, J. and Watson, G. S. (1951) Testing for Serial Correlation in Least Squares Regression, *Biometrika* 38, 159–71

Efron, B. (1979) Bootstrap Methods: Another Look at the Jackknife, *Annals of Statistics* 7(1), 1–26

– (1982) *The Jackknife, the Bootstrap and Other Resampling Plans*, Society for Industrial and Applied Mathematics, Philadelphia, PA

Embrechts, P., Klüppelberg, C. and Mikosch, T. (2013) *Modelling Extremal Events: for Insurance and Finance* (*Stochastic Modelling and Applied Probability Volume* 33), Springer Verlag, Heidelberg

Embrechts, P., Lindskog, P. and McNeil, A. J. (2003) Modelling Dependence with Copulas amd Applications to Risk Management, in S. T. Rachev (ed.) *Handbook of Heavy Tailed Distributions in Finance*, Elsevier, Amsterdam

Engel, C. and Hamilton, J. D. (1990) Long Swings in the Dollar: Are they in the Data and Do Markets Know It?, *American Economic Review* 80(4), 689–713

Engle, R. F. (1982) Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of United Kingdom Inflation, *Econometrica* 50(4), 987–1007

– (2002) Dynamic Conditional Correlation. A Simple Class of Multivariate GARCH Models, *Journal of Business and Economic Statistics* 20, 339–50

Engle, R. F. and Granger, C. W. J. (1987) Co-Integration and Error Correction: Representation, Estimation and Testing, *Econometrica* 55, 251–76

Engle, R. F. and Kroner, K. F. (1995) Multivariate Simultaneous Generalised GARCH, *Econometric Theory* 11, 122–50

Engle, R. F., Lilien, D. M. and Robins, R. P. (1987) Estimating Time Varying Risk Premia in the Term Structure: The ARCH-M Model, *Econometrica* 55(2), 391–407

Engle, R. F. and Manganelli, S. (2004) CAViaR: Conditional Autoregressive Value at Risk by Regression Quantile, *Journal of Business and Economic*

Statistics 22(4), 367–81

Engle, R. F. and Ng, V. K. (1993) Measuring and Testing the Impact of News on Volatility, *Journal of Finance* 48, 1749–78

Engle, R. F., Ng, V. K. and Rothschild, M. (1990) Asset Pricing with a Factor-ARCH Covariance Structure: Empirical Estimates for Treasury Bills, *Journal of Econometrics* 45, 213–38

Engle, R. F. and Russell, J. R. (1998) Autoregressive Conditional Duration: A New Model for Irregularly Spaced Transaction Data, *Econometrica* 66(5), 1127–62

Engle, R. F. and Yoo, B. S. (1987) Forecasting and Testing in Cointegrated Systems, *Journal of Econometrics* 35, 143–59

Fabozzi, F. J. and Francis, J. C. (1980) Heteroscedasticity in the Single Index Model, *Journal of Economics and Business* 32, 243–8

Fair, R. C. and Shiller, R. J. (1990) Comparing Information in Forecasts from Econometric Models, *American Economic Review* 80, 375–89

Fama, E. F. (1998) Market Efficiency, Long-Term Returns and Behavioral Finance, *Journal of Financial Economics* 49, 283–306

Fama, E. F., Fisher, L., Jensen, M. C. and Roll, R. (1969) The Adjustment of Stock Prices to New Information, *International Economic Review* 10, 1–21

Fama, E. F. and French, K. R. (1992) The Cross-Section of Expected Stock Returns, *Journal of Finance* 47, 427–65

– (1993) Common Risk Factors in the Returns on Stocks and Bonds, *Journal of Financial Economics* 33, 3–53

Fama, E. F. and MacBeth, J. D. (1973) Risk, Return and Equilibrium: Empirical Tests, *Journal of Political Economy* 81(3), 607–36

Fase, M. M. G. (1973) A Principal Components Analysis of Market Interest Rates in the Netherlands, 1962–1970, *European Economic Review* 4(2), 107–34

Ferson, W.E. and Foerster, S.R. (1994) Finite Sample Properties of the Generalized Method of Moments Tests of Conditional Asset Pricing Models, *Journal of Financial Economics* 36, 29–55

Fisher, R. A. (1932) *Statistical Methods for Research Workers*, 4th ed, Oliver and Boyd, Edinburgh

Fisher, R.A. and Tippett, L.H.C. (1928) Limiting Forms of the Frequency Distribution of the Largest or Smallest Member of a Sample, *Mathematical Proceedings of the Cambridge Philosophical Society* 24(2), 180–90

Franses, P. H. and van Dijk, D. (1996) Forecasting Stock Market Volatility Using Non-Linear GARCH Models, *Journal of Forecasting* 15, 229–35

– (2000) *Non-Linear Time Series Models in Empirical Finance*, Cambridge University Press, Cambridge, UK

French, K. R. (1980) Stock Returns and the Weekend Effect, *Journal of Financial*

Economics 8(1), 55–69

Fuller, W. A. (1976) *Introduction to Statistical Time Series*, Wiley, New York

Gençay, R. and Selçuk, F. (2004) Extreme Value Theory and Value-at-Risk: Relative Performance in Emerging Markets, *International Journal of Forecasting* 20, 287–303

George, T. J. and Longstaff, F. A. (1993) Bid–Ask Spreads and Trading Activity in the S&P 100 Index Options Market, *Journal of Financial and Quantitative Analysis* 28, 381–97

Gerlow, M. E., Irwin, S. H. and Liu, T.-R. (1993) Economic Evaluation of Commodity Price Forecasting Models, *International Journal of Forecasting* 9, 387–97

Ghosh, S. K. (1991) *Econometrics: Theory and Applications*, Prentice-Hall, Englewood Cliffs, NJ

Ghysels, E., Harvey, A. C. and Renault, E. (1995) Stochastic Volatility, in G. S. Maddala and C. R. Rao (eds.) *Handbook of Statistics Volume 14*, Elsevier, Amsterdam, 119–91

Giannone, D., Lenza, M. and Primiceri, G. (2014) Prior Selection for Vector Autoregressions, *Review of Economics and Statistics* 97, 436–51

Gibbons, M. R. and Hess, P. (1981) Day of the Week Effects and Asset Returns, *Journal of Business* 54(4), 579–96

Gibson, M. S. and Boyer, B. H. (1998) Evaluating Forecasts of Correlation Using Option Pricing, *Journal of Derivatives*, Winter, 18–38

Gilbert, C. (1986) Professor Hendry's Methodology, *Oxford Bulletin of Economics and Statistics* 48, 283–307

Glosten, L. R., Jagannathan, R. and Runkle, D. E. (1993) On the Relation Between the Expected Value and the Volatility of the Nominal Excess Return on Stocks, *The Journal of Finance* 48(5), 1779–801

Gnedenko, B. (1943) Sur la distribution limite du terme maximum d'une série aléatoire, *Annals of Mathematics* 44, 423–453

Goldfeld, S. M. and Quandt, R. E. (1965) Some Tests for Homoskedasticity, *Journal of the American Statistical Association* 60, 539–47

Granger, C. W. J. (1969) Investigating Causal Relations by Econometric Models and Cross-Spectral Methods, *Econometrica* 37, 424–38

Granger, C. W. J. and Newbold, P. (1986) *Forecasting Economic Time Series* 2nd ed, Academic Press, San Diego, CA

Greene, W. H. (2002) *Econometric Analysis*, 5th ed, Prentice-Hall, Upper Saddle River, NJ

Gregory, A. Tharyan, R. and Chistidis, A. (2013) Constructing and Testing Alternative Versions of the Fama-French and Carhart Models in the UK, *Journal of Business Finance and Accounting* 40(1) and (2), 172–214

Gregory, A. W. and Hansen, B. E. (1996) A Residual-Based Test for Cointegration in Models with Regime Shifts, *Journal of Econometrics* 70, 99–126

Gujarati, D. N. (2003) *Basic Econometrics*, 4th ed, McGraw-Hill, New York

Hadri, K. (2000) Testing for Stationarity in Heterogeneous Panel Data, *Econometrics Journal* 3, 148–61

Halcoussis, D. (2005) *Understanding Econometrics*, Thomson South Western, Mason, OH

Hall, A.R. (2005) *Generalised Method of Moments* Oxford University Press, Oxford

Hamilton, J. D. (1989) A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle, *Econometrica* 57(2), 357–84

– (1990) Analysis of Time Series Subject to Changes in Regime, *Journal of Econometrics* 45, 39–70

– (1994) *Time Series Analysis*, Princeton University Press, Princeton, NJ

Handa, P. and Tiwari, A. (2006) Does Stock Return Predictability Imply Improved Asset Allocation and Performance? Evidence from the US Stock Market (1954–2002), *Journal of Business* 79, 2423–68

Hansen, B. E. (1996) Inference When a Nuisance Parameter is not Identified Under the Null Hypothesis, *Econometrica* 64, 413–30

Hansen, L. P. (1982) Large Sample Properties of Generalised Method of Moments Estimators, *Econometrica* 50, 1029–54

Hansen, P. R. and Lunde, A. (2006) Consistent Ranking of Volatility Models, *Journal of Econometrics* 131, 97–21

Harris, L. (2002) *Trading and Exchanges: Market Microstructure for Practitioners*, Oxford University Press, New York

Harris, R. I. D. (1995) *Cointegration Analysis in Econometric Modelling*, Prentice-Hall, Harlow, UK

Harris, R. D. F. and Tzavalis, E. (1999) Inference for Unit Roots in Dynamic Panels where the Time Dimension is Fixed, *Journal of Econometrics* 91, 201–26

Harvey, A.C. (1989) *Forecasting, Structural Time Series Models and the Kalman Filter* Cambridge University Press, Cambridge, UK

Harvey, A. C., Ruiz, E. and Shephard, N. (1994) Multivariate Stochastic Variance Models, *Review of Economic Studies* 61, 247–64

Harvey, C. R. and Siddique, A. (1999) Autoregressive Conditional Skewness, *Journal of Financial and Quantitative Analysis* 34(4), 465–77

– (2000) Conditional Skewness in Asset Pricing Tests, *Journal of Finance* 55, 1263–95

Hasbrouck, J. (2007) *Empirical Market Microstructure: The Institutions, Economics, and Econometrics of Securities Trading*, Oxford University Press, New York

Haushalter, G. D. (2000) Financing Policy, Basis Risk and Corporate Hedging: Evidence from Oil and Gas Producers, *Journal of Finance* 55(1), 107–52

Heckman, J. J. (1976) The Common Structure of Statistical Models of Truncation, Sample Selection and Limited Dependent Variables and a Simple Estimator for Such Models, *Annals of Economic and Social Measurement* 5, 475–92

– (1979) Sample Selection Bias as a Specification Error, *Econometrica* 47(1), 153–61

Helwege, J. and Liang, N. (1996) Is there a Pecking Order? Evidence from a Panel of IPO Firms, *Journal of Financial Economics* 40, 429–58

Hendry, D. F. (1980) Econometrics – Alchemy or Science?, *Economica* 47, 387–406

Hendry, D. F. and Juselius, K. (2000) Explaining Cointegration Analysis: Part I, *Energy Journal* 21, 1–42

Hendry, D. F. and Mizon, G. E. (1978) Serial Correlation as a Convenient Simplification, Not a Nuisance: A Comment on a Study of the Demand for Money by The Bank of England, *Economic Journal* 88, 549–63

Hendry, D. F. and Richard, J. F. (1982) On the Formulation of Empirical Models in Dynamic Econometrics, *Journal of Econometrics* 20, 3–33

Hill, B. M. (1975) A Simple General Approach to Inference About the Tail of a Distribution, *Annals of Statistics* 3, 1163–74

Hill, C. W., Griffiths, W. and Judge, G. (1997) *Undergraduate Econometrics*, Wiley, New York

Hinich, M. J. (1982) Testing for Gaussianity and Linearity of a Stationary Time Series, *Journal of Time Series Analysis* 3(3), 169–76

– (1996) Testing for Dependence in the Input to a Linear Time Series Model, *Journal of Nonparametric Statistics* 6, 205–21

Hinich, M. J. and Patterson, D. M. (1985) Evidence of Nonlinearity in Daily Stock Returns, *Journal of Business and Economic Statistics* 3(1), 69–77

Hodgson, D. J., Linton, O. B. and Vorkink, K. (2004) Testing Forward Exchange Rate Unbiasedness Efficiently: A Semiparametric Approach, *Journal of Applied Economics* 7, 325–53

Hollstein, F. and Prokopczuk, M. (2016) Estimating beta, *Journal of Financial and Quantitative Analysis* 51(4), 1437–66

Hosking, J.R.M. and Wallis, J.R. (1987) Parameter and Quantile Estimation for the Generalised Pareto Distribution, *Technometrics* 29, 339–49

Hsiao, C. (2003) *Analysis of Panel Data*, 2nd ed, Cambridge University Press, Cambridge, UK

Hsieh, D. A. (1993) Implications of Nonlinear Dynamics for Financial Risk Management, *Journal of Financial and Quantitative Analysis* 28(1), 41–64

Huisman, R., Koedijk, K.G., Kool, C.J.M. and Palm, F. (2001) Tail Index Estimates in Small Samples, *Journal of Business and Economic Statistics* 19, 208–16

Hull, J. C. (2017) *Options, Futures and Other Derivatives*, 9th edn, Pearson, Harlow, UK

Hull, J. C. and White, A. D. (1987) The Pricing of Options on Assets with Stochastic Volatilities, *Journal of Finance* 42(2), 281–300

Hung, C.-H., Shackleton, M. and Xu, X. (2004) CAPM, Higher Co-Moment and Factor Models of UK Stock Returns, *Journal of Business Finance and Accounting* 31(1–2), 87–112

Im, K. S., Pesaran, M. H. and Shin, Y. (2003) Testing for Unit Roots in Heterogeneous Panels, *Journal of Econometrics* 115, 53–74

Ito, T. (1988) Use of (Time-Domain) Vector Autoregressions to Test Uncovered Interest Parity, *Review of Economics and Statistics* 70(2), 296–305

Jacquier, E., Polson, N. G. and Rossi, P. (1995) Stochastic Volatility: Univariate and Multivariate Extensions, *Mimeo*, Cornell University

Jaffe, J. and Westerfield, R. (1985) Patterns in Japanese Common Stock Returns: Day of the Week and Turn of the Year Effects, *Journal of Financial and Quantitative Analysis* 20(2), 261–72

Jaganathan, R., Skoulakis, G. and Wang, Z. (2002) Generalized Method of Moments: Applications in Finance, *Journal of Business and Economic Statistics* 20(4), 470–81

– (2010) *The Analysis of the Cross-Section of Security Returns*, in Y. Ait-Sahalia, and L. Hansen, (eds.), *Handbook of Financial Econometrics, Volume 2*, Elsevier, Amsterdam.

Jalles, J.T. (2009) Structural Time Series Models and the Kalman Filter: A Concise Review, *FEUNL Working Paper Series* WP541, Universidade Nova de Lisboa, Faculdade de Economia, also available on SSRN

Jensen, M. C. (1968) The Performance of Mutual Funds in the Period 1945–1964, *Journal of Finance* 23, 389–416

– (1978) Some Anomalous Evidence Regarding Market Efficiency, *Journal of Financial Economics* 6, 95–101

Johansen, S. (1988) Statistical Analysis of Cointegrating Vectors, *Journal of Economic Dynamics and Control* 12, 231–54

Johansen, S. and Juselius, K. (1990) Maximum Likelihood Estimation and Inference on Cointegration with Applications to the Demand for Money, *Oxford Bulletin of Economics and Statistics* 52, 169–210

Jorion, P. (2006) *Value at Risk*, 3rd ed, McGraw-Hill, New York

Kalman, R.E. (1960) A New Approach to Linear Filtering and Prediction Problems, *Journal of Basic Engineering* 82(1), 35–45

Kalman, R.E. and Bucy, R.S. (1961) New Results in Linear Filtering and Prediction Theory, *Journal of Basic Engineering* 83(1), 95–108

Kao, C. D. (1999) Spurious Regression and Residual-Based Tests for Cointegration in Panel Data, *Journal of Econometrics* 90, 1–44

Keim, D. B. and Stambaugh, R. F. (1984) A Further Investigation of the Weekend Effect in Stock Returns, *Journal of Finance* 39(3), 819–35

Kennedy, P. (2003) *Guide to Econometrics*, 5th ed, Blackwell, Malden, MA

Kim, S.-J., Moshirian, F. and Wu, E. (2005) Dynamic Stock Market Integration Driven by the European Monetary Union: An Empirical Analysis, *Journal of Banking and Finance* 29(10), 2475–502

Koenker, R. (2005) *Quantile Regression*, Cambridge University Press, Cambridge, UK

Koenker, R. and Bassett, G. (1978) Regression Quantiles, *Econometrica* 46, 33–50

Koenker, R. and Hallock, K. F. (2001) Quantile Regression, *Journal of Economic Perspectives* 15(4), 143–56

Koopmans, T. C. (1937) *Linear Regression Analysis of Economic Time Series*, Netherlands Economics Institute, Haarlem

Kräger, H. and Kugler, P. (1993) Nonlinearities in Foreign Exchange Markets: A Different Perspective, *Journal of International Money and Finance* 12, 195–208

Kroner, K. F. and Ng, V. K. (1998) Modelling Asymmetric Co-movements of Asset Returns, *Review of Financial Studies* 11, 817–44

Kroner, K. F. and Sultan, S. (1993) Time-Varying Distributions and Dynamic Hedging with Foreign Currency Futures, *Journal of Financial and Quantitative Analysis* 28(4), 535–51

Kwaitkowski, D., Phillips, P. C. B., Schmidt, P. and Shin, Y. (1992) Testing the Null Hypothesis of Stationarity Against the Alternative of a Unit Root, *Journal of Econometrics* 54, 159–78

Larsson, R., Lyhagen, J. and Lothgren, M. (2001) Likelihood-Based Cointegration Tests in Heterogeneous Panels, *Econometrics Journal* 4, 109–42

Leamer, E. E. (1978) *Specification Searches*, John Wiley, New York

– (1985) Vector Autoregressions for Causal Interference, in K. Brunner and A. Meltzer (eds.), *Understanding Monetary Regimes*, Cambridge University Press, Cambridge, UK, 255–304

Leitch, G. and Tanner, J. E. (1991) Economic Forecast Evaluation: Profit Versus the Conventional Error Measures, *American Economic Review* 81(3), 580–90

Levin, A., Lin, C. and Chu, C. (2002) Unit Root Tests in Panel Data: Asymptotic and Finite-Sample Properties, *Journal of Econometrics* 108, 1–24

Levine, D. (2009) Modelling Tail Behavior with Extreme Value theory, *Risk Management*, Society of Actuaries 17, 15–18

Leybourne, S. J., Mills, T. C. and Newbold, P. (1998) Spurious Rejections by Dickey–Fuller Tests in the Presence of a Break under the Null, *Journal of Econometrics* 87, 191–203

Ljung, G. M. and Box, G. E. P. (1978) On a Measure of Lack of Fit in Time Series Models, *Biometrika* 65(2), 297–303

Lo, A. W. and MacKinlay, C. A. (1990) Data-Snooping Biases in Tests of Financial Asset Pricing Models, *Review of Financial Studies* 3, 431–67

Lumsdaine, R. L. and Papell, D. H. (1997) Multiple Trend Breaks and the Unit Root Hypothesis, *Review of Economics and Statistics* 79 (2), 212–18

Lütkepohl, H. (1991) *Introduction to Multiple Time Series Analysis*, Springer-Verlag, Berlin

Lyon, J., Barber, B. and Tsai, C. (1999) Improved Methods of Tests of Long-Horizon Abnormal Stock Returns, *Journal of Finance* 54, 165–201

MacKinlay, A. C. (1997) Event Studies in Economics and Finance, *Journal of Economic Literature* 55, 13–39

Maddala, G. S. (1983) *Limited-Dependent and Quantitative Variables in Econometrics*, Cambridge University Press, Cambridge, UK

Maddala, G. S. and Kim, I-M. (1999) *Unit Roots, Cointegration and Structural Change*, Cambridge University Press, Cambridge

Maddala, G. S. and Wu, S. (1999) A Comparative Study of Unit Root Tests with Panel Data and a New Simple Test, *Oxford Bulletin of Economics and Statistics* 61, 631–52

Madhavan, A. (2000) Market Microstructure: A Survey, *Journal of Financial Markets* 3, 205–58

Makridakis, S. (1993) Accuracy Measures: Theoretical and Practical Concerns, *International Journal of Forecasting* 9, 527–9

Makridakis, S. and Hibon, M. (1995) Evaluating Accuracy (or Error) Measures, *INSEAD Working Paper* 95/18/TM

Matthews, K., Murinde, V. and Zhao, T. (2007) Competitive Conditions among the Major British Banks, *Journal of Banking and Finance* 31(7), 2025–42

McCue, T. E. and Kling, J. L. (1994) Real Estate Returns and the Macroeconomy: Some Empirical Evidence from Real Estate Investment Trust Data, 1972–1991, *Journal of Real Estate Research* 9(3), 277–87

McCulloch, J. H. (1987) US Government Term Structure Data, *Mimeo*. Ohio State

University

McNees, S. K. (1986) Forecasting Accuracy of Alternative Techniques: A Comparison of US Macroeconomic Forecasts, *Journal of Business and Economic Statistics* 4(1), 5–15

McNeil, A.J. (1998) Calculating Quantile Risk Measures for Financial Return Series Using Extreme Value Theory, *Working Paper*, ETHZ Zentrum, Zurich

McNeil, A.J. and Frey, R. (2000) Estimation of Tail-Related Risk Measures for Heteroscedastic Financial Time Series: An Extreme Value Approach, *Journal of Empirical Finance* 7, 271–300

Mills, T. C. and Markellos, R. N. (2008) *The Econometric Modelling of Financial Time Series*, 3rd ed, Cambridge University Press, Cambridge, UK

Mills, T. C. and Mills, A. G. (1991) The International Transmission of Bond Market Movements, *Bulletin of Economic Research* 43, 273–82

Mitchell, M. and Stafford, E. (2000) Managerial Decisions and Long-Term Stock Price Performance, *Journal of Business* 73, 287–329

Myers, R. J. and Thompson, S. R. (1989) Generalized Optimal Hedge Ratio Estimation, *American Journal of Agricultural Economics* 71(4), 858–68

Myers, S. C. (1984) The Capital Structure Puzzle, *Journal of Finance* 39, 575–92

Nelsen, R. B. (2006) *An Introduction to Copulas*, Springer-Verlag, New York

Nelson, C. R. and Plosser, C. I. (1982) Trends and Random Walks in Macroeconomic Time Series, *Journal of Monetary Economics* 10, 139–62

Nelson, D. B. (1991) Conditional Heteroskedasticity in Asset Returns: A New Approach, *Econometrica* 59(2), 347–70

Newey, W. K. and West, K. D. (1987) A Simple Positive-Definite Heteroskedasticity and Autocorrelation-Consistent Covariance Matrix, *Econometrica* 55, 703–8

O'Connell, P. G. J. (1998) The Overvaluation of Purchasing Power Parity, *Journal of International Economics* 44, 1–20

O'Hara, M. (1995) *Market Microstructure Theory*, Blackwell, Malden, MA

Osborn, D. (1990) A Survey of Seasonality in UK Macroeconomic Variables, *International Journal of Forecasting* 6(3), 327–36

Osterwald-Lenum, M. (1992) A Note with Quantiles of the Asymptotic Distribution of the ML Cointegration Rank Test Statistics, *Oxford Bulletin of Economics and Statistics* 54, 461–72

Ouliaris, S., Pagan, A.R. and Restrepo, J. (2016) *Quantitative Macroeconomic Modeling with Structural Vector Autoregressions. An EViews Implementation*, Unpublished e-book available at www.eviews.com

Pagan, A. R. and Schwert, G. W. (1990) Alternative Models for Conditional Stock Volatilities, *Journal of Econometrics* 45, 267–90

Panzar, J. C. and Rosse, J. N. (1982) Structure, Conduct and Comparative Statistics, *Bell Laboratories Economics Discussion Paper*

– (1987) Testing for 'Monopoly' Equilibrium, *Journal of Industrial Economics* 35(4), 443–56

Pearson, K. (1895) Contributions to the Mathematical Theory of Evolution II: Skew Variation, *Philosophical Transactions of the Royal Society of London, Series A* 186, 343–414.

Pedroni, P. (1999) Critical Values for Cointegration Tests in Heterogeneous Panels with Multiple Regressors, *Oxford Bulletin of Economics and Statistics* 61, 653–70

– (2004) Panel Cointegration: Asymptotic and Finite Sample Properties of Pooled Time Series Tests with an Application to the PPP Hypothesis, *Econometric Theory* 20, 597–625

Perron. P. (1989) The Great Crash, the Oil Price Shock and the Unit Root Hypothesis, *Econometrica* 57, 1361–401

– (1997) Further Evidence on Breaking Trend Functions in Macroeconomic Variables, *Journal of Econometrics* 80, 355–85

Pesaran, M. H. and Timmerman, A. (1992) A Simple Non-Parametric Test of Predictive Performance, *Journal of Business and Economic Statistics* 10(4), 461–5

Pickands III, J. (1975) Statistical Inference Using Extreme Order Statistics. *Annals of Statistics* 3, 119–31

Poon, W. P. H. (2003) Are Unsolicited Credit Ratings Biased Downward?, *Journal of Banking and Finance* 27, 593–614

Prabhala, N. R. (1997) Conditional Methods in Event-Studies and an Equilibrium Justification for Standard Event-Study Procedures, *Review of Financial Studies* 10(1), 1–38

Press, W. H., Teukolsy, S. A., Vetterling, W. T. and Flannery, B. P. (1992) *Numerical Recipes in Fortran*, Cambridge University Press, Cambridge, UK

Prokopczuk, M. and Wu, Y.Y. (2013) Estimating Term Structure Models with the Kalman Filter. In Bell, A.R., Brooks, C., and Prokopczuk, M. (eds.), *Handbook of Research Methods in Empirical Finance*, Edward Elgar, Cheltenham, UK

Quandt, R. (1960) Tests of the Hypothesis that a Linear Regression System Obeys Two Different Regimes, *Journal of the American Statistical Association* 55, 324–30

Ramanathan, R. (1995) *Introductory Econometrics with Applications*, 3rd ed, Dryden Press, Fort Worth, TX

Ramsey, J. B. (1969) Tests for Specification Errors in Classical Linear Least-Squares Regression Analysis, *Journal of the Royal Statistical Society B* 31(2), 350–71

Rao, B. B. (ed.) (1994) *Cointegration for the Applied Econonist*, Macmillan, Basingstoke, UK

Refenes, A.-P. (1995) *Neural Networks in the Capital Markets*, John Wiley, Chichester, UK

Renshaw, G. (2016) *Maths for Economics* 4th ed., Oxford University Press, Oxford

Rocco, M. (2011) Extreme Value Theory for Finance: A Survey, *Banca D'Italia Occasional Papers* 99

Ross, S. A. (1976) The Arbitrage Theory of Capital Asset Pricing, *Journal of Economic Theory* 13(3), 341–60

Runkle, D. E. (1987) Vector Autoregressions and Reality, *Journal of Business and Economic Statistics* 5(4), 437–42

Scheinkman, J. A. and LeBaron, B. (1989) Nonlinear Dynamics and Stock Returns, *Journal of Business* 62(3), 311–37

Schwarz, G. (1978) Estimating the Dimension of a Model, *Annals of Statistics* 6, 461–4

Scott, R. C., and Horvath, P. A. (1980). On the Direction of Preference for Moments of Higher Order than the Variance, *Journal of Finance* 35(4), 915–19

Shaffer, S. and DiSalvo, J. (1994) Conduct in a Banking Duopoly, *Journal of Banking and Finance* 18, 1063–82

Shanken, J. (1992) On the Estimation of Beta-Pricing Models, *Review of Financial Studies* 5, 1–33

Shea, G. (1984) Pitfalls in Smoothing Interest Rate Term Structure Data: Equilibrium Models and Spline Approximations, *Journal of Financial and Quantitative Analysis* 19(3), 253–69

– (1992) Benchmarking the Expectations Hypothesis of the Interest Rate Term Structure: An Analysis of Cointegrating Vectors, *Journal of Business and Economic Statistics* 10(3), 347–66

Shephard, N. (1996) Statistical Aspects of ARCH and Stochastic Volatility, in D. R. Cox, D. V. Hinkley and O. E. Barndorff-Nielsen (eds.), *Time Series Models: In Econometrics, Finance, and Other Fields*, Chapman and Hall, London, 1–67

Siegel, A. F. (1997) International Currency Relationship Information Revealed by Cross-Option Prices, *Journal of Futures Markets* 17, 369–84

Sims, C. A. (1972) Money, Income, and Causality, *American Economic Review* 62(4), 540–52

– (1980) Macroeconomics and Reality, *Econometrica* 48, 1–48

Stock, J. H. and Watson, M. W. (1988) Testing for Common Trends, *Journal of the American Statistical Association* 83, 1097–1107

– (2011) *Introduction to Econometrics*, 3rd ed, Pearson, Boston, MA

Sullivan, R., Timmermann, A. and White, H. (1999) Data-Snooping, Technical

Trading Rule Performance, and the Bootstrap, *Journal of Finance* 54, 1647–91

Sutcliffe, C. (1997) *Stock Index Futures: Theories and International Evidence*, 2nd ed, International Thompson Business Press, London

Swift, L. and Piff, S. (2014) *Quantitative Methods: For Business, Management and Finance*, 4th ed, Palgrave Macmillan Basingstoke, UK

Swinkels, L. and Van Der Sluis, P.J. (2006) Return-Based Style Analysis with Time-Varying Exposures, *European Journal of Finance* 12(6–7), 529–52

Taylor, M. P. (1987) Risk Premia and Foreign Exchange. A Multiple Time Series Approach to Testing Uncovered Interest Parity, *Weltwirtschaftliches Archiv* 123(4), 579–91

– (1989) Covered Interest Arbitrage and Market Turbulence, *Economic Journal* 99, 376–91

Taylor, M. P. and Sarno, L. (1998) The Behavior of Real Exchange Rates During the Post-Bretton Woods Period, *Journal of International Economics* 46(2), 281–312

Taylor, M. P. and Tonks, I. (1989) The Internationalisation of Stock Markets and the Abolition of UK Exchange Controls, *Review of Economics and Statistics* 71, 332–6

Taylor, S. J. (1986) Forecasting the Volatility of Currency Exchange Rates, *International Journal of Forecasting* 3, 159–70

– (1994) Modelling Stochastic Volatility: A Review and Comparative Study, *Mathematical Finance* 4, 183–204

Theil, H. (1966) *Applied Economic Forecasting*, North-Holland, Amsterdam

Tobin, J. (1958) Estimation of Relationships for Limited Dependent Variables, *Econometrica* 26(1), 24–36

Tong, H. (1983) *Threshold Models in Nonlinear Time Series Analysis*, Springer-Verlag, New York

– (1990) *Nonlinear Time Series: A Dynamical Systems Approach*, Oxford University Press, Oxford

Trippi, R. R. and Turban, E. (1993) *Neural Networks in Finance and Investing*, McGraw-Hill, New York

Tse, Y. K. (1995) Lead–Lag Relationship between Spot Index and Futures Price of the Nikkei Stock Average, *Journal of Forecasting* 14, 553–63

– (2000) A Test for Constant Correlations in a Multivariate GARCH Model, *Journal of Econometrics* 98, 107–27

Tse, Y. K. and Tsui, A. K. C. (2002) A Multivariate GARCH Model with Time-Varying Correlations, *Journal of Business and Economic Statistics* 20, 351–62

Van der Weide, R. (2002) GO-GARCH: a Multivariate Generalised Orthogonal GARCH Model, *Journal of Applied Econometrics* 17, 549–64

Van Eyden, R. J. (1996) *The Application of Neural Networks in the Forecasting of Share Prices*, Finance and Technology Publishing, Haymarket, Chicago, IL

Vrontos, I. D., Dellaportas, P. and Politis, D. N. (2000) Full Bayesian Inference for GARCH and EGARCH Models, *Journal of Business and Economic Statistics* 18(2), 187–98

Walter, C. and Lopez, J. (2000) Is Implied Correlation Worth Calculating? Evidence from Foreign Exchange Options, *Journal of Derivatives*, Spring, 65–81

Wang, G. H. K. and Yau, J. (2000) Trading Volume, Bid–Ask Spread and Price Volatility in Futures Markets, *Journal of Futures Markets* 20(10), 943–70

Wang, G. H. K., Yau, J. and Baptiste, T. (1997) Trading Volume, Transactions Costs in Futures Markets, *Journal of Futures Markets* 17(7), 757–80

Watsham, T. J. and Parramore, K. (2004) *Quantitative Methods in Finance*, 2nd ed, International Thompson Business Press, London

West, K. D. and Cho, D. (1995) The Predictive Ability of Several Models of Exchange Rate Volatility, *Journal of Econometrics* 69, 367–91

White, H. (1980) A Heteroskedasticity-Consistent Covariance Matrix Estimator and a Direct Test for Heteroskedasticity, *Econometrica* 48, 817–38

– (1992) *Artificial Neural Networks: Approximation and Learning Theory*, Blackwell, Malden, MA

– (2000) A Reality Check for Data Snooping, *Econometrica* 68, 1097–126

Wisniewski, M. (2013) *Mathematics for Economics*, Palgrave Macmillan, Basingstoke, UK

Wooldridge, J. M. (2010) *Econometric Analysis of Cross-section and Panel Data*, 2nd ed, MIT Press, MA

Yadav, P. K., Pope, P. F. and Paudyal, K. (1994) Threshold Autoregressive Modelling in Finance: The Price Difference of Equivalent Assets, *Mathematical Finance* 4, 205–21

Zarowin, P. (1990) Size, Seasonality and Stock Market Overreaction, *Journal of Financial and Quantitative Analysis* 25, 113–25

Zellner, A. (1962) An Efficient Method of Estimating Seemingly Unrelated Regressions and Tests for Aggregation Bias, *Journal of the American Statistical Association* 57, 348–68

Zivot, E. and Andrews, K. (1992) Further Evidence on the Great Crash, the Oil Price Shock, and the Unit Root Hypothesis, *Journal of Business and Economic Statistics* 10, 251–70

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